HIDDEN PROPERTIES OF THE EQUILATERAL TRIANGLE

Moshe Stupel [*], Ruti Segal [†], Alfinio Flores [‡]

Abstract

The equilateral triangle provides a rich context for students and teachers to explore and discover geometrical relations using GeoGebra. In this paper, we provide teachers with interactive applets to use in their classrooms to support student conjecturing regarding properties of the equilateral triangle. Proofs of the properties are then presented. Proofs make use of theorems in geometry, trigonometry, coordinate geometry, as well as inequalities about numbers.

Keywords: geometry, proof, inquiry-based teaching

1 INTRODUCTION

We present surprising and often overlooked properties of a familiar shape, the equilateral triangle. By definition, an equilateral triangle is a triangle with three sides of equal length. From this definition, in Euclidean geometry, we can derive the property that its angles are also equal, each being of $60°$. At first glance, secondary level students may regard equilateral triangles to be simple shapes, yet upon further inspection, they soon realize that this is not the case. As their study deepens, students uncover many surprising, lesser-known properties. We illustrate this idea through a discussion of seven classroom-ready explorations.

The properties presented in the seven explorations were discovered and further developed by pre-service high school teachers. We presented initial ideas to launch the teachers' exploration, and they explored and tested their correctness by combining different mathematical tools with known theorems. The "new" properties were found as part of four advanced courses for teachers, namely, *Selected problems in Euclidean geometry*, *Strategies for solving problems in mathematics*, *Combining fields in mathematics*, and *Mathematical research seminar*. To prove the results for the equilateral triangle, pre-service teachers used axioms and basic results of geometry and theorems - many which are accessible to secondary-level students. These include the law of cosines, the extended law of sines, the Pythagorean theorem, Ptolemy's theorem for cyclical quadrilaterals, Heron's formula for the area of a triangle, and a trigonometric version of Ceva's theorem. In addition, pre-service teachers used inequalities for numbers such as the geometric mean - arithmetic mean inequality, and the formula for the area of a triangle in a coordinate plane to justify conjectures that they formulate in GeoGebra. The use of dynamic mathematics software to motivate proof is well-known [1] and one that we

[*]Gordon College, Shaanan College, Hifa, Israel
[†]Oranim College, Kiryat tivon, Shaanan College, Haifa, Israel
[‡]University of Delaware, United States

promote in our work with preservice teachers. The more one considers the equilateral triangle and thinks multi-directionally, the more one can find additional properties and relations. The process of investigation and discovery contributes much to the improvement of mathematical education, both of pre-service teachers and of students in high school education.

Before we present the "additional" properties, we give a reminder of several known properties of the equilateral triangle:

- The angle bisector, the median, and the attitude from the same vertex coincide with each other and with the perpendicular bisector on the side opposite the vertex.
- All the angle bisectors, the medians, the altitudes and the perpendicular bisectors of the triangle intersect at a single point.
- This point of intersection is both the center of the circle circumscribing the triangle and the center of the circle inscribed in the triangle. It is also the center of gravity of the triangle, and the Fermat point of the triangle.
- The three medians of the triangle divide it into six congruent triangles (this property also holds for the other lines: angle bisectors, altitudes etc.). It is important to note that in an arbitrary triangle we would obtain six triangles of equal area, which in general are not congruent.
- When connecting the feet of the altitudes through the sides of the triangle, we obtain four congruent equilateral triangles.
- The area of an equilateral triangle is $\frac{a^2\sqrt{3}}{4}$, where a denotes the side length of the triangle.
- The length of the radius of the circle circumscribing the triangle is $R = \frac{a\sqrt{3}}{3}$, and the length ofthe inscribed circle is half the size of the radius of the circumcircle ($r_{inscribed} = \frac{1}{2}R$).
- The sum of the distances of any point in an equilateral triangle to the sides of the triangle is constant and is equal to the height h of the triangle ($h = \frac{a\sqrt{3}}{2}$) (Viviani's theorem).

Figure 1 suggests a proof without words of this result. A dynamic sketch of the situation is provided at `https://www.geogebra.org/material/simple/id/2890821`.

