# EXPLORING CONSERVATION OF MOMENTUM IN INELASTIC AND ELASTIC COLLISIONS AND EXPLOSIONS

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**Abstract**

*A GeoGebra applet was created to supplement lectures and experiments on collisions for high school or introductory college physics. With this simulation students can explore two-body one-dimensional conservation of momentum problems by manipulating the masses and velocities of two colliding objects. The applet compares both vectorial and graphical momentum representations of inelastic and elastic collisions as well as the novel addition of explosions, a special form of inelastic collision. The authors explore the importance of representing momentum as a graphical area along with the mathematical features of GeoGebra and provide examples for using the applet in physics classrooms.*

Keywords: Physics, elastic collisions, inelastic collisions, undergraduate, GeoGebra

## 1 INTRODUCTION

### *1.1 Elastic Collisions, Inelastic Collisions and Explosions*

Many students at the high school and college level have difficulties separating nuances of momentum and kinetic energy in inelastic and elastic collisions [1]. Conservation of momentum is essential for students in introductory physics courses to understand because of its application to a broad range of physical and biological processes [2]. Using visual applications and simulations such as those provided by the Ck-12 Exploration Series [3] and collected in the blog of Frank Noschese [4], information from textbooks can be supplemented to help improve understanding of this challenging concept [5]. Using GeoGebra [6], an "applet" has been created to show side-by-side comparisons of inelastic/elastic collisions and explosions giving students more insight into various momentum scenarios.

For all collisions the total momentum $\Sigma\mathbf{p}$ (sum of all masses $m$ multiplied by their vector velocities $\mathbf{v}$) is conserved in a closed system before (initial "$i$") and after (final "$f$") the interaction, in the absence of external forces. Mathematically this is written as:

$$\Sigma\mathbf{p}_f = \Sigma\mathbf{p}_i.$$

Inelastic collisions result in objects colliding and "sticking" together, or alternatively "exploding" apart, and are identified by having different amounts of "mechanical energy" before and after the interaction. This mechanical energy is usually in the form of kinetic energy or gravitational energy. Energy analysis can be further simplified by excluding vertical motion so that the only conserved mechanical energy is kinetic. A classic example of an inelastic collision scenario is a moving train car coupling with another car at rest. The final velocity of both cars will be the same because they are traveling together with energy transferred from kinetic to thermal. On the other hand, during elastic collisions, all mechanical energy is conserved, i.e. no thermal energy is generated. But final velocities may vary, because "bouncing" can occur where the momentum from one object is passed on to the other, changing speed and even direction of the objects after the collision. Units are important in so far as the momentum and energy units must be consistent with units chosen for mass and velocity. In the introduction and math discussion below units are ignored to simplify analysis.

Explosions are like the time reversal of collisions: instead of coming together the objects separate, but as always total momentum is conserved. Typically mechanical energy increases at the expense of some type of stored energy, like chemical energy stored in a firework. For example, if two objects each have a starting mass of $1$ (summing to a total mass of $2$) and initial velocity is $+1$, then the system has a total initial momentum of $+2$. After the explosion, the total momentum is still $+2$, but the velocity of each object will differ, such as $-1$ for one object and $+3$ for the other. No matter the changes of the variables, the difference in total momentum between the starting scenario and the ending scenario is $0$, which complies with the law of conservation of momentum.

### *1.2 Introducing the Applet*

The applet (URL: `http://www.geogebra.org/m/2603873`) represents one-dimensional momentum conservation of two colliding objects (blue and red) both in vector form and as the area of each object's velocity versus mass before (initial) and after (final) a collision (Figure 1). This representation allows students to explore momentum as the geometric construct, the mass (widths of rectangles) times velocity (heights of rectangles) as the latter changes after a collision process. Students can then compare the areas to a traditional initial and final momentum vector representation for collisions or an explosion (vector arrows). In the traditional vector representation the total momentum ($\Sigma\mathbf{p}$) is simply the vector sum of momenta and are presented in "tip-to-tail" fashion to emphasize that the total momentum is indeed a sum of components. The total momentum remains constant but does not provide students transparent information about velocity changes of the objects. On the other hand the total area (which is the total momentum) before and after the collisions must remain constant thus the velocities differ transparently depending on collision type as seen in the elastic collision example shown in Figure 1. The sliders at bottom left control the masses of each object (in arbitrary mass units) and their initial velocities (in arbitrary speed units). The "*Collision Type*" enables the users to transition between three collision processes. In the case of "Explosions" the initial velocity of the combined object and a final velocity of one of the two objects is controlled.

The logical "*Collision Type*" selector is controlled through either the slider or input box with options $C = -1, 0, +1$, corresponding to inelastic collisions, explosions, or elastic collisions, respectively. The equation below was used in the calculation of the vertices of each rectangle based on solutions to Equations 1-4 described in section 1.3. The three values of $C$ specify collision type, making a single linear combination that zeros out the other two solutions depending on the slider selection.

