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of the 3rd Annual Southern Connecticut GeoGebra Conference

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ACKNOWLEDGMENTS

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FOSTERING GROWTH IN TEACHER CONTENT KNOWLEDGE THROUGH THE CONSTRUCTION OF SKETCHES FOR STUDENTS

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Abstract: In this paper, the author investigates connections between teacher content knowledge and the construction of sketches through an examination of teacher-constructed applets. Specifically, the author analyzes the mathematics that teachers needed to know to design sketches in GeoGebra to support student understanding of a proportional reasoning problem aligned to 6th grade Common Core math standards.

Keywords: pedagogical content knowledge, teacher knowledge, problem solving, proportional thinking, modeling

Introduction

A good deal of research has explored the use of dynamic geometry software as a tool for strengthening student understanding of various secondary school topics [1, 3, 5]. Less common are studies that investigate connections between teacher content knowledge and the construction of sketches. What mathematics content do teachers learn as they create sketches for their students? How does this content compare to the content that teachers convey to students through such sketches? In the following discussion, we respond to these questions through an examination of teacher-constructed applets that support student work on Ty's Escalator Task [2]:
Ty took the escalator to the second floor. The escalator is 12 meters long, and he rode the escalator for 30 seconds. Which statements are true? Select all that apply.

1. He traveled 2 meters every 5 seconds.
2. Every 10 seconds he traveled 4 meters.
3. He traveled 2.5 meters per second.
4. He traveled 0.4 meters per second.
5. Every 25 seconds, he traveled 7 meters.

Note that the task requires students to draw various conclusions about Ty's speed - a concept that is well suited for visualization with dynamic mathematics software. Through the analysis of two teacher-constructed models, we explore how the construction of sketches intended to support student exploration may strengthen teacher understanding of proportions and various other topics within the school mathematics curriculum.

Although the teacher-constructed applets we explore in this paper provide nearly identical end-user experiences for students, the underlying mathematics that teachers used to create the applets are remarkably different. We posit that providing teachers with time to construct their own applets and share construction strategies with each other provides a novel way to foster content knowledge growth among teachers. Such explorations provide teachers with opportunities to see connections between various areas of the school mathematics curriculum that are too often overlooked.

Finished Sketch: End-User Experience

In a summer professional development experience, specifically a keynote presentation at the 2015 Midwest GeoGebra Conference, Ty's Escalator Task was shared with a group of approximately 100 secondary-level school teachers. In addition, teachers were shown a dynamic sketch that accompanied the problem. The sketch, constructed by the authors of this paper, is illustrated in Figure 1.1.

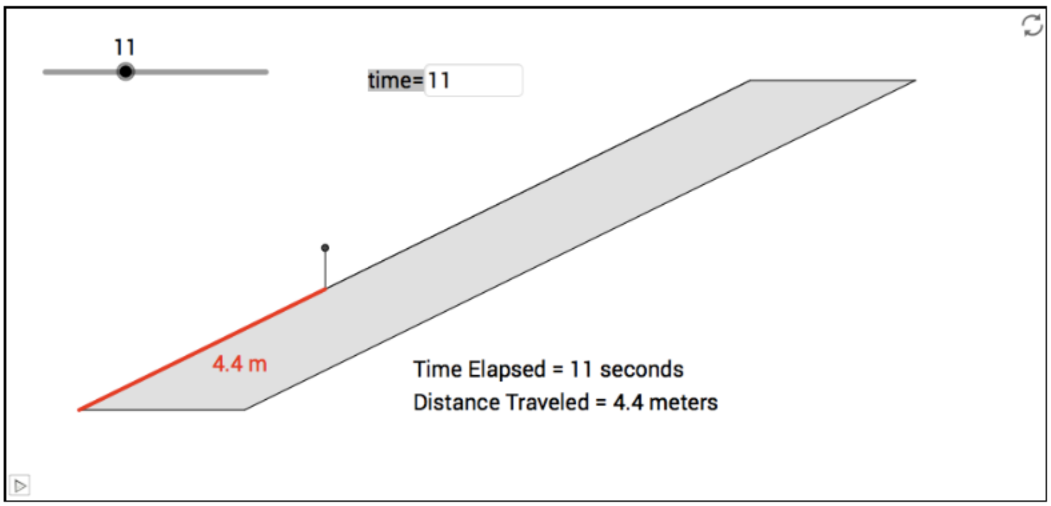

Figure 1.1 Sketch of Ty's Escalator Task (available on-line at http://tube.geogebra.org/m/1297645)

In small groups, teachers were asked to examine the task and the sketch, then brainstorm features of the applet they felt would help students. Ideas for enhancing the sketch were also solicited. Through audience participation, teachers noted the following positive features.

- The variable, `time`, is controlled by dragging the slider in the upper left corner of the sketch. Students may also enter a specific time in the input box to the immediate right of the slider.
- Specific values of `time` and `distance` are shown below the escalator. These values depend upon the value of time and are updated automatically as the slider is manipulated.
- The total distance that Ty has traveled up the escalator is highlighted in red as a line segment.
- The slider changes values of time in increments of 1 second. Pressing the play button in the bottom left corner of the sketch automates the movement of the slider (and, therefore, the movement of Ty).
- Including automation enables students to visualize the escalator movement in real-time, focusing full attention on number relationships in the sketch without the distraction of having to move the slider.

The following were offered as possible enhancements:

- Provide students with a means to analyze captured time and distance data within a graph or table. Students will likely benefit from seeing multiple representations of the same data.
- Provide the option for students to modify the length and speed of the escalator. This would encourage students to generalize their findings.

The underlying strategies used to construct the sketch in Figure 1.1 were not shared with teachers. Instead, teachers were encouraged to build their own versions of the sketch using GeoGebra. We encouraged teachers to enhance the design of the sketch in any way they saw fit. At the end of the day-long meeting during an optional debriefing session, participants shared their sketches.

In the sections that follow, we explore the design approaches of two teachers. These were selected because of their relative simplicity and because the underlying mathematics used to construct the two applets was quite different. Discussions of these (and other) strategies during the debriefing provided teachers with opportunities to make connections between various areas of mathematics as they learned more about the GeoGebra software.

Construction Method 1: Coordinates and Component Vectors

The first teacher modeled the escalator ride by considering Ty's horizontal and vertical movement separately within the x-y coordinate plane. Figures 1.2-1.6 provide step-by-step screenshots of such a construction. First, the teacher constructed line segment $\overline{AB}$, with A fixed at the origin, as shown in Figure 1.2. The placement of B was arbitrary.

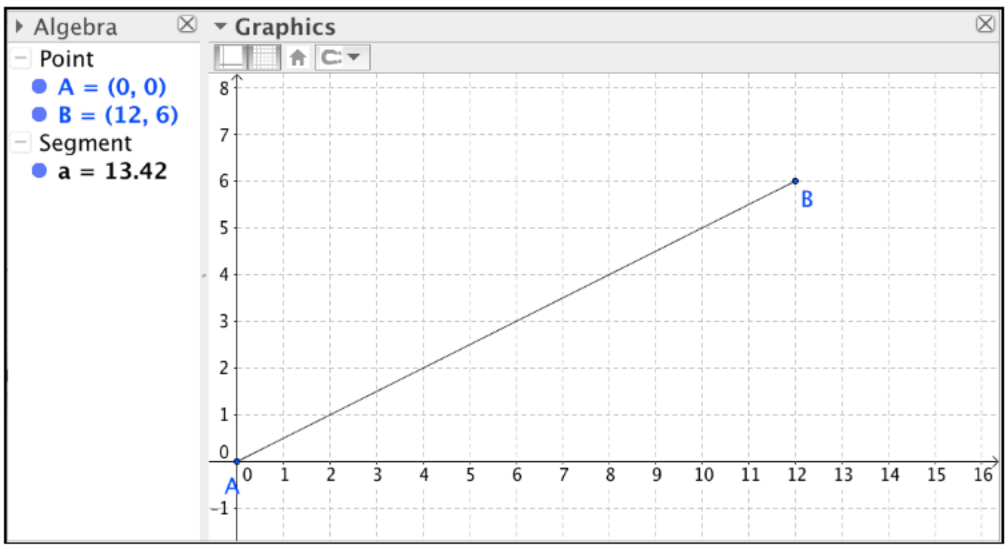

Figure 1.2 Segment AB with A fixed at the origin.

Next, the teacher constructed a slider with values ranging from 0 to 30 seconds, incremented by 1 unit. Slider settings are depicted in Figure 1.3.

Figure 1.3 Settings for time slider.

With `time` defined, the teacher defined Ty's horizontal and vertical movement as separate variables – `hdist` and `vdist`, respectively.