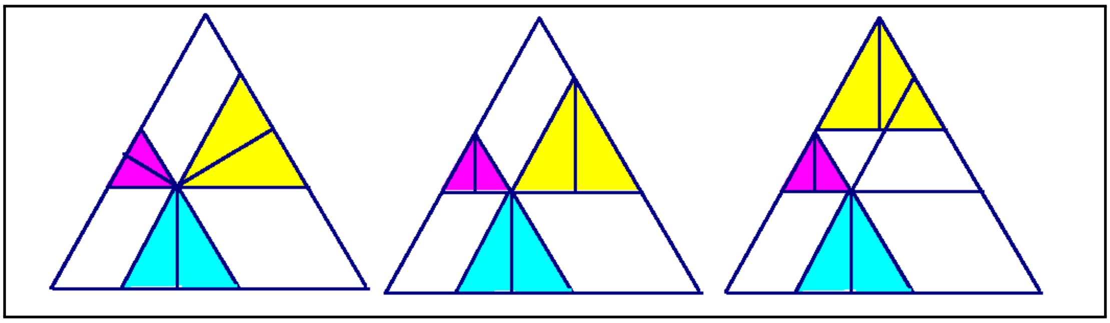

Figure 1. Sum of distances to the sides (adapted from [3]).

Also known are some relations between the sides of triangles and some relations between the angles of triangles, which, if they hold, imply the triangle is equilateral:

- If the measure of one angle $\gamma = 60°$ and $sin(\alpha) \cdot sin(\beta) = \frac{3}{4}$ for the remaining angles, α and β, then the triangle is equilateral.
- If sides a, b, and c satisfy the relation $a^2 - ab + b^2 = c^2$, and the angles satisfy the relation $sin(\alpha) \cdot sin(\beta) = \frac{3}{4}$, then the triangle is equilateral.
- If interior angles α, β, and γ satisfy the relation $cos(\alpha - \beta) \cdot cos(\alpha - \gamma) \cdot cos(\beta - \gamma) = 1$, then the triangle is equilateral.

2 Exploring Hidden Properties of Equilateral Triangles

For each of the properties in this article we first present an exploratory activity to students (in our case, preservice teachers) using GeoGebra. This initial activity encourages learners to explore and make conjectures. Students can construct their own figures or they can use the corresponding interactive figures in GeoGebraTube. To foster the exploration, we direct students' attention towards interesting relations or invariants. Students are encouraged to state conjectures in their own words and prove the results. Following each exploratory activity, "obvious" properties (i.e., those which we intended students to uncover) are stated and proved along with unanticipated results.

2.1 Exploration 1

In an equilateral triangle, put a movable point D on the side opposite to vertex A. Construct $\overline{AD}$. See `http://tube.geogebra.org/m/2804121` or `https://www.geogebra.org/apps/?id=kzLwaGXm`. Measure $\overline{AB}$, $\overline{AD}$, $\overline{BD}$, and $\overline{DC}$, as well as $\angle BDA$. Then consider the following questions.

1. What can you say about the length of $\overline{AD}$ and the side of the triangle?
2. What can you say about the length of $\overline{AD}$ compared to the lengths of $\overline{BD}$ and $\overline{DC}$?
3. What can you say about $m(\angle BDA)$?

State a conjecture about the length of $\overline{AD}$ compared to the length of $\overline{AB}$ and prove it. State conjectures about the length of $\overline{AD}$ compared to the lengths of $\overline{BD}$ and $\overline{DC}$ and prove them.

2.1.1 Property 1 (Conjecture)

Given equilateral triangle $\triangle ABC$ and D, a point on side $\overline{BC}$ (see Fig. 2), the following inequalities hold: $AD > BD$, $AD > DC$, and $AD < AB$.

