VISUALIZING SOLIDS OF REVOLUTION IN GEOGEBRA

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Abstract: Surfaces can be graphed in GeoGebra by either using the `Surface` command, defining a function $f(x, y)$, or inputting an equation of at most three variables. Some surfaces can be easily graphed using a function of two variables or an equation, but we have more control over the surface if the `Surface` command is used. We start by graphing surfaces of revolution encountered in single variable calculus class. A surface of revolution can be graphed in GeoGebra provided we can parameterize the circular cross-section. This method of graphing a surface of revolution can be applied to model objects with circular cross-section, like a Hershey's Kiss, in GeoGebra.

Keywords: solids of revolution, modeling, multivariable, GeoGebra: `Surface`

Introduction

Students encounter solids of revolution during their study of integral calculus in a single-variable calculus class. A solid of revolution is created by revolving a planar region R about a line. Integration can be used to find the volume of a solid of revolution, but the solid may be difficult for some students, or even teachers, to visualize. Unfortunately, GeoGebra cannot graph a solid of revolution, but there is a way for GeoGebra to help. The idea is to use GeoGebra to graph the surface of the solid of revolution.

The graphing of a surface of revolution can be accomplished by describing the surface as a function $f(x, y)$, but we will use parametric equations to describe the surface and graph using the `Surface` command. As it turns out, the parametric equations used to describe a surface of revolution are simple and easy to manipulate. The difficulty with using parametric equations lies in creating the equations to describe the surface. After learning how to graph a surface of revolution, we apply our method to model the surface of a Hershey's Kiss.

Graphing a surface of revolution

Suppose a region R in the xy-plane is bounded by $x = a$, $x = b$, the x-axis, and f, a function that is continuous on $[a, b]$. Let S be the solid of revolution created by revolving R about the x-axis. To visualize S, we graph the surface of S, which is created by revolving the graph of f on $[a, b]$ about the x-axis. The surface of S can be described by the equation

$$y^2 + z^2 = [f(x)]^2 \tag{3.1}$$

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with $x \in [a,b]$. One way to graph the surface of revolution is to solve equation (3.1) for z. Doing so gives the functions

$$g(x,y) = \sqrt{[f(x)]^2 - y^2}$$
$$h(x,y) = -\sqrt{[f(x)]^2 - y^2}$$

which can be put into GeoGebra without trouble. This method will actually graph more than the desired surface, though, causing problems in visualizing the solid. Even though f is defined on the interval $[a,b]$, GeoGebra will show the function on its whole domain, creating an inaccurate picture of the surface.

To create a better graph of the surface of revolution, the surface will be plotted using *parametric equations.* The benefit of graphing with parametric equations is the amount of control over the curve/surface. When we use parametric equations to graph a surface of revolution, we can set the domain of the equations explicitly. Also, parametric equations provide enough flexibility to create myriad curves and surfaces with ease.

To parameterize a surface of revolution, we need three equations in two-variables. One variable, t, parametrizes the interval $[a,b]$ on the x-axis, and another variable, u, parametrizes the cross-sections parallel to the yz-plane.

For a fixed $t \in [a,b]$, the equation of the cross-section of the surface of revolution is given by

$$y^2 + z^2 = [f(t)]^2,$$

a circle centered on the x-axis with radius $f(t)$. In the xy-plane, a circle with radius r and center (h,k) can be parameterized by

$$x = h + r\cos(u), \qquad y = k + r\sin(u), \qquad u \in [0, 2\pi].$$

Since the circular cross-section of a surface of revolution has radius $f(t)$ and center on the x-axis, it can be parameterized by

$$y(u) = f(t)\cos(u), \qquad z(u) = f(t)\sin(u), \qquad u \in [0, 2\pi].$$

Hence, the equations that describe the surface of the solid of revolution are

$$x(t,u) = t,$$
$$y(t,u) = f(t)\cos(u),$$
$$z(t,u) = f(t)\sin(u),$$

for $t \in [a,b]$ and $u \in [0, 2\pi]$.