Next, the teacher defined Ty's location on the x-y plane, P, in terms of these variables. Since Ty's starting point is placed at the origin, point A, the x-coordinate of point B, `x(B)`, describes the total horizontal distance Ty travels. Likewise, `y(B)` describes the entire distance Ty travels in the vertical direction. Multiplying horizontal and vertical components of distance traveled by the fraction of the total time elapsed, `time/30`, yields Ty's horizontal and vertical location at the current time. Hence, `P=(hdist,vdist)` describes Ty's current location.

These steps, illustrated in Figure 1.4, consider Ty's movement as component vectors.

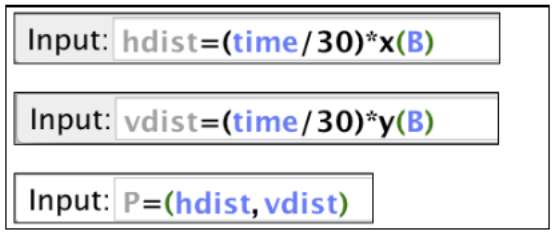

Figure 1.4 Definition of `hdist`, `vdist`, and point `P` (Ty's position on the escalator).

Interestingly, the teacher who shared this construction was surprised when others suggested her work implemented a vectors-based approach. As a geometry teacher, she did not teach her students about vectors and had not worked with vectors formally "since college." The teacher commented that she "hadn't thought of it that way" and that her motivation for using x- and y- components separately stemmed from a desire to break down the construction into "bite-sized" steps. Clearly, through our discussion of this sketch, a number of teachers strengthened their understanding of vectors and connected them to the content typically found in entry-level courses.

Figure 1.5 illustrates the teacher's sketch with the aforementioned variables defined. Figure 1.6 shows a completed sketch with gridlines and the Algebra View hidden.

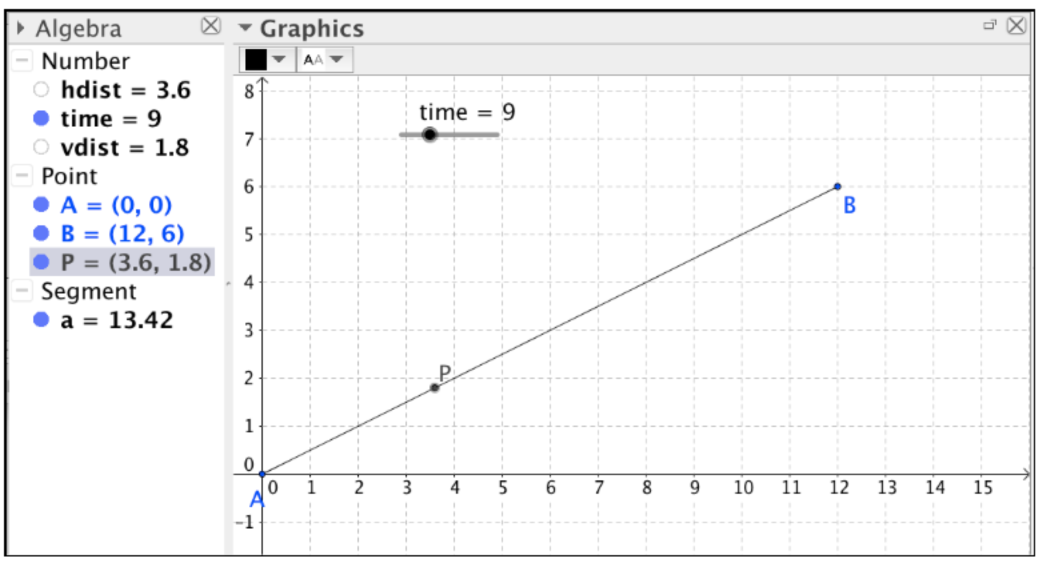

Figure 1.5 Sketch including Ty's position on the escalator with respect to time.

Construction Method 2: Similar Circles

While the component vector approach described in the previous section has a decidedly algebraic flavor, geometry yields an alternative construction that is arguably more elegant and more accessible to students in entry-level courses. Figures 1.7-1.11 provide step-by-step screenshots of a teacher-generated approach offered by a second teacher. First, using the "Circle with Center and Radius" tool, the teacher constructed a circle with fixed radius of 12 units centered at point A. The results of this are shown in Figure 1.7.

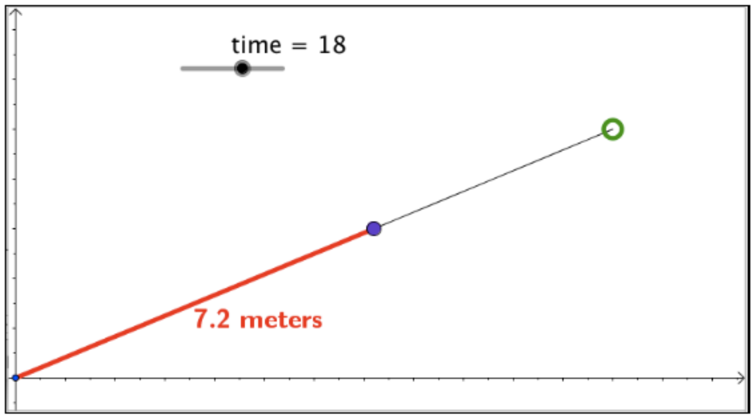

Figure 1.6 Completed sketch implementing a component vector approach.

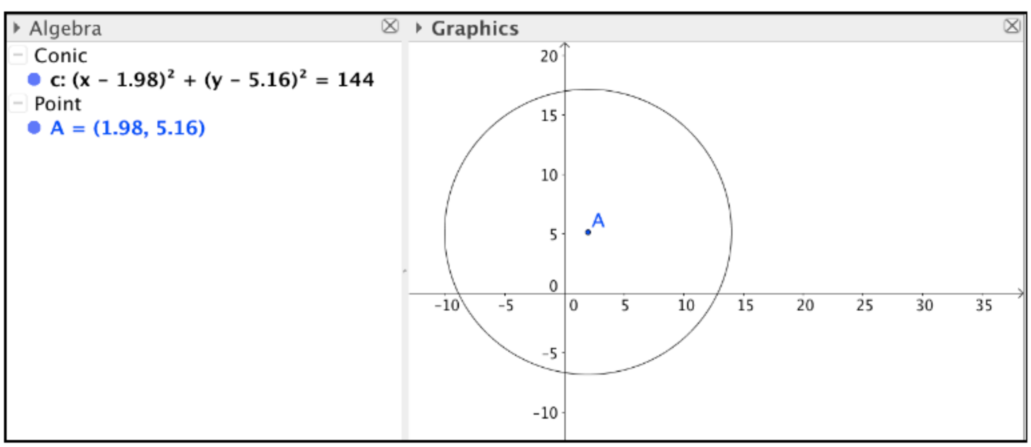

Figure 1.7 Construction of a circle with fixed radius of 12 units.

Next, the teacher constructed a point B on the circle followed by radius AB. This radius, shown in Figure 1.8, represents the entire length of the escalator.

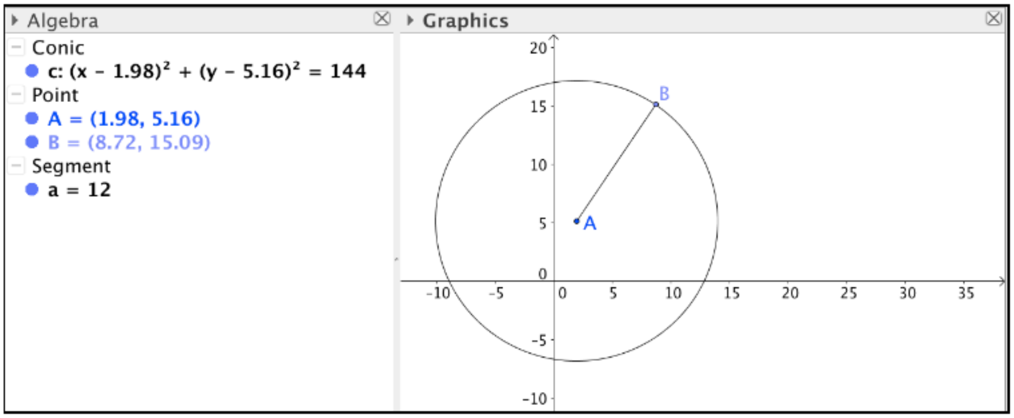

Figure 1.8 Segment $\overline{AB}$ representing the escalator in Ty's Escalator Task.

Sitting at a table in a common meeting area, the teacher explained to us that she used the radius of a scaled circle to represent the distance Ty traveled up the escalator at time, t. After creating slider t, she constructed a second circle centered at A with scaled radius. She defined the radius length as shown in Figure 1.9. Note that a is the length of segment AB while t/30 is the fraction of the total time that has elapsed during Ty's trip.