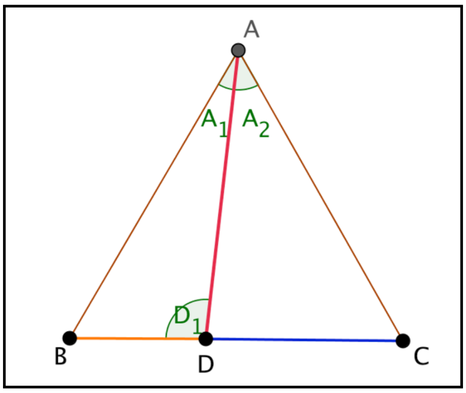

Figure 2. Cevian segment in equilateral triangle.

2.1.2 *Proof*

$m(\angle D_1) > m(\angle C) = 60°$, therefore $m(\angle D_1) > m(\angle B) = 60°$. Hence $AB > AD$, because in the triangle $\triangle ABD$ the larger side lies opposite the larger angle. Each of the angles, $\angle A_1$ and $\angle A_2$, has a measure smaller than $60°$. Therefore in $\triangle ABD$ we have $BD < AD$ because $m(\angle B) > m(\angle A_1)$, and in $\triangle ADC$ we have $DC < AD$ because $m(\angle C) > m(\angle A_2)$.

2.1.3 *Exercise*

Find a counterexample for Property 1 for a triangle that is *not* equilateral.

2.2 Exploration 2

Chose a movable point M inside an equilateral triangle and connect it to each of the vertices. See `http://ggbtu.be/m2804279` or `https://www.geogebra.org/apps/?id=exgeUmug`. Measure the lengths of segments $\overline{AM}$, $\overline{BM}$ and $\overline{CM}$. Compare the length of each segment with the sum of the other two segments. State a conjecture in your own words.

2.2.1 *Property 2 (Conjecture)*

Given equilateral triangle $\triangle ABC$ and a point M inside the triangle (Fig. 3), segments $\overline{AM}$, $\overline{BM}$, and $\overline{CM}$ satisfy triangle inequalities, that is, $AM+CM > BM$, $AM+BM > CM$, $BM+CM > AM$.

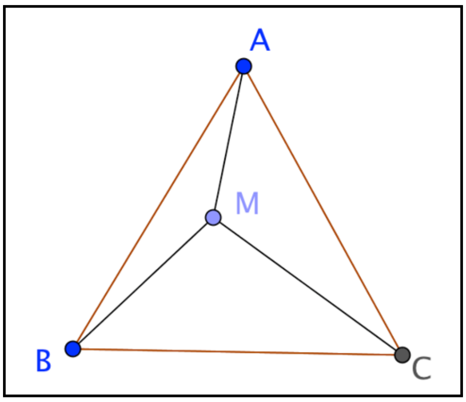

Figure 3. Distances to the vertices of an equilateral triangle.

2.2.2 *Proof*

Through point M we draw a line parallel to side $\overline{BC}$ that intersects $\overline{AB}$ and $\overline{AC}$ at points N and P. It is clear that $MB + MC > BC$ (the sum of two sides in a triangle $\cdots$). From property 1 we have that $NP > AM$ (because $\triangle ANP$ is equilateral). It is also clear that $BC > NP$, and therefore $BM + MC > AM$.

2.2.3 *Exercise*

Find a counterexample for Property 2 for a triangle that is *not* equilateral.

2.3 Exploration 3

Use $\overline{AD}$, $\overline{BD}$, and $\overline{DC}$ as in the equilateral triangle of Exploration 1. Construct a triangle with segments of lengths AD, BD, and DC (or, alternately, refer to `https://www.geogebra.org/material/simple/id/OEoihkhU`). Observe the angles of the new triangle. What can you say about the size of the biggest angle of the new triangle? State a conjecture and prove it.

2.3.1 *Property 3 (Conjecture)*

Given equilateral triangle $\triangle ABC$ and D a point on side BC (see Fig. 4), a triangle may be constructed from segments AD, BD and DC such that the measure of one interior angle equals $120°$.

