HINGED TILINGS

Alfinio Flores

University of Delaware

Abstract

In this article, we explore dynamical tilings formed by rigid regular polygons that are hinged to each other at common vertices, leaving polygonal empty spaces to allow for rotations. As one of the hinged tiles is rotated, the other tiles rotate, too, and the global shape of the tiling is changed. Students gain experience with rotations and translations of regular polygons as the tiling is morphed into another tiling and have opportunities to develop their visual thinking, spatial sense, and kinesthetic sense.

Keywords: GeoGebra, hinged tilings, uniform tilings, dynamical morphing

1 INTRODUCTION

For this article we form dynamic tilings using rigid polygons that are hinged together at their vertices and separated by empty spaces. Two rigid polygons are considered hinged when they have one vertex in common, and both are free to rotate around that vertex. In a tiling where the polygons are hinged to each other, when one of the rigid pieces moves, all the other rigid shapes move also around hinges at their vertices, and the whole tiling adjusts its global shape. The colored tiles are rotated and translated, maintaining the same shape and size all the time; they just change their positions and angles with respect to each other. To adjust for the movements of the rigid pieces, the shapes of the empty spaces change but remain congruent to each other.

Experimenting with hinged tilings provides students with opportunities to develop ideas about how distances and angles behave under rotations and translations, an important concept emphasized by the Common Core Standards for students in the middle grades and high school [1, p. 55, 76]. Interacting with the dynamical figures, students also learn ways in which dynamical geometry software can be used as an appropriate tool for learning mathematics. Using appropriate tools strategically is one of the standards for mathematical practice emphasized in the Common Core Standards [1, p. 8].

The kinds of tilings presented in this article are called hinged tilings or hinged tessellations [2, 7]. In the last three examples given in this article, rigid polygons are linked to each other with rigid rods that are hinged to vertices of the polygons. Early work in the kinds of transformations illustrated in this article was developed by Stuart [5]. To go beyond static printed figures, he used finger movies to convey a dynamical feeling of hinged transformations for polyhedra and mosaics.

We use GeoGebra to represent dynamic hinged tilings. Students can drag the polygons to make them rotate around the hinges so they can get a kinesthetic feeling and visually follow the changes as tilings morph into other tilings. As part of the exploration, we invite students to focus on specific angles of the empty shapes, as some will generate uniform tilings [3, 4] or other interesting arrangements. We ask students to guess what shapes will be formed for different angles of the empty shapes as they interact with the figures. Interesting examples of tilings can be reached at different stages as one tiling continuously transforms into another. Students can reverse the transformations by dragging in the opposite direction. In addition, they can see how different kinds of regular or uniform tilings are related to each other, and to other tilings that are not uniform, via these transformations. The discussions following each activity and the static figures we provide focus student attention on interesting tilings that are generated for specific angles. The discussions also provide answers to questions we pose during the activities.

2 Dynamical hinged tilings

Instructions for students

1. Open the file at `http://bit.ly/hinged1`.
2. Click and drag on red point `B` to move the yellow squares
3. The squares are hinged around a rhombus.
4. As you move `B`, the rigid squares move around but stay the same size. The empty space changes shape as the squares move.
5. When the angle at `B` is $90°$ empty squares are formed.
6. When the angle at `B` is $0°$ or $180°$ a tiling of yellow squares is formed.
7. You will interact with the other figures in similar ways, by dragging the indicated point.

2.1 Hinged squares

Activity for students

1. Open the file at `http://bit.ly/hinged2`.
2. Drag `B` and focus at special angles of the rhombus at `B`.
3. What kind of tiling do you get when the angle of the rhombus is $0°$ or $180°$?
4. What if the angle of the rhombus is $90°$?
5. What kind of tiling do you get when the angle of the rhombus is $60°$ or $120°$?
6. For an angle of $60°$ imagine each rhombus subdivided into two equilateral triangles.
7. What kind of tiling would you get?

Discussion of activity: This first tiling is formed by rigid squares hinged together. The empty spaces form rhombuses that can be changed in shape (Figure 1a). Look at the angle of the empty rhombus at the vertices of the hinges. A tiling consisting of only squares can be obtained in two ways. First, when the angle of the rhombus is $90°$, the empty space becomes a square (Figure 1b). And when the angle is $180°$ or $0°$, the empty space disappears (Figure 1c). You can drag the point to move the squares until the rhombuses have a $60°$ angle. If you divide each rhombus into two equilateral triangles you would have a uniform tiling – that is, a tessellation of the plane by regular polygons with the same configuration at each vertex – formed by squares and equilateral triangles (Figure 1d).