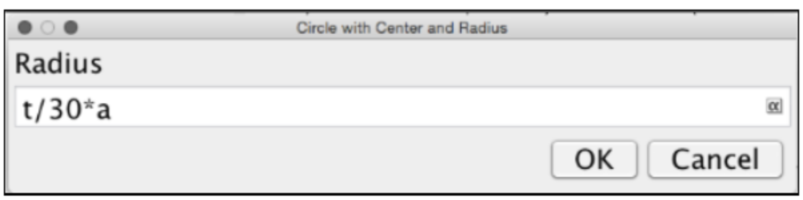

Figure 1.9 Constructing a second circle with scaled radius.

The results of this construction are illustrated in Figure 1.10. The point at which the scaled circle intersects the original radius AB represents Ty's location on the escalator at time, t.

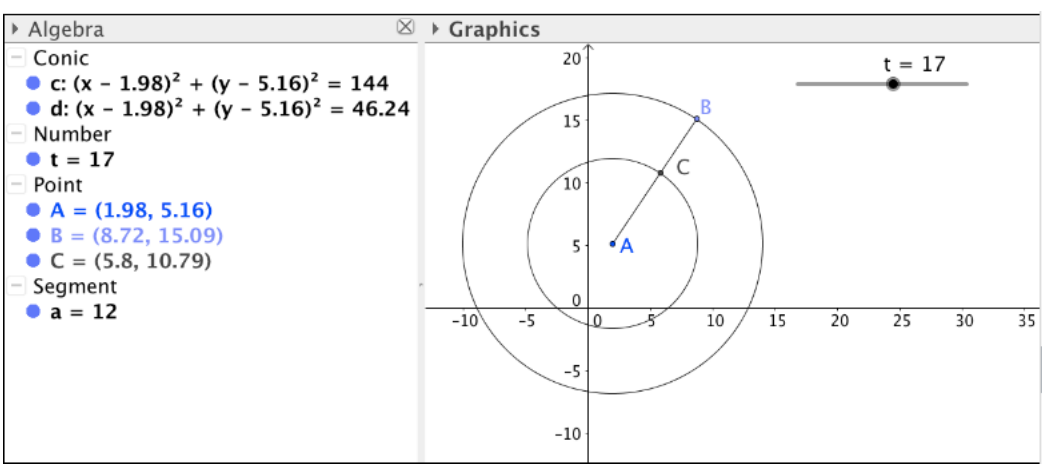

Figure 1.10 Ty's location on the escalator as represented by the intersection of a scaled circle and radius AB.

Figure 1.11 shows a completed sketch with the grid, axes, circles, and Algebra View hidden.

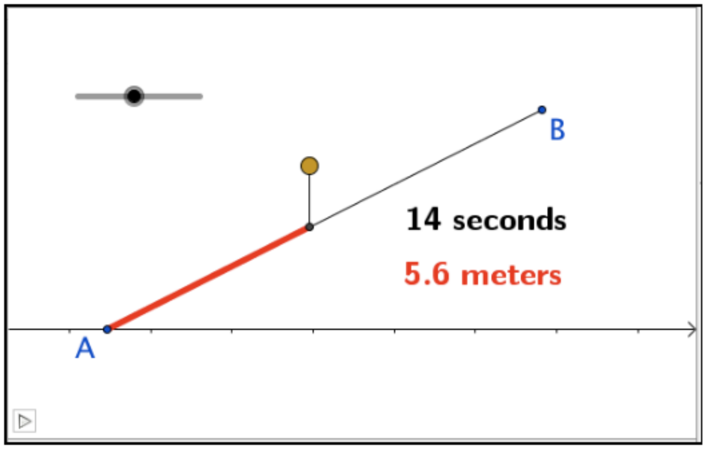

Figure 1.11 Completed sketch implementing similar circles approach.

In this second approach, the teacher essentially constructed a scaled circle using a size change transformation with scale factor t/30. This observation was an epiphany for a handful of our participants, particularly those who did not teach geometry and had a tendency to think of proportional situations numerically rather than geometrically.

Observations and Implications

Although the construction approaches illustrated in the preceding discussion are markedly different—one uses the underlying idea of component vectors; the other uses scaled circles—the resulting sketches are remarkably similar. For instance, both final sketches depict the escalator as a line segment. Both use a slider to control time. Both display time elapsed and distance traveled as dynamic text. When teachers shared their construction methods during the debriefing session, they were surprised by the variety of construction ideas that their colleagues implemented. As one teacher commented, "I had no idea there were so many ways to do the same thing with the software."

In general, four basic approaches were used by teachers to construct a model of Ty's Escalator Task, namely (a) geometric scaling, (b) numerical scaling, (c) function and equation scaling, and (d) tabular scaling (using GeoGebra's spreadsheet features, these teachers diverged somewhat from the original task of duplicating the escalator, focusing on determining escalator speeds instead). Sharing various construction strategies appeared to deepen teacher understanding of ratios and proportionality, particularly for those who tended to think of proportional situations either as numerical or as geometric, but not both. In addition to the usual task of finding ratios and their equivalents, which can be accomplished numerically, the modeling activity engaged teachers in more interesting discoveries beyond calculating ratios. As teachers shared strategies, they connected content from throughout the K-12 curriculum, including vector algebra, transformational geometry, and advanced algebra.

It is interesting to note that the content that students explore with the finished sketches focuses primarily on rates. The finished sketches in Figures 1.6 and 1.11 do not immediately suggest radii of concentric circles or vectors, yet this content was instrumental in a number of teachers' construction strategies. This suggests that designers may learn mathematics content that is above and beyond that which is conveyed to students through finished sketches. This has implications for ways in which teachers use dynamic sketches with their students. For instance, rather than engaging students with ready-made sketches in all cases, teachers should consider providing their students with opportunities to construct their own sketches. Doing so encourages students to connect content that they are currently learning to content from other courses in meaningful and creative contexts.

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USING GEOGEBRA TO PRESENT KINETIC DATA AND LIGAND BINDING DATA TO A BIOCHEMISTRY CLASS

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Abstract: There are two hyperbolic functions commonly presented in Biochemistry classes. Both of these functions, the Michaelis-Menten enzyme activity curve and the Scatchard ligand binding isotherm, are routinely re-expressed using a linear transform in order to determine the rate and binding constants inherent in each model. We will demonstrate how these transforms, when presented in GeoGebra [1], make useful and interactive [1] [2] [3] teaching tools.

Introduction to the Michaelis-Menten enzyme activity curve

One of the most common mathematical presentations given in a Biochemistry class is that of the simple Michaelis-Menten enzyme activity curve, [2]. This curve is important because it illustrates a mathematical expression of the rate at which enzymes create product from a substrate molecule. This presentation usually derives the following hyperbolic expression from first principles involving a fixed concentration of the enzyme and various concentrations of the substrate:

$$V = \frac{V_{max} s}{K_m + s} \tag{2.1}$$

where V is the velocity of an enzyme converting substrate to product, V_{max} is a constant that describes the maximum rate of this conversion, K_m is the Michaelis-Menten constant specific for the enzyme in question, and s is the substrate concentration. The resulting hyperbolic function is no longer a challenge to modern computer fitting programs, but in the old days it was useful to linearize the hyperbolic Michaelis-Menten curve into the double reciprocal Lineweaver-Burk space, [3]. In that space it is straightforward to determine K_m and V_{max}.

In Figure 2.1, the Lineweaver-Burk plots are the dashed lines, with $1/V_{max}$ as the intercept on the $1/V$ axis and V_{max}/K_m is the slope. Further, the intersection of the line with the $-1/s$ axis in the second quadrant is $-1/K_m$. For the purpose of instruction, it is useful to show the effects on the Lineweaver-Burk plot due to three different types of enzyme inhibition by a second molecule. These three common types of inhibition are:

1. Competitive inhibition, where the second molecule also binds at the enzyme's active site. This produces a Lineweaver-Burk plot with a higher slope but the same V_{max}.