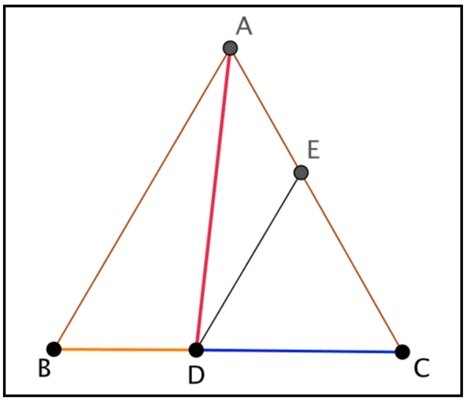

Figure 4. A triangle with an angle of $120°$.

2.3.2 *Proof*

We draw $\overleftrightarrow{DE}$ parallel to $\overline{AB}$. It is clear that $\triangle DEC$ is equilateral, and therefore $DC = DE$. Trapezoid $AEDB$ is isosceles, and therefore $BD = AE$. Hence it follows that the sides of $\triangle AED$ are congruent to segments $\overline{AD}$, $\overline{BD}$ and $\overline{CD}$. In $\triangle AED$ we have $m(\angle AED) = 120°$ (supplementary angle to an angle of $60°$). Thus the property has been proved.

2.4 Exploration 4

From an arbitrary point M within equilateral triangle ABC draw perpendicular segments to each side. These determine the points N, P, and Q (see `http://ggbtu.be/m2804329` or `https://www.geogebra.org/apps/?id=hA4KszYB`). Measure the lengths of segments $\overline{AQ}$, $\overline{BN}$, and $\overline{CP}$. Compute the sum of these lengths. Drag M. What do you observe about the sum of the three lengths, $AQ + BN + CP$? State a conjecture and prove it [2].

2.4.1 *Property 4 (Conjecture)*

Given equilateral triangle $\triangle ABC$ and interior point M, from which altitudes $\overline{MQ}$, $\overline{MP}$, and $\overline{MN}$ are drawn (see Fig. 5, left). Then $AQ + BN + CP$ equals half the perimeter of $\triangle ABC$.

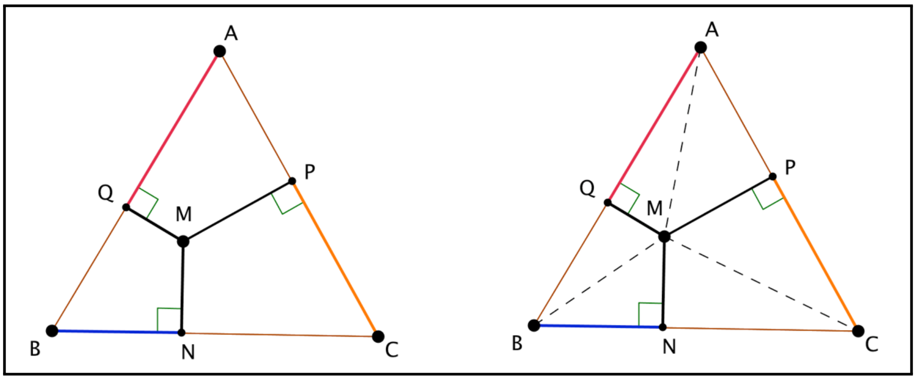

Figure 5. Half the perimeter.

2.4.2 *Proof*

Applying the Pythagorean Theorem to triangles $\triangle AQM and \triangle\ BQM$ we have

$$\begin{aligned} QM^2 &= AM^2 - AQ^2 \\ &= BM^2 - BQ^2 \implies \\ AM^2 - BM^2 &= AQ^2 - BQ^2 \end{aligned} \tag{1}$$

In the same manner, for the other two pairs of triangles, we obtain

$$BM^2 - MC^2 = BN^2 - NC^2 \tag{2}$$
$$MC^2 - AM^2 = PC^2 - AP^2 \tag{3}$$

Combining (1), (2) and (3) yields $AQ^2 + BN^2 + PC^2 = BQ^2 + NC^2 + AP^2$. Expressing segments $\overline{BQ}$, $\overline{NC}$, and $\overline{AP}$ in terms of a, the side length of $\triangle ABC$, we obtain $3a^2 - 2a(AQ + BN + PC) = 0$ or $AQ + BN + CP = \frac{3}{2}a$ (half the perimeter of $\triangle ABC$).