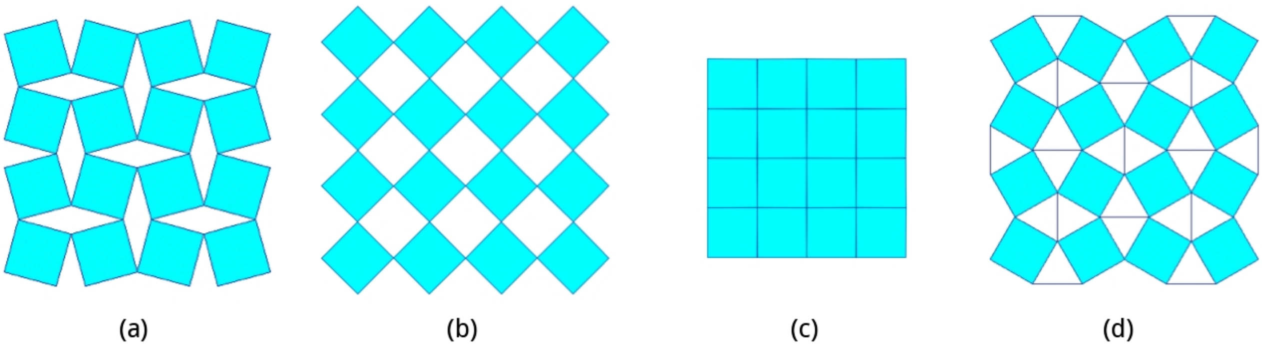


Figure 1. Hinged squares.

2.2 *Hexagons and triangles*

Activity for students

1. Open the file at `http://bit.ly/hinged3`.
2. Drag C and focus at special angles of the rhombus at C.
3. What kind of tiling do you get when the angle of the rhombus is $90°$?
4. What kind of tiling do you get when the angle of the rhombus is $60°$ or $120°$?
5. For these angles, imagine each rhombus divided into two equilateral triangles. What kind of tiling do you get?
6. What if the angle is $0°$ degrees or $180°$?

Discussion of activity: This second tiling consists of rigid regular hexagons and equilateral triangles, leaving rhombuses as empty spaces (Figure 2a). A special case of the rhombus is when the angle is $90°$, in which case the rhombus will be a square, and a uniform tiling of the plane with hexagons, squares, and triangles is obtained (Figure 2b). When the angles of the rhombus are $60°$ and $120°$, the rhombus can be divided into two equilateral triangles (Figure 2c). This will give rise to a uniform tiling of the plane formed by hexagons surrounded and separated by triangles. Another such tiling with hexagons and triangles – a mirror image of the one shown – may also be obtained. When the angle is zero, the rhombuses disappear, and a new uniform tiling is formed with only hexagons and triangles. Three hexagons surround each triangle (Figure 2d).