[1] Michaeis Menten: `http://tube.geogebra.org/material/show/id/1491085`
[2] Scatchard: `http://tube.geogebra.org/material/show/id/1491063`
[3] SCATSUM: `http://tube.geogebra.org/material/show/id/1498145`

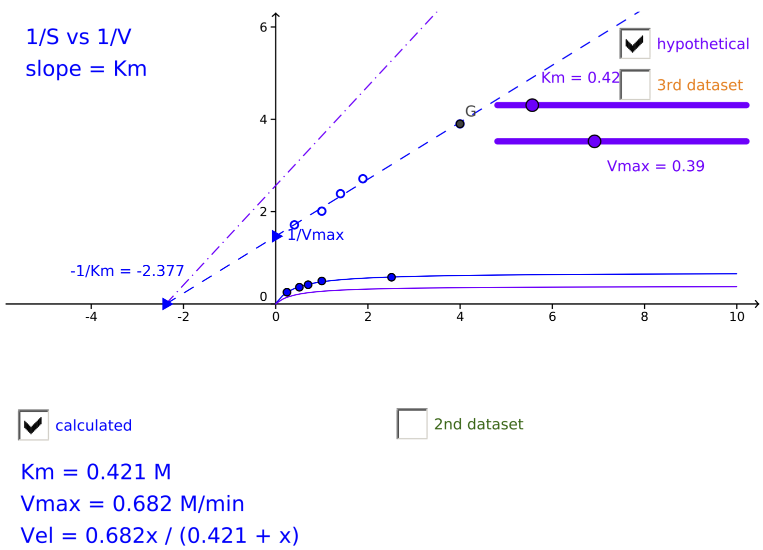

Figure 2.1 Both Michaelis-Menten and Lineweaver-Burk data are plotted in the same GeoGebra space

2. Noncompetitive, where the second molecule binds at a different site on the enzyme but also causes inhibition, which produces a Lineweaver-Burk plot with lower slope but the same K_m.

3. Uncompetitive; where the second molecules only binds to the enzyme after the substrate binds to the active site, which produces a Lineweaver-Burk plot parallel to the original plot with changes in both V_{max} and K_m.

The changes in both the Michaelis-Menten and the Lineweaver-Burk lines are readily apparent on the GeoGebra plot when used in the interactive mode. The two hyperbolic Michaelis-Menten curves seen at the bottom of Figure 2.1 appear only slightly different. When these are transformed into the Lineweaver-Burk space (dashed lines) the difference in the two K_m and V_{max} values are much more readily apparent. Note that Figure 2.1 shows an example of Competitive inhibition since K_m is different in the two dashed lines but V_{max} is the same.

Method 1

How the GeoGebra charts were made: Given a set of data in the spreadsheet, we can use a GeoGebra command to create a list of points. This list will have the property that the points will update when we change values in the spreadsheet. The traditional biochemistry technique for the data is to plot the point that has absicssa equal to the reciprocal of the x-value and ordinate equal to the y-value. In GeoGebra this is done through the `Sequence` command to create a new list of points:

```
list2=Sequence[(1/x(Element[list1,i]),1/y(Element[list1,i])),i,1,Length[list1]].
```

From that list, we can use the `FitLine` command to do the least squares regression.

Mathematically, this is interesting for a few reasons. First, it is an excellent real world application of rational functions, which seem to be few and far between. This even has the bonus of a contextually significant horizontal asymptote. The asymptote is the maximum velocity of the enzyme's ability to convert substrate to product at the given conditions of temperature and pH. Also, the traditional biochemist's technique of linear transformation is a nice bit of algebra. If $y = ax/(b+x)$ then $1/y = (b+x)/(ax) = (b/a)(1/x) + (1/a)$. where $1/y$ is linear with respect to $1/x$. Then we can see the y-intercept of this line is the reciprocal of the maximum velocity, and the slope of the line lets us derive the K_m parameter from knowing the maximum velocity.

Introduction to the Scatchard binding isotherm

The second type of hyperbolic function concerns the binding interactions of macromolecules and smaller ligand molecules and is a branch of biochemical kinetics. This type of kinetics is often treated in advanced undergraduate biochemistry courses or in qualifying graduate courses. The goal of these kinetic studies is to determine both the binding affinity of the smaller ligand molecule for the macromolecule and the number of potential binding sites available to the ligand often using radiolabelled ligands. The binding kinetics are described mathematically as

$$B = \frac{nFK_a}{1+FK_a} \tag{2.2}$$

where
B is the amount of ligand bound to the macromolecule,
n is a constant that describes the maximum number of ligand binding sites on the macromolecule,
Ka is the association constant specific for the ligand in question, and
F is the free substrate concentration.

The resulting hyperbolic binding function is also no longer a challenge to modern computer fitting programs, but in the old days it was useful to linearize the binding function into the well-studied Scatchard binding isotherm [4] by plotting Bound/Free on the y-axis and Bound on the x-axis.

$$B/F = nKa - BKa \tag{2.3}$$

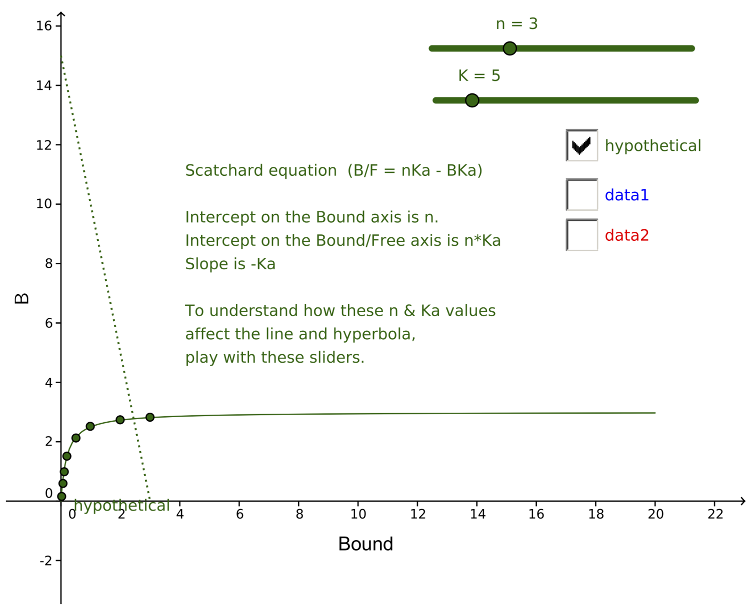

Figure 2.2 Example of a linear Scatchard plot with 1 class of equivalent binding sites

In those biochemistry lectures, it is usually pointed out that a linear Scatchard plot results when binding involves a single class of equivalent and independent sites. In that case it is straightforward to determine Ka and n, since $-Ka$ is the slope of the line, nKa is the intercept on the B/F axis and n is the intercept on the B axis, as shown in Figure 2.2. Often examples are given to illustrate the effects of variation in Ka and n on the Scatchard line shape.

In more advanced classes, examples of ligand binding that produce non-linear Scatchard plots are examined. In contrast to the simple linear plot, a non-linear Scatchard plot is produced when two or more non-equivalent binding sites exist on the macromolecule for the same ligand. The non-linear Scatchard plot is the vector summation of a number of straight line Scatchard plots[5]. Each line corresponds to an appropriate Equation 2.3. The total concentration of bound ligand (B) is the sum of concentrations of all the species of bound ligand, where each kind of bound ligand is characterized by its intrinsic binding constant to its own binding site.

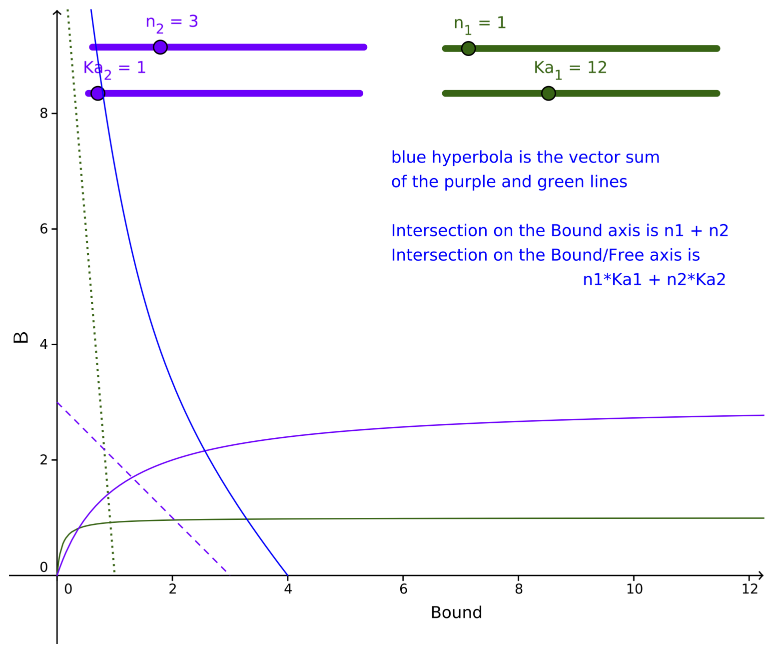

Figure 2.3 Example of a non-linear Scatchard plot with 2 classes of equivalent binding