2.5 Exploration 5

A small equilateral triangle, $\triangle DEF$, is inscribed in equilateral triangle $\triangle ABC$ with D on $\overline{BC}$, E on $\overline{AC}$, and F on $\overline{AB}$ so that $AF = BD = CE$ (see `http://ggbtu.be/m2804353` or `https://www.geogebra.org/apps/?id=gH9o5BmD`). Inscribed circles are constructed in triangles $\triangle DEF$ and $\triangle AFD$. Measure the radii of these two circles and compare them to the radius of the inscribe circle in the original equilateral triangle. State a conjecture and prove it.

For the next proof we need to remember that the length of segment d from vertex A to the point of tangency of the inner circle can be expressed as $2d = b + c – a$ (see Fig. 6).

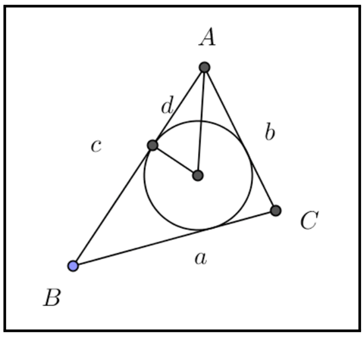

Figure 6. Distance to point of tangency.

2.5.1 *Property 5 (Conjecture)*

Given equilateral triangle $\triangle ABC$, mark points F, D, E on sides $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, respectively, so that $AF = BD = CE$ (see Fig. 7). Let $r_{\triangle AFE}$, $r_{\triangle DEF}$, and $r_{\triangle ABC}$ be the radii of the circles inscribed in triangles $\triangle AFE$, $\triangle DEF$, and $\triangle ABC$, respectively, then $r_{\triangle AFE} + r_{\triangle DEF} = r_{\triangle ABC}$.

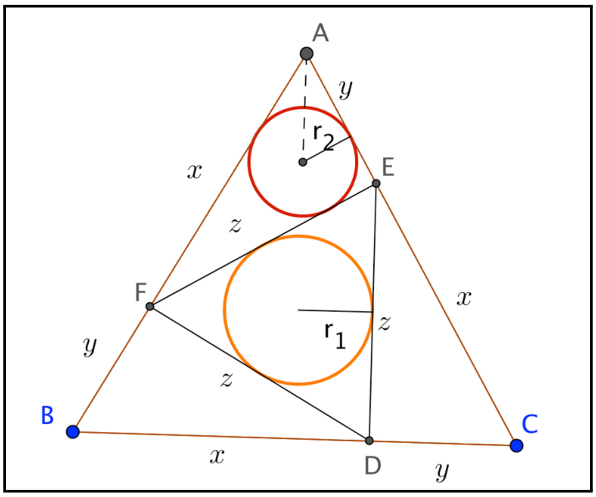

Figure 7. Sums of radii.

2.5.2 *Proof*

We denote $AF = BD = CE = x$, $BF = DC = EA = y$, and $EF = ED = DE = z$. With this notation, one can write:

$$r_{\triangle AFE} = \frac{x + y - z}{2} \cdot tan(30^\circ) \tag{4}$$

$$r_{\triangle DEF} = \frac{z}{2} \cdot tan(30^\circ) \tag{5}$$

$$r_{\triangle ABC} = \frac{x + y}{2} \cdot tan(30^\circ) \tag{6}$$

and by adding the relations (4) and (5), we obtain relation (6).

2.6 Exploration 6

An equilateral triangle $\triangle ABC$ with side length a is inscribed within a circle with M a point on the circle (see `https://www.geogebra.org/material/simple/id/b5EiTbJ1`).

1. Measure the lengths of $\overline{AM}$ and $\overline{MC}$ and compare these to the length of $\overline{BM}$. State a conjecture and prove it.
2. Compute MA^2, MB^2, and MC^2 and compare the sum, $MA^2 + MB^2 + MC^2$, to the square of the side of the triangle, a^2. State a conjecture and prove it.