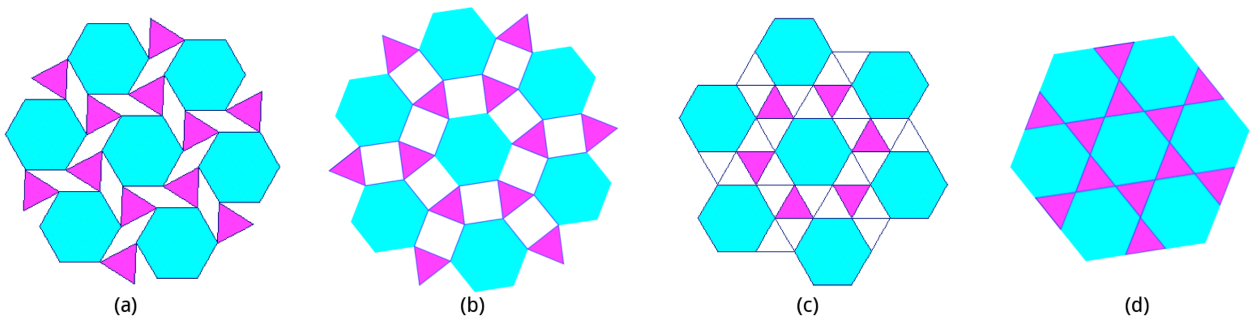


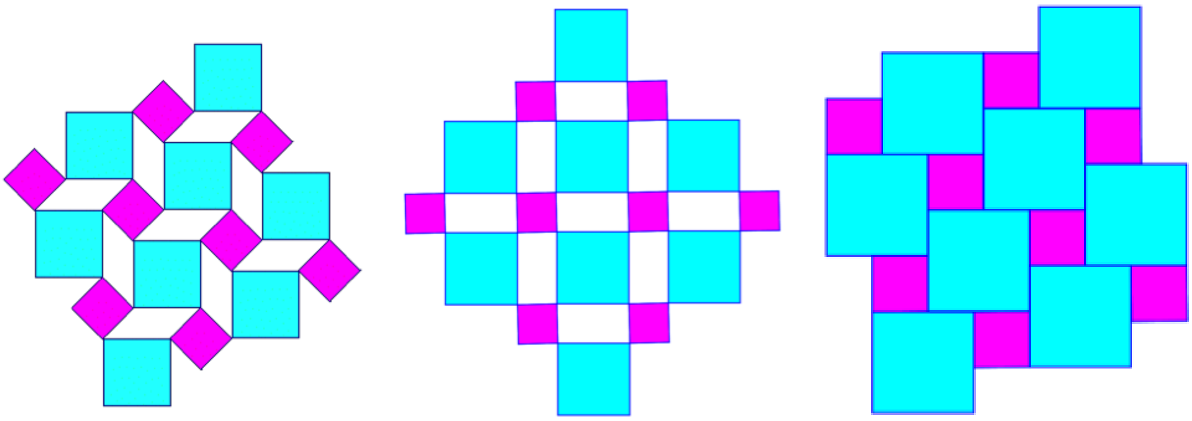
Figure 2. Hinged hexagons and triangles.

2.3 Squares of two sizes

Activity for students

1. Open the file at `http://bit.ly/hinged4`.
2. Drag `B` and experiment with different angles of the parallelogram at `B`.
3. What happens if the angle of the parallelogram is $90°$?
4. What happens if the angle is $0°$ or $180°$?

Discussion of activity: Rigid squares of two sizes form the third hinged tiling. The empty spaces are parallelograms (Figure 3a). When the angles of the empty parallelogram are $90°$, we have rectangles separating the squares (Figure 3b). When the angle is $0°$ we have a tiling of squares of two different sizes (Figure 3c).



(a) **(b)** **(c)**

Figure 3. Hinged squares of different sizes.

2.4 Centers of squares around a parallelogram

Activity for students

1. Open the file at `http://bit.ly/hinged5`.
2. Drag `B`.
3. Look at the centers of squares surrounding a parallelogram.
4. What shapes do the centers of the squares form around a parallelogram?
5. Show the grid to verify your guess.

Discussion of activity: The grid formed by the centers of the four squares on the sides of each parallelogram allows us to see that the centers of the squares around a parallelogram form a square (Figure 4). This is true for any shape of the parallelogram.

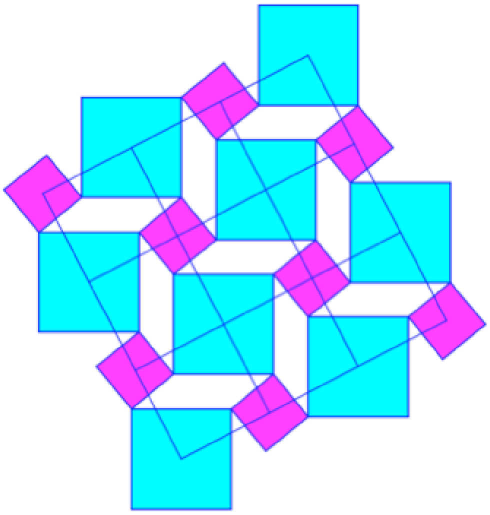


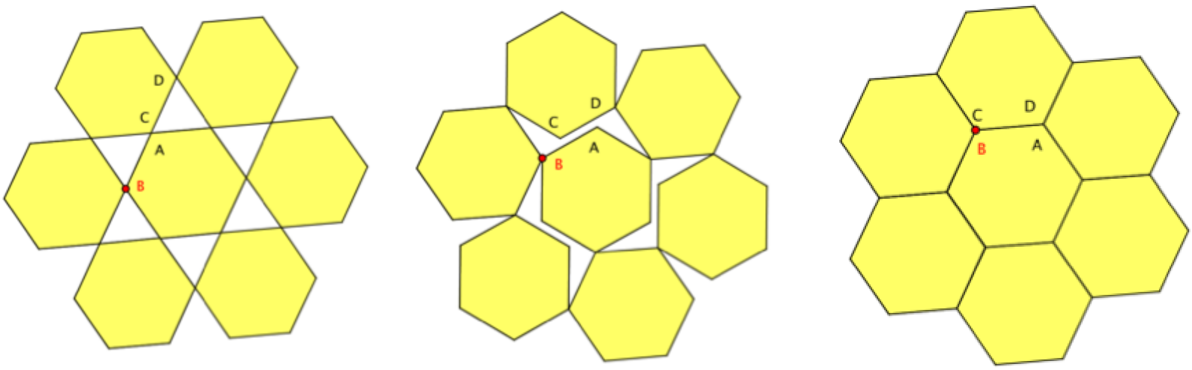
Figure 4. Centers of squares.