In the non-linear type of Scatchard plot as shown in Figure 2.3, the intercept on the Bound (B) axis is the sum of the individual ligand binding numbers, n values, $(n_1 + n_2 + \ldots + n_j)$ and the intercept on the bound/free (B/F) axis is the sum of the individual nKa products $(n_1Ka_1 + n_2Ka_2 + \cdots + n_jKa)$. More directly, in Figure 2.3, the blue hyperbolic line represents the experimentally derived, non-linear Scatchard line. The intercept on the Bound axis is 4 and the intercept on the bound/free axis is 15. Since this curve is non-linear, we know it is formed by two or more straight lines; in this case two straight lines. By definition, the simple Scatchard plot is linear with a slope of $-Ka$, a Bound intercept of n and a bound/free intercept of nKa, as in Figure 2.2. So the program allows the user to define the two component Scatchard lines and then does the vector addition to draw the non-linear composite line which would be found experimentally given the conditions of the two component lines. In Figure 2.3 the first simple Scatchard line is defined by $n_1 = 1, Ka_1 = 12$. The second simple Scatchard line is defined by , $n_2 = 3$, and $Ka_2 = 1$. This represents in nature the idea that the macromolecule has two types (classes) of response to the same binding ligand. The first class is a strong binding response with a Ka value of 12. This response is specific with an n_1 value of 1 and usually is employed when ligand availability is low. The second class of response is weaker with a Ka_2 value of 1 and is non-specific with an n_2 value of 3. This weak class of response is employed when ligand availability is high. These conditions are often seen in circulating blood proteins when there is a single, high affinity binding site represented by the $n_1 = 1$ line and also a number of low affinity, non-specific sites represented by the $n_1 = 3$ line. Note that in the hyperbolic summation line, the intercept on the Bound axis is $n_1 + n_2 = 4$ and the intercept on the bound/free axis is $n_2Ka_1 + n_2Ka_2 = 15$ as would be expected by theory.

Second Method

The simplest example of a non-linear Scatchard plot involves the case of two classes of independent sites. To produce this type of plot precisely, the sheet, SCATSUM, was written in GeoGebra. SCATSUM uses the curve peeling method of Rosenthal [5] as a model to construct the resultant Scatchard curve from two straight Scatchard lines. This task is achieved by vector addition of points on each straight line. Rosenthal's original method offers a full explanation of deconvoluting a curved Scatchard plot into its straight line components. Here, we are doing the reverse, starting with the known, n_1, Ka_1, n_2 and Ka_2 values that form straight lines and creating the resulting curvilinear Scatchard line.

Biochemists making a multiple variable Scatchard plot decompose the data into lines fit to data in (Bound, Bound/Free) form. These are the decomposition of a hyperbola that is the vector sum of these lines. This is a very unusual construction in mathematics and quite interesting! In order to do this in the GeoGebra sketch, the lines were converted to a common parametric form, $y = ax + b$ to $(b/(t-a), bt/(t-a))$ where $t = 0$ corresponds to the x-intercept. This allows us to easily add the lines. Parametric curves are graphed in GeoGebra using the Curve function. For example, the curve in the SCATSUM sketch is

```
Curve[Ka_1 n_1/(t+Ka_1)+Ka_2 n_2/(t+Ka_2),t(Ka_1 n_1/(t+Ka_1)+Ka_2 n_2/(t+Ka_2)),t,0,100]
```

Remember, in the Scatchard context, Ka is the opposite of the slope, and n is the x-intercept. So the y-intercept of the Scatchard lines is nKa.

In SCATSUM, the sliders allow the user to create their own n_1, Ka_1, n_2 and Ka_2 values. These values are used to create the two linear Scatchard lines, then these lines are summed by vector addition to form the resultant non-linear curve. SCATSUM output utilizes the Ka and n values as well as a column of fixed Free ligand values, column A, in the spreadsheet portion. Column B shows the list of generated values on the Bound axis and column C shows the generated values on the Bound/Free axis associated with one set of n and Ka values. Columns D and E respectively show the Bound and Bound/Free values associated with the other set of n and Ka constants. Because SCATSUM will produce precise Bound and Bound/Free values for the curvilinear Scatchard plot, the student can use those values and a different curve fitting program to determine the constants using the summation equation

$$B = \frac{n_1 F Ka_1 + n_2 F Ka_2}{(1 + Ka_1 F)(1 + Ka_2 F)}. \tag{2.4}$$

Discussion

In GeoGebra, the `Function` command can be used to restrict the domain of a function.

```
Function[<Expression>,<Parameter Variable 1>,<Start Value>,<End Value>].
```

This is nice in modeling contexts, to have the start value and the end value be physically reasonable limits. Using this function and the Sliders in GeoGebra, one can quickly and dramatically present the effect of changing parameters on changes in line shapes. We hope to further develop a suite of line transformations that would be useful to biochemistry students such as irreversible/covalent inhibitors, cooperativity and Hill coefficients and mixed inhibitors.

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VISUALIZING SOLIDS OF REVOLUTION IN GEOGEBRA

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Abstract: Surfaces can be graphed in GeoGebra by either using the `Surface` command, defining a function $f(x,y)$, or inputting an equation of at most three variables. Some surfaces can be easily graphed using a function of two variables or an equation, but we have more control over the surface if the `Surface` command is used. We start by graphing surfaces of revolution encountered in single variable calculus class. A surface of revolution can be graphed in GeoGebra provided we can parameterize the circular cross-section. This method of graphing a surface of revolution can be applied to model objects with circular cross-section, like a Hershey's Kiss, in GeoGebra.

Keywords: solids of revolution, modeling, multivariable, GeoGebra: `Surface`

Introduction

Students encounter solids of revolution during their study of integral calculus in a single-variable calculus class. A solid of revolution is created by revolving a planar region R about a line. Integration can be used to find the volume of a solid of revolution, but the solid may be difficult for some students, or even teachers, to visualize. Unfortunately, GeoGebra cannot graph a solid of revolution, but there is a way for GeoGebra to help. The idea is to use GeoGebra to graph the surface of the solid of revolution.

The graphing of a surface of revolution can be accomplished by describing the surface as a function $f(x,y)$, but we will use parametric equations to describe the surface and graph using the `Surface` command. As it turns out, the parametric equations used to describe a surface of revolution are simple and easy to manipulate. The difficulty with using parametric equations lies in creating the equations to describe the surface. After learning how to graph a surface of revolution, we apply our method to model the surface of a Hershey's Kiss.

Graphing a surface of revolution

Suppose a region R in the xy-plane is bounded by $x=a$, $x=b$, the x-axis, and f, a function that is continuous on $[a,b]$. Let S be the solid of revolution created by revolving R about the x-axis. To visualize S, we graph the surface of S, which is created by revolving the graph of f on $[a,b]$ about the x-axis. The surface of S can be described by the equation

$$y^2+z^2=[f(x)]^2 \tag{3.1}$$

with $x \in [a, b]$. One way to graph the surface of revolution is to solve equation (3.1) for z. Doing so gives the functions

$$g(x, y) = \sqrt{[f(x)]^2 - y^2}$$

$$h(x, y) = -\sqrt{[f(x)]^2 - y^2}$$

which can be put into GeoGebra without trouble. This method will actually graph more than the desired surface, though, causing problems in visualizing the solid. Even though f is defined on the interval $[a, b]$, GeoGebra will show the function on its whole domain, creating an inaccurate picture of the surface.

To create a better graph of the surface of revolution, the surface will be plotted using *parametric equations.* The benefit of graphing with parametric equations is the amount of control over the curve/surface. When we use parametric equations to graph a surface of revolution, we can set the domain of the equations explicitly. Also, parametric equations provide enough flexibility to create myriad curves and surfaces with ease.

To parameterize a surface of revolution, we need three equations in two-variables. One variable, t, parametrizes the interval $[a, b]$ on the x-axis, and another variable, u, parametrizes the cross-sections parallel to the yz-plane.

For a fixed $t \in [a, b]$, the equation of the cross-section of the surface of revolution is given by

$$y^2 + z^2 = [f(t)]^2,$$

a circle centered on the x-axis with radius $f(t)$. In the xy-plane, a circle with radius r and center (h, k) can be parameterized by

$$x = h + r\cos(u), \qquad y = k + r\sin(u), \qquad u \in [0, 2\pi].$$

Since the circular cross-section of a surface of revolution has radius $f(t)$ and center on the x-axis, it can be parameterized by

$$y(u) = f(t)\cos(u), \qquad z(u) = f(t)\sin(u), \qquad u \in [0, 2\pi].$$

Hence, the equations that describe the surface of the solid of revolution are

$$x(t, u) = t,$$

$$y(t, u) = f(t)\cos(u),$$

$$z(t, u) = f(t)\sin(u),$$

for $t \in [a, b]$ and $u \in [0, 2\pi]$.