2.6.1 Property 6 (Conjecture)

Given equilateral triangle $\triangle ABC$ inscribed within a circle with M a point on the perimeter of the circle (on the small arc $\overset{\frown}{AC}$, as shown in Fig. 8). Then the following equations hold:

$$AM + CM = BM \tag{7}$$
$$MA^2 + MB^2 + MC^2 = 2a^2 \tag{8}$$

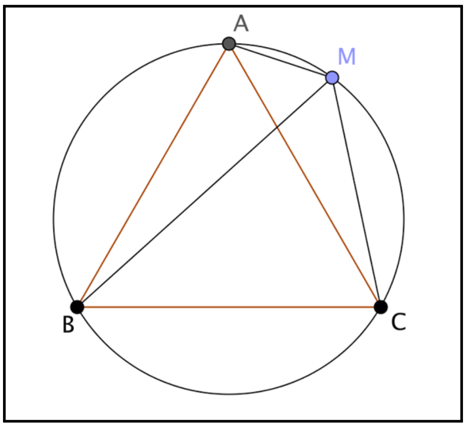

Figure 8. Sums of distances.

2.6.2 Proof

Part (a). From Ptolemy's Theorem, in the quadrilateral $AMCB$ we have $AM \cdot BC + MC \cdot AB = MB \cdot AC$. However, $AB = BC = CA = a$, and therefore $AM + CM = MB$.

Part (b). $m(\angle AMC = 120^\circ$ from the quadrilateral inscribed in the circle. The law of cosines applied to $\triangle AMC$ yields:

$$AM^2 + MC^2 + AM \cdot MC = a^2 \tag{9}$$

From the conclusion of part (a), we have

$$AM^2 + MC^2 + AM(MB - MA) = a^2 \tag{10}$$

and also

$$AM^2 + MC^2 + MC(MB - MC) = a^2 \quad (11)$$

Adding the equalities together, we obtain

$$MC^2 + AM \cdot MB + AM^2 + MC \cdot MB = 2a^2 \quad (12)$$

or

$$MC^2 + AM^2 + MB(AM + MC) = 2a^2 \quad (13)$$

and therefore

$$MA^2 + MB^2 + MC^2 = 2a^2 \quad (14)$$

2.7 Exploration 7

Construct equilateral triangles to the outside of an arbitrary triangle and find their centers (see `https://www.geogebra.org/material/simple/id/2810237`). Connect the centers of the equilateral triangles. What do you observe? State a conjecture and prove it.

2.7.1 Property 7 (Conjecture)

Outside equilateral triangles are constructed on the sides of $\triangle ABC$ (see Fig. 9). Then the centers of the circles circumscribing the three equilateral triangles are the vertices of an equilateral triangle.

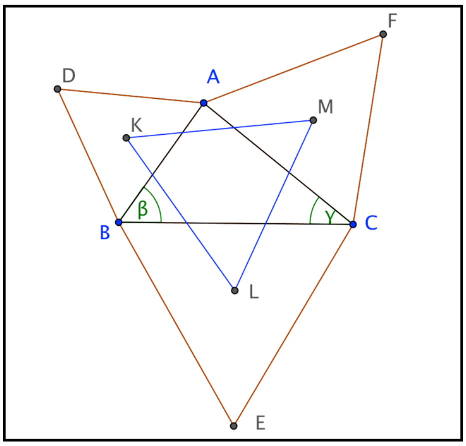

Figure 9. Napoleon's Theorem.

2.7.2 Proof

The three equilateral triangles are $\triangle CAF$, $\triangle CBE$, and $\triangle ABD$, and their centers are M, L, and K, respectively. It follows that

$$\angle CBL = \angle ABK = 30^\circ$$
$$BL = \frac{a}{\sqrt{3}}$$
$$BK = \frac{c}{\sqrt{3}}$$
$$CM = \frac{b}{\sqrt{3}}$$

Applying the law of cosines to $\triangle BKL$ yields

$$KL^2 = BK^2 + BL^2 - 2 \cdot BK \cdot BL \cdot cos(\beta + 60^\circ)$$
$$= \frac{c^2}{3} + \frac{a^2}{3} - \frac{2ab}{3} \cdot cos(\gamma + 60^\circ)$$