2.5 Hinged hexagons

Activity for students

1. Open the file at `http://bit.ly/hinged6`.
2. Drag `B`.
3. Form a regular tiling of hexagons only. Notice that hexagons can share different sides.
4. Form a uniform tiling of hexagons and triangles.
5. Observe that for the intermediate configurations you can obtain tilings that are mirror images from each other, depending on the direction of rotation.

Discussion of activity: Figure 5 shows a uniform tiling of hexagons and triangles that can be transformed to a regular tiling of hexagons. In this case, every other vertex of the hexagons is hinged. When non-hinged vertices of two hexagons touch each other, we get equilateral triangles (Figure 5a). As the figure rotates, the vertex `A` of the central hexagon that was touching vertex `C` of another hexagon moves closer to the next vertex `D`, as `B` moves closer to `C`. (Figure 5b). Finally, the empty space is closed and a regular tiling of hexagons is obtained (Figure 5c).



(a) **(b)** **(c)**

Figure 5. Hinged hexagons.

2.6 Octagons and squares

Activity for students

1. Open the file at `http://bit.ly/hinged7`.
2. Drag `C` to close and open the empty spaces.
3. Notice how rotating the central octagon clockwise or counter-clockwise will generate configurations that are mirror images of each other.
4. When the empty spaces vanish, octagons and squares form a tiling.
5. Notice how squares and octagons can be made to share different sides.

Discussion of activity: Figure 6 shows how a tiling of octagons and squares can be opened to leave some empty spaces and then be closed again into another tiling of the same kind, however, one in which polygons share different sides. Every other vertex of an octagon is hinged. For intermediate stages, the empty spaces may form configurations that have rotational symmetry, but no mirror symmetry. When the central octagon is rotated in the opposite direction, a corresponding configuration that is the mirror image is formed.