To graph the surface in GeoGebra, we use the `Surface` command,

```
Surface[x(t,u), y(t,u), z(t,u), t, a, b, u, c, d]
```

where $x(t, u)$, $y(t, u)$, and $z(t, u)$ are the coordinate functions, t is the first parameter, ranging from a to b, and u is the second parameter, ranging from c to d. To graph a surface of revolution, the `Surface` command would look like

```
Surface[t, f(t) cos(u), f(t) sin(u), t, a, b, u, 0, 2 pi].
```

For example, suppose the region R in the xy-plane is bounded by the equations $x = 1, y = 0$, and $y = x^2$, and let S be the solid created by revolving R about the x-axis. The surface of S can be described by the equations

$$x(t, u) = t,$$

$$y(t, u) = f(t)\cos(u) = t^2\cos(u),$$

$$z(t, u) = f(t)\sin(u) = t^2\sin(u)$$

with $t \in [0, 1]$ and $u \in [0, 2\pi]$.

The command to graph the surface of S would be

```
Surface[t, t^2 cos(u) , t^2 sin(u), t, 0, 1, u, 0, 2 pi].
```

Figure 3.1 shows the region R in the xy-plane, and Figure 3.2 shows the surface of S created by revolving the graph of x^2 on $[0, 1]$ about the x-axis.

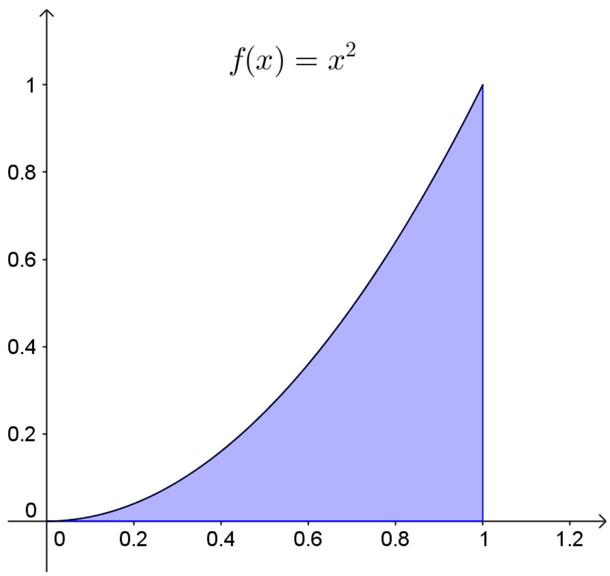

Figure 3.1 The Region R

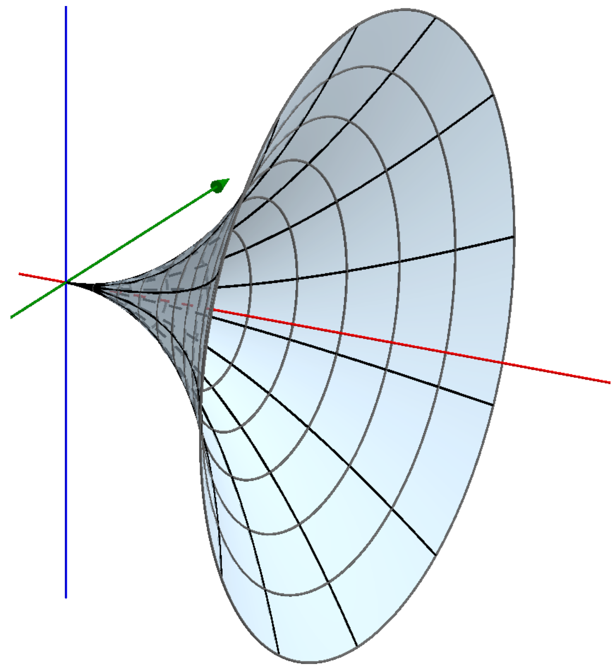

Figure 3.2 The Surface of S

Application: Modeling a Hershey's Kiss

One interesting feature of GeoGebra is its ability to insert images directly into the Graphics view. When a picture is inserted into the Graphics view, the user can manipulate the picture and apply transformations like translations, rotations, and dilations. This allows the user to investigate properties of an image, like the types of symmetry it possess, or use the image for modeling.

When graphing a surface of revolution, GeoGebra can be set up so the surface can be easily altered just by changing the boundary function $f(x)$ and/or the bounds, which allows students to easily manipulate a solid. To see how GeoGebra can be used to model a 3-dimensional object, we will create a 3-D model of a Hershey's Kiss. A Hershey's Kiss can be viewed as a solid of revolution where the axis of revolution goes through the tip and the center of its circular base, so its surface can be graphed in GeoGebra if we can find a boundary function. This is where we use GeoGebra's ability to insert images onto the Graphics view.

To create a model of a Hershey's Kiss, we get a picture of the Kiss and insert the picture into GeoGebra. Then, we move the picture around so that the axis of revolution of the Kiss coincides with the x-axis and the center of the base matches up with the origin. Once the image is in place, we do not want it to move around, so we will put it in the background. To do this, go to `Object Properties` under the `Basic` tab, click on the checkbox for `Background Image`.

With the image in GeoGebra, the boundary function can be created. A fifth-degree polynomial approximates the boundaryof this shape quite well. Six points are plotted on the boundary of the kiss, and a polynomial is created through those six points using the command

```
f(x)=Polynomial[C, D, E, F, G, H].
```

See Figure 3.3.

Figure 3.4 shows the model of the surface of the Hershey's Kiss created from the command

```
Surface[t, f(t) cos(u), f(t) sin(u), t, a, b, u, 0, 2 pi],
```

where a is the x-coordinate of C and b is the x-coordinate of H. The boundary curve can be changed by moving the points around, giving us the option to make the boundary function fit better to the picture. By setting the image to the background, the image will remain fixed even when the boundary points are moved.

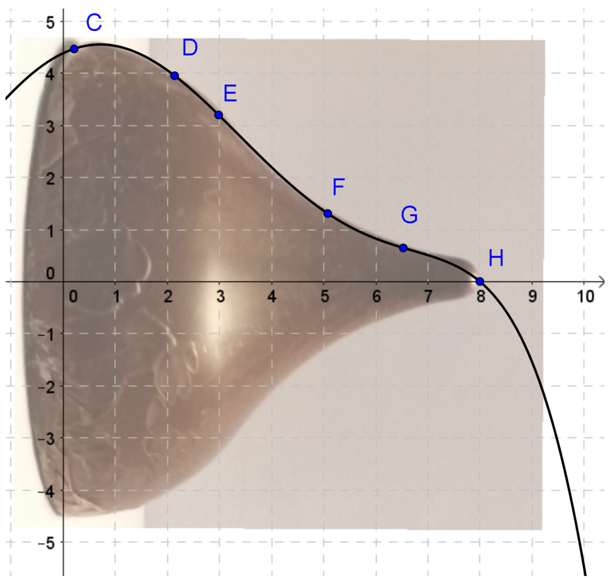

Figure 3.3 Hershey's Kiss [1] With Boundary Function and Points

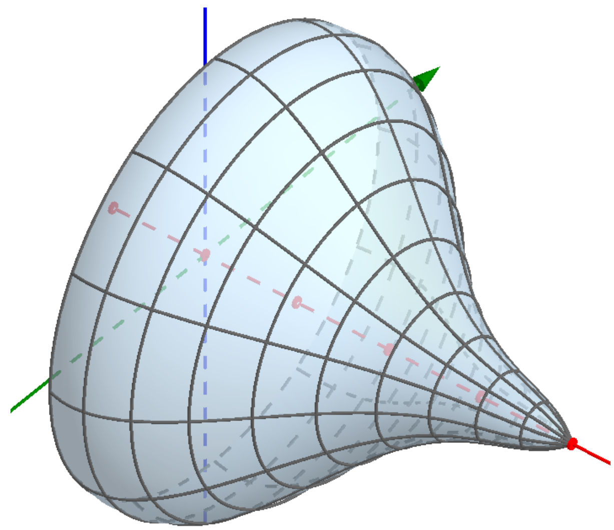

Figure 3.4 Model of the Surface of a Hershey's Kiss

Conclusion

GeoGebra can graph surfaces through the use of the `Surface` command. The biggest hurdle with graphing surfaces with nice cross-sections is parameterizing the cross-section. Surfaces of revolution have circular cross-sections which make their parameterization simple and easy to change. Surfaces with different cross-sections can be graphed as long as the cross-sections can be parameterized. Graphing surfaces using the `Surface` command can be used to help create models of everyday objects like a Hershey's Kiss. With some experimentation and a little skill, students and teachers can not only create new and exotic surfaces but see them too.

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AN INFORMAL APPROACH TO LINEAR LEAST SQUARES

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Abstract: Modeling data using the least squares method is used extensively in practice, and is therefore an essential topic for students of data science. This paper describes a GeoGebra applet used to facilitate understanding of the objective and the underlying mathematics of the least squares method. Additionally, lists, one of the most robust and valuable GeoGebra features is highlighted through the use of the `Sequence` command. The discussion includes ideas for enhancing the applet.