Similarly, in $\triangle CML$ we obtain

$$ML^2 = \frac{a^2}{3} + \frac{b^2}{3} - \frac{2ab}{3} \cdot cos(\gamma + 60^\circ)$$

We must prove that $KL^2 = ML^2$. It remains to be proved that

$$c^2 - 2ac \cdot cos(\beta + 60^\circ) = b^2 - 2ab \cdot cos(\gamma + 60^\circ)$$

or

$$c^2 - b^2 = 2a\left[c \cdot cos(\beta + 60^\circ) - b \cdot cos(\gamma + 60^\circ)\right]$$

Using the extended law of sines, we obtain:

$$a = 2Rsin(\alpha), b = 2Rsin(\beta), c = 2Rsin(\gamma)$$

We wish to prove the following equality:

$$sin^2(\gamma) - sin^2(\beta) = 2sin(\alpha)\left[sin(\gamma) \cdot cos(\beta + 60^\circ) - sin(\beta) \cdot cos(\gamma + 60^\circ)\right]$$

Using the trigonometric identity $sin^2(\gamma) - sin^2(\beta) = sin(\gamma+\beta) \cdot sin(\gamma-\beta)$ and the triangle identity $sin(\alpha) = sin(\beta+\gamma)$, the following remains to be proved:

$$sin(\gamma - \beta) = 2\left[sin(\gamma) \cdot cos(\beta + 60^\circ) - sin(\beta) \cdot cos(\gamma + 60^\circ)\right]$$

We take care of the right hand side using the formula for transition from a product to a sum:

$$2 \cdot \frac{1}{2}[sin(\gamma + \beta + 60^\circ) + sin(\gamma - \beta - 60^\circ)$$
$$-sin(\beta + \gamma + 60^\circ) - sin(\beta - \gamma - 60^\circ)] =$$
$$= sin(\gamma - \beta - 60^\circ) - sin(\beta - \gamma - 60^\circ)$$
$$= 2sin(\gamma - \beta) \cdot cos(-60^\circ)$$
$$= sin(\gamma - \beta) \implies KL = LM$$

Similarly we prove that $KL = KM$, and therefore $\triangle KLM$ is equilateral.

This property is attributed to Napoleon, who was fond of mathematics. A triangle with three equilateral triangles around it is known as the Torricelli configuration who used it to find the Fermat point of the triangle.

3 FINAL REMARKS

We would like to discuss two complementary aspects of the explorations and proofs presented in this article. On one hand, the explorations do not require much previous mathematical knowledge. Students can discover properties of the equilateral triangle interacting with the applets by observing, measuring and experimenting. They can state their conjectures in simple geometrical terms. On the other hand, the proofs presented make use of previous knowledge from many topics in mathematics, including geometry, trigonometry, coordinate geometry, and inequalities of numbers. This offers the opportunity for future teachers to see how mathematical ideas are interconnected. Thus the explorations on the equilateral triangle presented here can offer preservice teachers a capstone experience in Euclidean geometry.

4 ACKNOWLEDGEMENT

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Ruti Segal taught high school mathematics for 26 years. Currently, she teaches mathematics and didactics of mathematics at Shaanan College and Oranim College in Israel, focusing on junior and senior high school Curriculum. She routinely integrates technology in mathematics instruction and research and spends significant time in various public service projects related to teaching and learning of mathematics.

Moshe Stupel, Professor, Head of Mathematics Department, a researcher, and a pre-service teacher educator in two academic colleges, Shaanan Academic College and Gordon College. In the past, he was a high school principal. He has published and presented 40 papers on mathematics and mathematics education. Recently, his research is focused on various methods of problem solving and on variance and invariance property in dynamic geometrical environments (DGEs) in mathematics.

Alfinio Flores teaches mathematics and mathematics methods courses *con ganas* at the University of Delaware. He uses technology to provide opportunities for students and teachers to explore mathematics on their own. He has conducted activities for students in schools ranging for Kindergarten to 12th grade, and conducted professional development sessions for teachers of grades K-16.