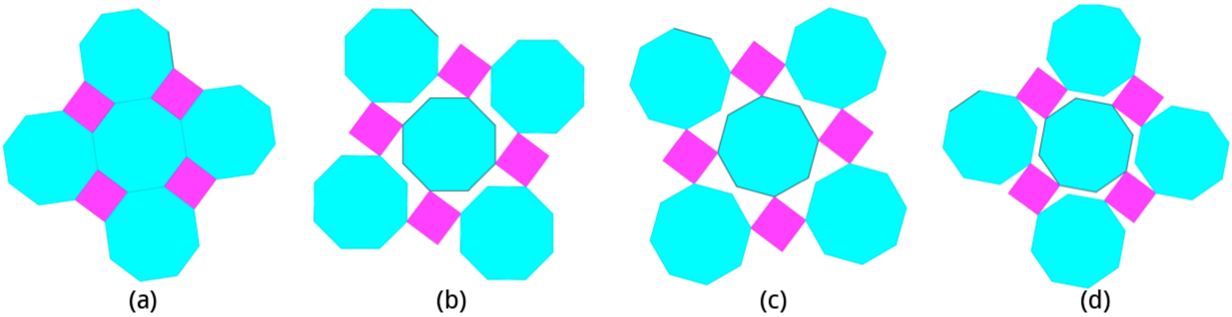


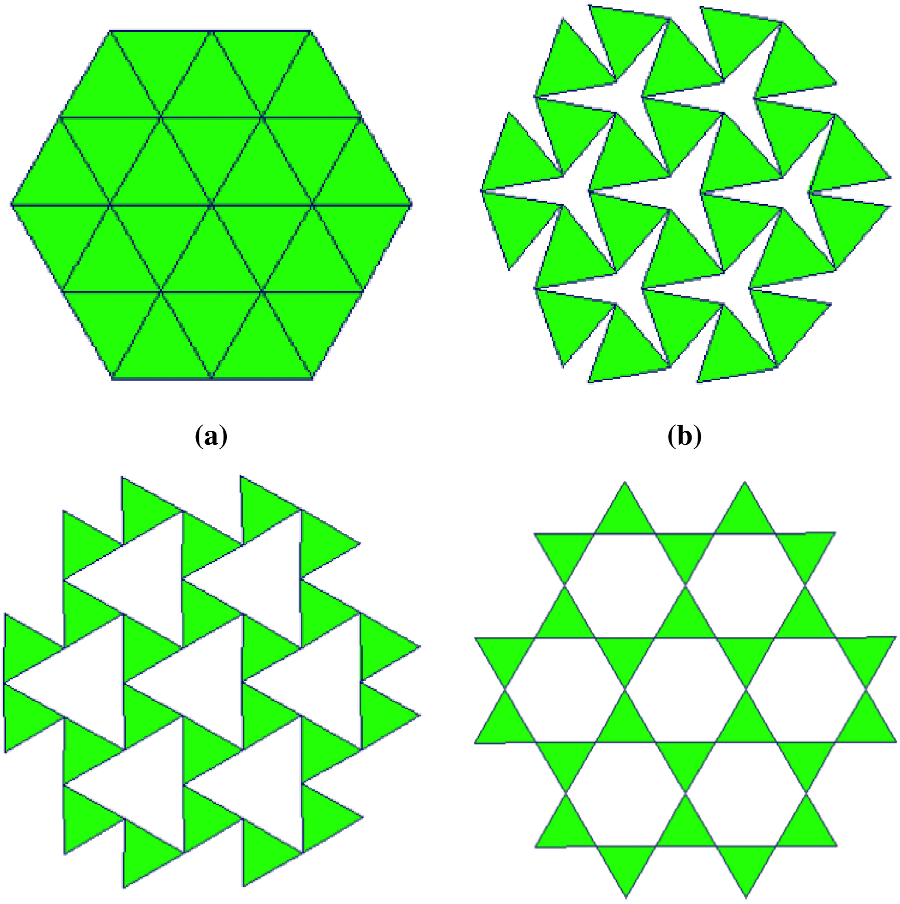
Figure 6. Hinged octagons and squares.

2.7 Hinged triangles

Activity for students

1. Open the file at `http://bit.ly/hinged8`.
2. Drag `B` to form a regular tiling of triangles.
3. Open the empty spaces to form a tiling of green triangles and three-pointed stars.
4. For what range of angles will you get three-pointed stars?
5. Form a tiling of green triangles and big white triangles.
6. What angle will give you white equilateral triangles?
7. Form a uniform tiling of green triangles and white hexagons.
8. What angle will give you regular hexagons?

Discussion of activity: In Figure 7 we transform a tiling using only triangles (Figure 7a) into a tiling of empty hexagons surrounded by triangles (Figure 7d).



(a) **(b)** **(c)** **(d)**

Figure 7. Hinged triangles.

2.8 Hexagons linked by rods

Activity for students

1. Open the file at `http://bit.ly/hinged9`.
2. Drag `B`.
3. Form a regular tiling of yellow hexagons.
4. Form a tiling of hexagons and three-pointed stars.
5. Form a tiling of hexagons and equilateral triangles.
6. Imagine each of the triangles subdivided into four smaller triangles.
7. Where have you seen this tiling of hexagons surrounded by triangles before?
8. Form a tiling of white and yellow hexagons.

Discussion of activity: In Figure 8, hexagons are linked to other hexagons with rigid rods. A regular tiling consisting of yellow hexagons (Figure 8a) can be transformed into a tiling of yellow hexagons surrounded by empty hexagons (Figure 8e). An interesting intermediate case is when the empty spaces are equilateral triangles (Figure 8c). If we subdivide these into four smaller equilateral triangles, we obtain a uniform tiling of hexagons and equilateral triangles (Figure 8d).