Keywords: least squares, approximation, GeoGebra: lists, `Sequence`

Introduction

Students often struggle, when using software, to understand concepts that underlie mathematics because the intermediate steps are hidden from them in a decidedly "black-box" fashion. Buchberger [1] recommended that the black-box software be replaced with a "glass-box" in which steps that promote student understanding are included. The current topic is a prime example. While students have probably encountered the Least Squares Method (LSM) on several occasions, possibly even on a graphing calculator as far back as 7th grade, the underlying mathematics has been hidden from them.

Case in point, a project assigned in an undergraduate computer programming course asked approximately 15 to 20 students to code the LSM. The project requirements included reading the data set and solving the normal equations, which were to be derived offline using calculus, a prerequisite of the course, to produce the best linear model for the data. In the first semester the project was assigned, the concepts that underlie the method were described in a handout as well as covered and discussed during class. Additionally, students were given several references (for example, [6]) in the handout that provided in-depth coverage of the technique. However, less than one quarter of the students were able to successfully complete the assignment without direct assistance from the instructor or other students.

In response to these difficulties, a GeoGebra [3] applet was created for subsequent semesters affording students the opportunity to interact with most of the underlying concepts and experience the method dynamically. After using the applet in conjunction with the other resources, all but a few students were able to successfully code the project without much instructor assistance.

This article demonstrates how GeoGebra can be used to investigate and understand the method of least squares to fitting data. First, we describe the applet including the interface objects and commands needed. Additionally we describe its use in understanding the LSM. Next, we discuss ways to improve the applet, provide some in-depth ideas for students to explore, and highlight calls by prominent organizations to include its study in school mathematics. We point out the incredible opportunity this topic affords teachers to improve their technological skills, sharpen their mathematics, and prove their pedagogical knowledge.

Least Squares Method and GeoGebra Applet

The GeoGebra applet used to demonstrate the underlying processes of the LSM is composed of two windows, *Graphics* and *Graphics 2*. See Figure 4.1. *Graphics* contains the scatterplot of the data (X_i, Y_i) in green, the linear predictive model ($\hat{Y}_i = aX_i + b$) in blue, and the residuals in red. *Graphics 2* contains standard user interface controls like checkboxes and sliders as well as a text object showing the computational results of the sum of the squared errors (SSE).

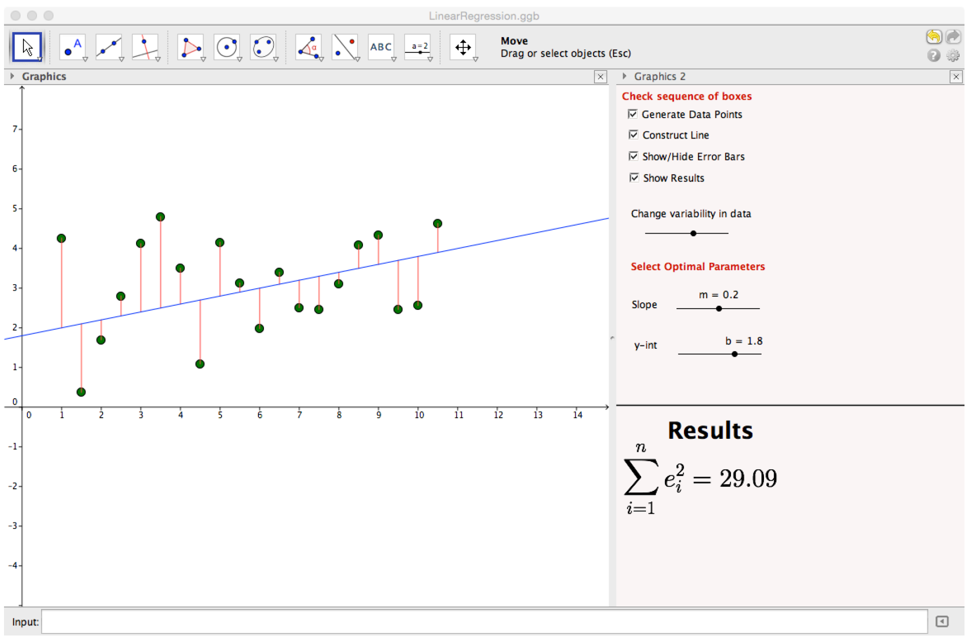

Figure 4.1 GeoGebra applet for understanding the Least Squares Method

Checkboxes are used to show/hide the model, the residuals, and the SSE. A slider is used to adjust the variability in the data set. The order in which the checkbox controls are placed is important to the learning process. That is, first we have to generate a data set, then construct the linear model, then reveal the error in the model.

With or without the SSE shown, students can dynamically explore different models by moving the sliders associated with varying the two parameters, slope and y-intercept. If the SSE is shown, students observe that better fitting lines are associated with smaller numbers.

The crux of the matter in terms of developing the applet is knowledge of lists in GeoGebra. First, two lists are generated using the `Sequence` command, one for the X's and one for the Y's. The X values are selected at equally spaced points while the Y values incorporate normally distributed random errors. These values form a list of ordered pairs called, in GeoGebra, `data`. The residuals are a list of segments created with the `Sequence` command. See Appendix 1 for complete construction protocol.

There are several potential improvements to the GeoGebra applet that can further develop advanced students' conceptual understanding of LSM. The applet could have options to append an outlier to the data set as the method is sensitive to outliers, to include heteroskedasic data, allow for non-normality in residuals, and have options for alternative data models such as quadratic, logarithmic, and exponential.

Discussion

Because the objective of LSM is to minimize total error, while experimenting, it is natural for students to question why compute the SSE instead of just the sum of the residuals. Upon further examination and continued discussion among their peers, the applet allows them to observe that the model overestimates some values while underestimating others (in other words, some of the residuals are positive and some negative), thus realizing the potential for catastrophic cancellation. Moreover, the use of GeoGebra in this project can aid students in a number of other ways.

First, it may help reinforce previous material. For example, the use of a dynamic sliders controlling the model parameters serve as a reminder of the effects slope and intercept have on a line. Second, it may be a platform to introduce new topics such as summation notation. Third, it provides an opportunity to quickly and efficiently view some concepts such as variability in data with a dynamic graphical representation which would be difficult if not impossible in a static environment. There are many other notions to explore and discover with the help of GeoGebra, too. In particular,

1. the sum of the residuals is zero (this can be easily proven for students that derive the normal equations),
2. the sum of the observed values Y_i equals the sum of the fitted values $\hat{Y}_i = mX_i + b$,
3. the weighted sums, $\sum_{i=1}^{n} X_i e_i = 0$, and $\sum_{i=1}^{n} \hat{Y}_i e_i = 0$ hold, and
4. the regression line always goes through the point $(\bar{X}, \bar{Y})$.

Although originally developed to demonstrate the LSM process to mathematics students in an undergraduate programing course, the method and applet are appropriate for students in middle and high school. Both the National Governors Association and the National Council of Teachers of Mathematics call for students to have experience fitting data and deciding among many possible linear models. In particular, the *Common Core State Standards* recommend fitting data to a linear model and assessing its fit informally even at the middle school level [2]. The National Council of Teachers of Mathematics calls for students to understand the least squares regression line with a visual model and measure error in this model as well as determine the “goodness of fit" [5].

Lastly, developing and implementing technology into the curriculum can be a powerful means of teaching but requires significant combination of mathematical, technological, and pedagogical knowledge. In particular, teachers must have a deep understanding of the underlying mathematics, know multiple ways to represent mathematical objects, be aware of potential student difficulties, and be able to create helpful illustrations [4].

Conclusion

This interactive GeoGebra applet provides students at multiple levels with an opportunity to understand, at least informally, the LSM using a dynamic visual model. It quickly and effectively conveys the overall objective of the method of finding parameters (slope and y-intercept) that minimize the sum of the squared errors. Additionally, LSM provides a context for improving Technological, Pedagogical, and Content Knowledge (TPACK). The most important addition to any future versions of the applet is the idea of “goodness of fit".