To graph the surface in GeoGebra, we use the `Surface` command,

```
Surface[x(t,u), y(t,u), z(t,u), t, a, b, u, c, d]
```

where $x(t,u)$, $y(t,u)$, and $z(t,u)$ are the coordinate functions, t is the first parameter, ranging from a to b, and u is the second parameter, ranging from c to d. To graph a surface of revolution, the `Surface` command would look like

```
Surface[t, f(t) cos(u), f(t) sin(u), t, a, b, u, 0, 2 pi].
```

For example, suppose the region R in the xy-plane is bounded by the equations $x = 1, y = 0$, and $y = x^2$, and let S be the solid created by revolving R about the x-axis. The surface of S can be described by the equations

$$x(t,u) = t,$$
$$y(t,u) = f(t)\cos(u) = t^2\cos(u),$$
$$z(t,u) = f(t)\sin(u) = t^2\sin(u)$$

with $t \in [0,1]$ and $u \in [0, 2\pi]$.

The command to graph the surface of S would be

```
Surface[t, t2̂ cos(u) , t2̂ sin(u), t, 0, 1, u, 0, 2 pi].
```

Figure 3.1 shows the region R in the xy-plane, and Figure 3.2 shows the surface of S created by revolving the graph of x^2 on $[0,1]$ about the x-axis.

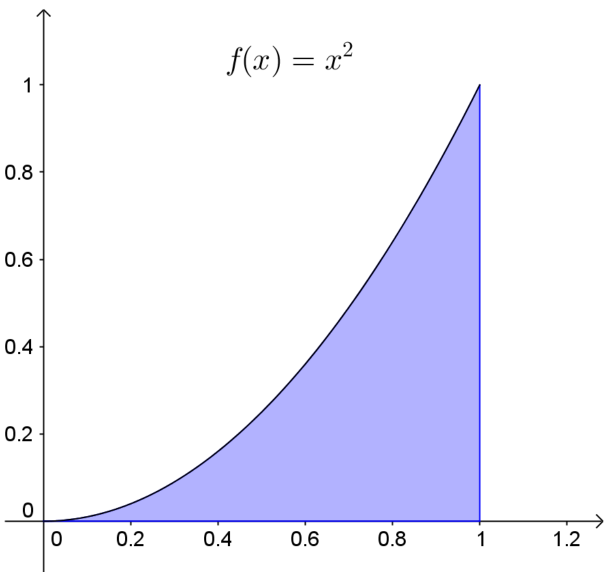

Figure 3.1 The Region R

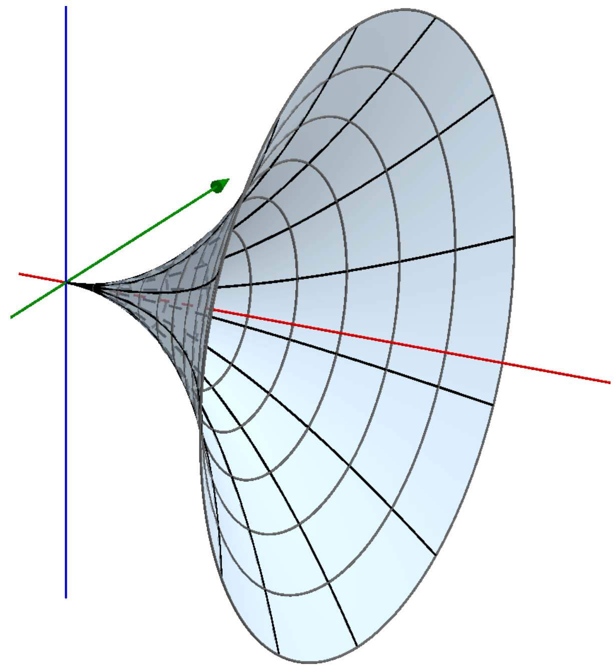

Figure 3.2 The Surface of S

Application: Modeling a Hershey's Kiss

One interesting feature of GeoGebra is its ability to insert images directly into the Graphics view. When a picture is inserted into the Graphics view, the user can manipulate the picture and apply transformations like translations, rotations, and dilations. This allows the user to investigate properties of an image, like the types of symmetry it possess, or use the image for modeling.

When graphing a surface of revolution, GeoGebra can be set up so the surface can be easily altered just by changing the boundary function $f(x)$ and/or the bounds, which allows students to easily manipulate a solid. To see how GeoGebra can be used to model a 3-dimensional object, we will create a 3-D model of a Hershey's Kiss. A Hershey's Kiss can be viewed as a solid of revolution where the axis of revolution goes through the tip and the center of its circular base, so its surface can be graphed in GeoGebra if we can find a boundary function. This is where we use GeoGebra's ability to insert images onto the Graphics view.