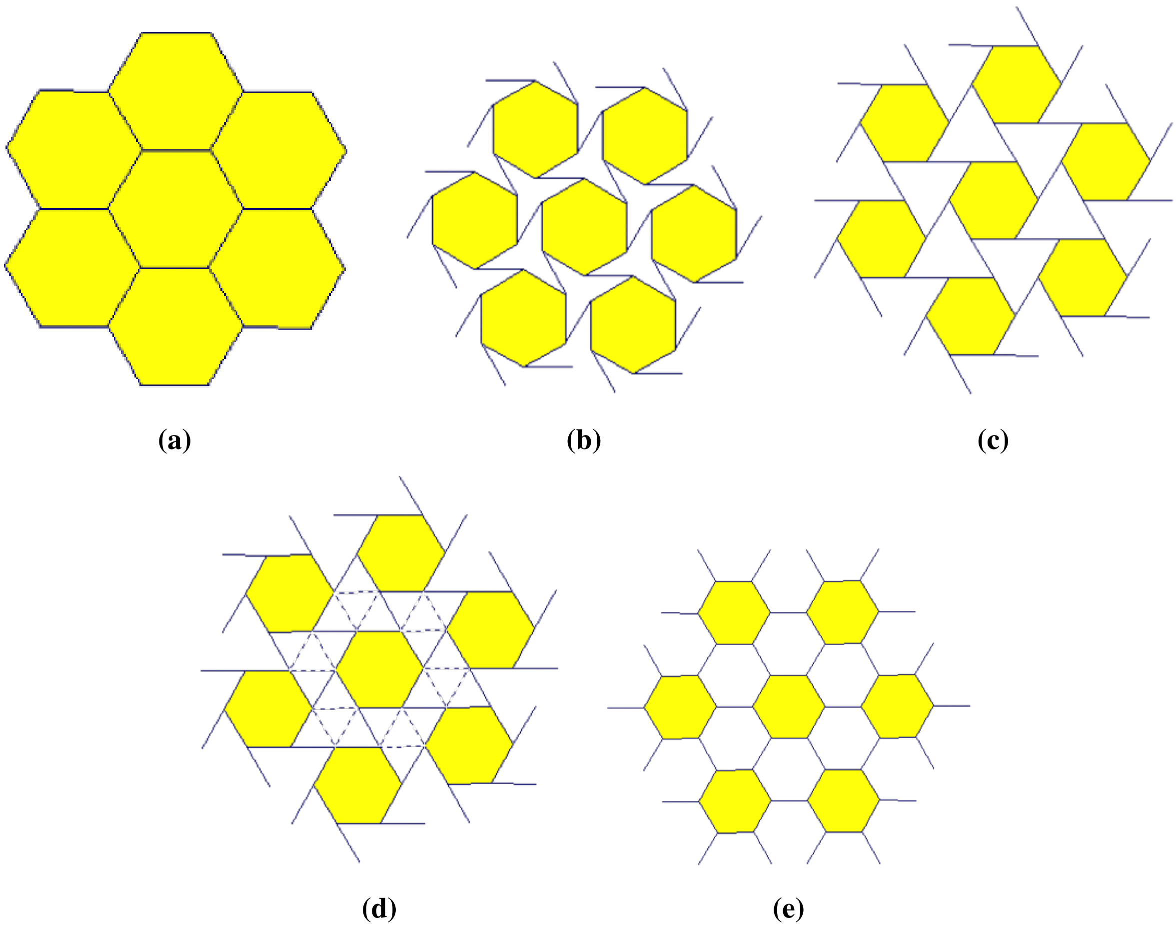


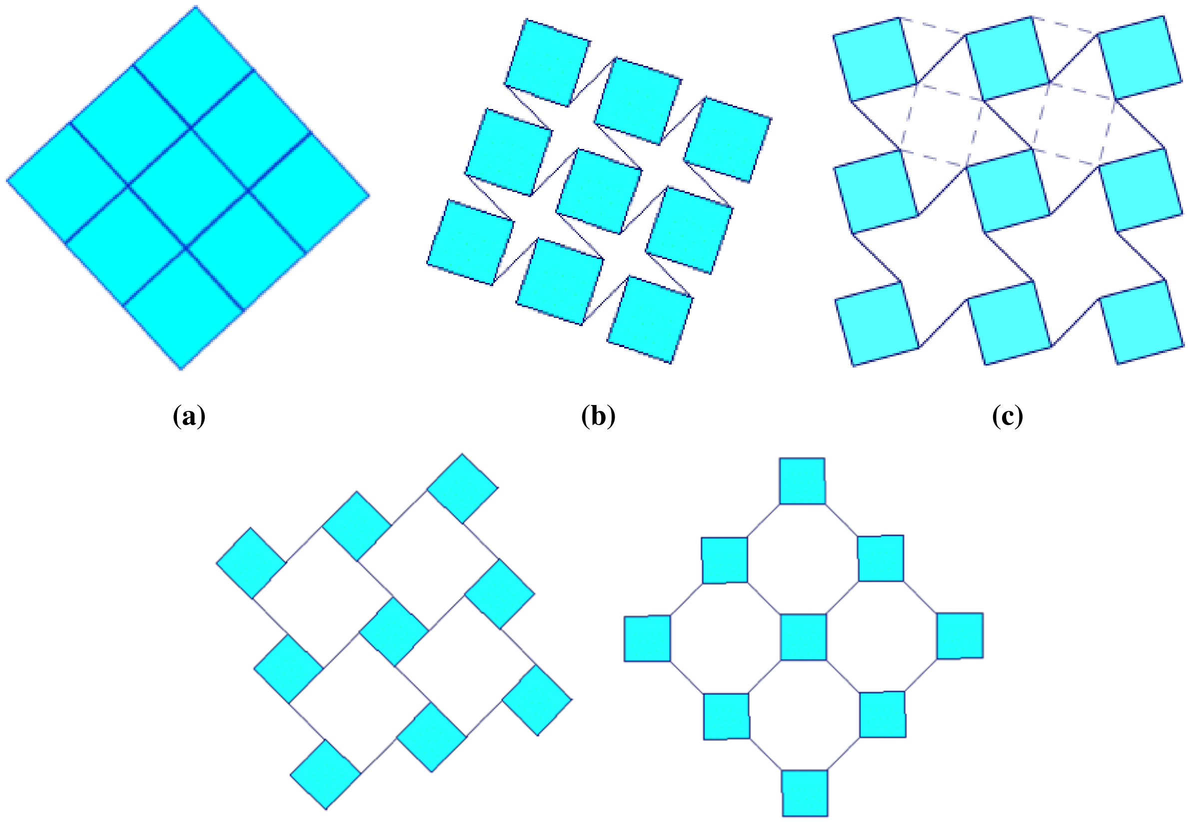
Figure 8. Hinged hexagons with rods.

2.9 Squares linked by rods

Activity for students

1. Open the file at `http://bit.ly/hinged10`.
2. Drag `C`.
3. Form a tiling of blue squares.
4. Form a tiling of blue squares and four-pointed stars. Make the angle of the point of the star 60°.
5. Mentally subdivide each star into a square and four equilateral triangles.
6. Make a tiling of blue squares and big white squares.
7. Make a tiling of squares and regular octagons.

Discussion of activity: In Figure 9, squares are linked to other squares with rigid rods. A tiling consisting only of squares (Figure 9a) can be transformed into a uniform tiling of squares and regular octagons (Figure 9e). For an angle of 60° (Figure 9c), we can imagine subdividing the four pointed star into four triangles surrounding a square. We would get the tiling formed by squares and triangles we saw in Figure 1d. An interesting intermediate case is when the empty spaces form bigger squares (Figure 9d).



(a) **(b)** **(c)**

(d) **(e)**

Figure 9. Hinged squares with rods.

2.10 Triangles linked by rods

Activity for students

1. Open the file at `http://bit.ly/hinged11`.
2. Drag `B`.
3. Form a tiling of green triangles.
4. Form a tiling of green triangles and six-pointed stars. Make the angle of the star $60°$.
5. Mentally divide each star into six equilateral triangles and a hexagon in the middle.
6. Form a tiling of triangles and big white hexagons.
7. Form a tiling of triangles and regular dodecagons.

Discussion of activity: In Figure 10, triangles are linked to other triangles with rigid rods. A tiling of equilateral triangles (Figure 10a) can be transformed into a uniform tiling of equilateral triangles and regular empty dodecagons (Figure 10e). What are some of the interesting intermediate special cases? You can divide the star with $60°$ angles (Figure 10c) into triangles and hexagons to obtain the tiling shown in Figure 8d.