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Appendix 1: Construction protocol

1. Define a number (the standard deviation of the data points), `std = 1.2`
2. Define a number (the slope), `m = 0.2`
3. Define a number (the y-intercept), `b = 0.2`
4. Define a linear function, `f(x) = m*x+b`
5. Define a number (of data points), `n = 20`
6. Define a list of x's, `xs = Sequence[1 + 0.5k, k, 0, n]`
7. Define a list of y's, `ys = Sequence[k / k RandomNormal[3, std], k, 1, n]`
8. Define a list of data points,
   `data = Sequence[(Element[xs, i], Element[ys, i]), i, 1, n]`
9. Define sequence of error bar segments,
   ```
   errors = Sequence[Segment[Element[data, i],
   (x(Element[data, i]), f(x(Element[data, i])))], i, 1, n]
   ```
10. Define a list of squared errors, `sqerror = errors^2`
11. Define a number, the sum of square errors (SSE), `SSE = Sum[sqerror]`
12. Define text to display results,
    `Results = FormulaText["$ \sum_i=1^n e_i^2 = $"] + FormulaText[SSE]`

SHORT PRESENTATIONS

ABSTRACTS

5.1 Measures of Center - Dr. Forest Fisher, Guttman Community College, NY

Abstract: Using GeoGebra to help students make sense of the mean and median as measures of center. The activity is designed according to APOS theory with the goal of developing an object understanding of the mean and median. In particular, students work with a dot plot in GeoGebra, and drag the dots around to dynamically manipulate the mean and median.

5.2 Investigating Parametric Equations on GeoGebra - Christa L. Fratto, Greens Farms Academy, CT

Abstract: This session focused on using GeoGebra to develop an understanding of parametric functions. We investigated the topic from a graphical, numerical and symbolic perspective with graphs and spreadsheets. We then used GeoGebra to model the motion of a swing

5.3 Polar Curves using Parametric Equations - Rasha Tarek, Greenwich H.S., CT

Abstract: This presentation showed how to use GeoGebra to investigate the connection between polar and rectangular equations with ease and learn how to utilize parametric equations to create some amazing polar curves.

5.4 Z-Scores - Dr. Audrey Nasar-Guttman, Community College, NY

Abstract: The presentation showed how to use GeoGebra to introduce z-scores as linear functions. This approach aims to help students make the connection between the normal distribution and the Cartesian plane in efforts to better understand relative standing and build upon students prior knowledge of linear functions. By visualizing the linear functions for a variety of normal distributions, students can interpret the domain, range, and x-intercepts in a dynamic environment.

5.5 Creating Transformations in GeoGebra - Dr. Adam Goldberg, SCSU

Abstract: The presentation will show how to create and use transformations with GeoGebra, starting with basic transformations such as rotations, reflections, and translations and moving to more advanced ones such as glide reflections and dilations.

INTERACTIVE POSTERS

ABSTRACTS

6.1 Varignon's theorem - Sandra Ollerhead, Mansfield High School, MA

Abstract: This activity explores Varignon's Theorem which states, "The midpoints of the sides of an arbitrary quadrangle form a parallelogram." It should be used after the students have learned the properties of quadrilaterals. As students answer each question they will be required to think about which properties are enough to guarantee that a quadrilateral is a parallelogram, rectangle, rhombus, or square.

6.2 The Ferris Wheel - Rasha Tarek, Greenwich HS, CT

Abstract: As you sit on a Ferris wheel your position varies sinusoidallly with respect to time. Your position can be traced using a set of parametric equations. In this animation you see yourself on the wheel and as the wheel turns two sinusoidal graphs are formed; one graph represents your horizontal distance from the origin, and another curve represents your vertical distance from the origin.

6.3 Reimann's Sums - Rasha Tarek, Greenwich HS, CT

Abstract: We use Reimann's sums to estimate the area under a given curve using rectangles. Depending on the concavity of the curve and the type of Reimann's sum used we may end up with an underestimate or an overestimate. This animation displays both the Left Reimann approximation and the Right Reimann approximation. The total area from each sum is displayed as is the difference between the two. As we increase the number of rectangles that error is diminished and the limit of the sum approaches the actual area under the curve.

6.4 The Ambiguous Case - Rasha Tarek, Greenwich HS, CT

Abstract: When solving a triangle using the law of sines we sometimes run into some ambiguity. This happens when we are given the lengths of two sides and the measure of a non-included angle. When solving such triangles the number of the solutions depends on the length of the height constructed across from the given angle. This animation shows the different cases of the ambiguous case; one triangle, two triangles, or no triangles.

6.5 Exploring the laws of Sines and Cosines in a Pre-Calc class - Dr. Jason Hardin, Worcester State University, MA

Abstract: The laws of sines and cosines are important topics in trigonometry, allowing one to solve non-right triangles given a side length along with two additional pieces of data (lengths of remaining sides, angle measures). These identities are usually presented as formulas for students to memorize. In this "poster" we look at an exploratory activity

in which students use GeoGebra to create triangles of various shapes and sizes, and then use the side lengths and angle measures to verify the laws of sines and cosines for the triangles they've created. This activity uses a simple GeoGebra applet which was easy to create and which students found enjoyable and enlightening.

6.6 Completing the squares - Brian Darrow, Jr., Southern Connecticut State University, CT

Abstract: Students are generally taught the process of completing the square in a very procedural manner where the reasons for, and the linkages between, each step are often not made clear. Observing the process visually (geometrically) might offer another perspective to the students who merely memorize the procedure without fully understanding why it works or how the process results in the answer. This poster presentation will explore the geometric analog to the algebraic process of completing the square through the actual "completion" of a two-dimensional square in the plane.

6.7 Taylor Polynomial Exploration - Dr. Leon Brin, SCSU/GISCT, CT

Abstract: Users will generate Taylor polynomials and their graphs with this worksheet. The degree of the polynomial is set using a slider. The center of expansion is set by dragging a point along the x-axis. The function to approximate is entered in a text box. The Taylor polynomial and its graph are displayed dynamically as changes are made.

6.8 Sierpinski's Gasket - Dr. Leon Brin, SCSU/GISCT, CT

Abstract: Sierpinski's triangle is a commonly depicted fractal. This worksheet generates an approximation one point or many points at a time using the random iteration algorithm. A point is given inside the triangle. A new point is created midway between the given point and a randomly selected triangle vertex. The next point is created midway between this new point and another randomly selected vertex. Successive points are created likewise. Using buttons to control the procession of the random iteration algorithm, the user can see Sierpinski's Triangle develop at whatever rate suits their fancy.

6.9 Level curves - Dr. Braxton Carrigan, SCSU/GISCT, CT

Abstract: In multivariate calculus, it is important for students to develop a good grasp of level curves before developing needed skills such as integration by slicing, gradient vectors, etc. This worksheet allows us to interact with the surface, graphed in the 3-D graphics window, simultaneously with its level curves, graphed in the 2-D graphics window. The user is asked to enter any two-variable polynomial, then uses a slider to vary a plane horizontal to the xy-plane. Simultaneously the level curve that is the intersection of the curve and the plane is graphed in the 2-D graphics window.

6.10 Special lines in a triangle - Dr. Marie Nabbout, SCSU/GISCT, CT

Abstract: The worksheet allows students to learn about the special lines in a triangle (medians, heights, bisectors, perpendicular bisectors). Students make conjectures about the location of the intersection points and study the cases of isosceles and equilateral triangles. They also learn about Euler's line.

6.11 Pythagorean Identities - Albert Navetta, Univ. New Haven, CT

Abstract: The worksheet provides a graphical approach to learning and discovering Pythagorean identities of trigonometric functions. Starting by identifying right triangles associated with the 3 familiar identities, we then identify other right triangles and find more Pythagorean identities. With a couple of examples, we then demonstrate how to construct new right triangles to find limitless Pythagorean identities. Along the way we discover geometric meanings for several less familiar trigonometric functions.

6.12 Shikaku Puzzle - Alex Briasco Brin, Freeport Middle School, ME

Abstract: The challenge is to completely cover a 10 by 10 grid with exactly 13 of the given colored rectangles without overlap. The user will simply drag and drop the rectangles into place. The catch is each rectangle must contain exactly one number, and that number must represent the rectangle's area.

6.13 How Fast Are You Spinning? - Tim Brzezinski, Berlin High School, CT

Abstract: The following applet allows students to see the circumference of the circle of latitude that he/she rotates around in one day. Our linear speed depends upon our latitude. In this applet, students use a slider to enter their latitude to the nearest tenth of a degree. Students are then asked to solve for one's linear speed at the equator and then solve for one's linear speed at his/her location on Earth. Any student with a solid knowledge of right-triangle trigonometry should be able to easily solve for his/her linear speed from the visual provide by the applet. After a student answers each of these 2 questions, a checkbox is provided for him/her to check his/her solution. The key introduction is contained in the worksheet & leading questions for students are provided in the applet itself.

6.14 Exterior Angles of Polygons: Proof Without Words - Tim Brzezinski, Berlin High School, CT

Abstract: This applet allows students to easily discover that the sum of the measures of the exterior angles of any convex polygon (with $n = 3, 4, 5, 6, 7$, or 8), is always $360°$. This applet can be used by teachers to either (a) have students quickly and informally discover this theorem during class or (b) have students who may have forgotten this theorem after learning it re-experience meaningful remediation. Key leading questions are provided in the worksheet.