To create a model of a Hershey's Kiss, we get a picture of the Kiss and insert the picture into GeoGebra. Then, we move the picture around so that the axis of revolution of the Kiss coincides with the x-axis and the center of the base matches up with the origin. Once the image is in place, we do not want it to move around, so we will put it in the background. To do this, go to `Object Properties` under the `Basic` tab, click on the checkbox for `Background Image`.

With the image in GeoGebra, the boundary function can be created. A fifth-degree polynomial approximates the boundaryof this shape quite well. Six points are plotted on the boundary of the kiss, and a polynomial is created through those six points using the command

```
f(x)=Polynomial[C, D, E, F, G, H].
```

See Figure 3.3.

Figure 3.4 shows the model of the surface of the Hershey's Kiss created from the command

```
Surface[t, f(t) cos(u), f(t) sin(u), t, a, b, u, 0, 2 pi],
```

where a is the x-coordinate of C and b is the x-coordinate of H. The boundary curve can be changed by moving the points around, giving us the option to make the boundary function fit better to the picture. By setting the image to the background, the image will remain fixed even when the boundary points are moved.

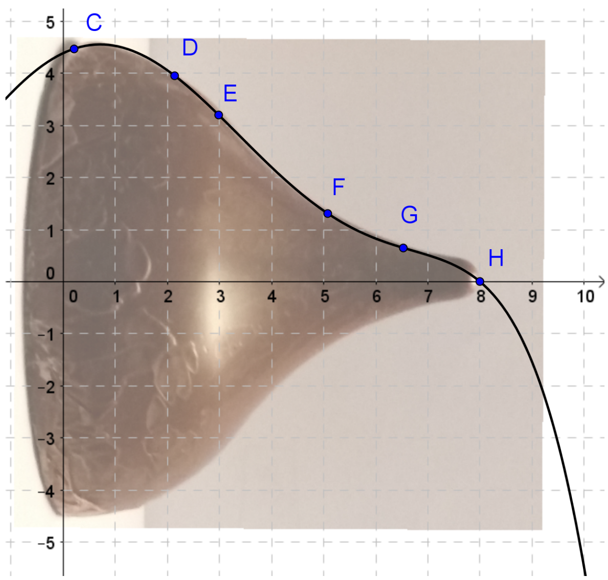

Figure 3.3 Hershey's Kiss [1] With Boundary Function and Points

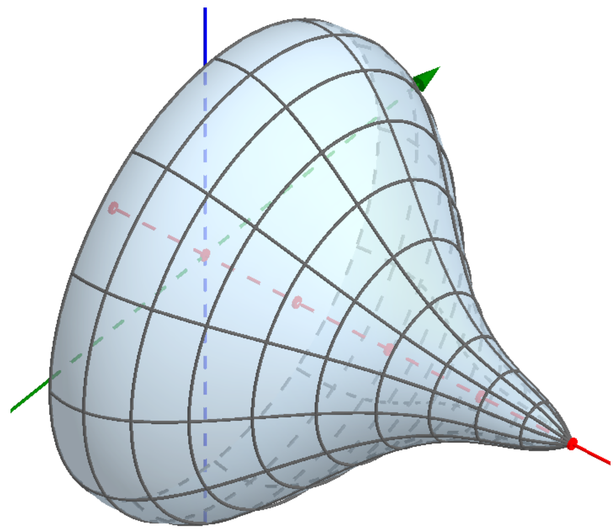

Figure 3.4 Model of the Surface of a Hershey's Kiss

Conclusion

GeoGebra can graph surfaces through the use of the `Surface` command. The biggest hurdle with graphing surfaces with nice cross-sections is parameterizing the cross-section. Surfaces of revolution have circular cross-sections which make their parameterization simple and easy to change. Surfaces with different cross-sections can be graphed as long as the cross-sections can be parameterized. Graphing surfaces using the `Surface` command can be used to help create models of everyday objects like a Hershey's Kiss. With some experimentation and a little skill, students and teachers can not only create new and exotic surfaces but see them too.

REFERENCES

1. Hoffman, Douglas. "Hershey's Kiss." 2015. PNG file.
2. Hohenwarter, M. (2002). *GeoGebra*. Available online at http://www.geogebra.org/cms/en/