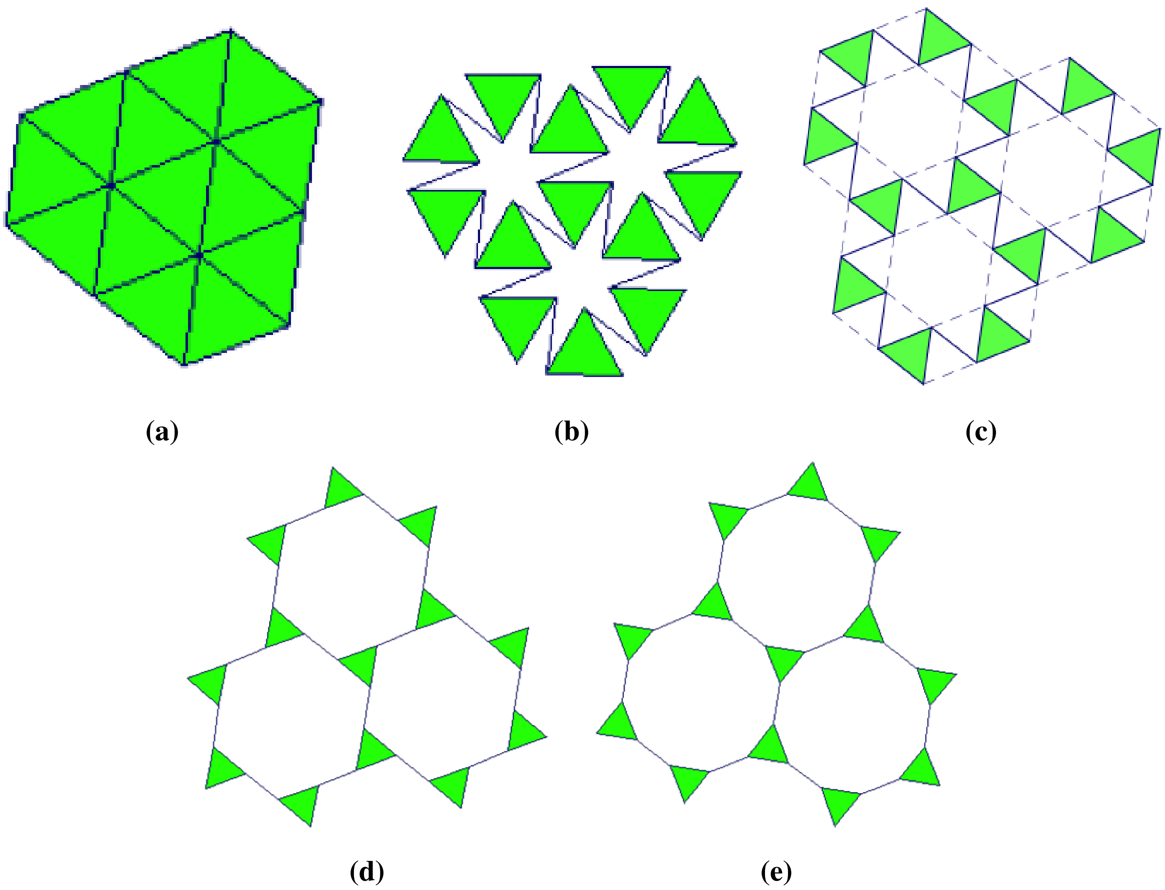


Figure 10. Hinged triangles with rods.

3 Implications for teaching

The activities discussed in the preceding sections primarily address visual thinking, spatial sense, and kinesthetic (motion) sense. Together, these form one of the six major divisions for mathematical thinking identified by Thurston [6]. The interactions with the dynamic tilings provide students with opportunities to practice and develop their abilities in this major division. Unfortunately, this way of thinking receives too little attention in the traditional school mathematics curriculum and is usually not developed in a systematic way. Although students may have previously encountered some of the regular and uniform tilings obtained in the preceding activities, it is unlikely that they have done so in a dynamic context. Uniform tilings are more commonly encountered through static figures in textbooks or static real life examples of tilings. In this sense, dynamic hinged tilings provide students with opportunities to engage a familiar topic, regular and uniform tilings, in a new way and from a different perspective, which Thurston identifies as one of the elements that contributes to the joy of doing mathematics.

The tilings used in this article could, in theory, be extended to the whole plane. The tilings will have symmetries of different types and thus be invariant with respect to specific rotations, reflections, or translations. An important step in the development of understanding of rigid transformations such as rotations is thinking about rotations of the plane, in addition to rotations of individual figures. By interacting with the dynamical tilings, students can sharpen their understanding of rotations and translations by comparing and contrasting rotations and translations of individual tiles with rotations and translations of the plane as a whole. One important element in the development of a concept is the use of examples and non-examples. Whereas the morphing of one tiling to another will leave distances and angles within each of the colored tiles invariant, it will not leave distances between different tiles invariant nor leave the angles of the empty spaces invariant and thus will not be a rigid transformation of the whole plane.

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Alfinio Flores teaches mathematics and mathematics methods courses *con ganas* at the University of Delaware. He uses technology to provide opportunities for students and teachers to explore mathematics on their own. He has conducted activities for students in schools ranging for Kindergarten to 12th grade, and conducted professional development sessions for teachers of grades K-16.