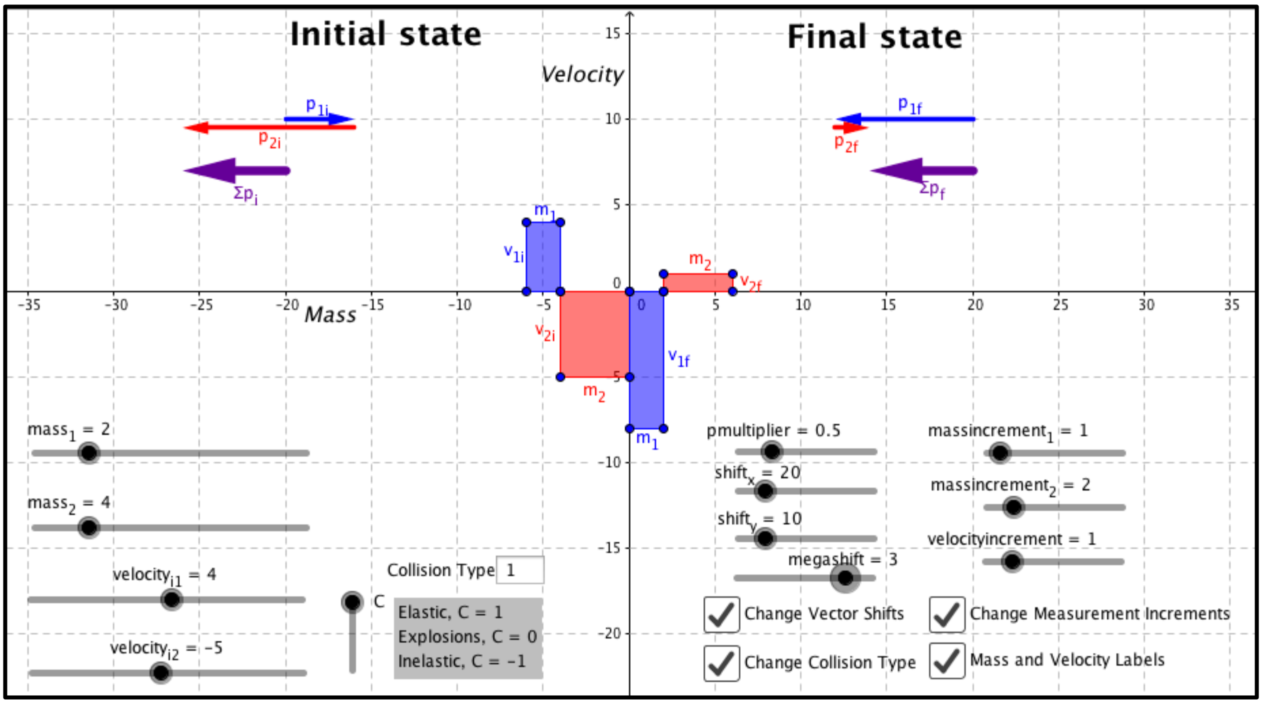


**Figure 1.** Complete applet with all features shown for the case of an elastic collision. Conservation of momentum is represented in the traditional vectorial fashion at top and the more novel total area of mass versus velocity before and after the collision below.

$$\frac{C(C-1)}{2}(\text{inelastic solution}) - (C+1)(C-1)(\text{explosion solution}) + \frac{C(C+1)}{2}(\text{elastic solution})$$

$$= \begin{cases} 1 & \text{if } C = -1 \\ 0 & \text{otherwise} \end{cases} \qquad = \begin{cases} 1 & \text{if } C = 0 \\ 0 & \text{otherwise} \end{cases} \qquad = \begin{cases} 1 & \text{if } C = 1 \\ 0 & \text{otherwise} \end{cases}$$

The momentum vectors were calculated in the same way. This decision process allows all the rectangles and vectors to appear on the same layer without conditional statements. However this means the equations to calculate the velocity vertices and momentum vectors are somewhat complicated as readers can see by viewing the property dialog boxes of the online applet.

Figure 1 also shows four hide/show check boxes in the fourth quadrant of the grid provided to simplify the screen, including one for "Change Collision Type" just described. Also included is "Change Vector Shifts" (four sliders directly above the respective checkbox in Figure 1) used to adjust the momentum vector size and locations to avoid overlap with the rectangles or vectors themselves. The "Change Measurement Increments" check box (three mass and velocity sliders directly above the respective checkbox in Figure 1) enables widely differing masses and velocities to handle such problems as an explosion of a bullet from a gun. Lastly the "Mass and Velocity Labels" checkbox allows viewing of the rectangle mass and velocity labels. The math behind the applet along with examples of three collision scenarios of varying imagination is provided next.

## 2 The Math Behind the Applet

### 2.1 Inelastic Collisions

The mathematical solution to inelastic collisions for a system consisting of two objects of mass $m_1$ and $m_2$ with initial known velocities $\mathbf{v}_{1i}$ and $\mathbf{v}_{2i}$ only requires the following conservation of momentum equation to determine the final velocity $\mathbf{v}_f$ of the two objects.

$$\Sigma_f = m\mathbf{v}_f = \Sigma\mathbf{p}_i = m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i}$$

Equation 1 is the solution of the final velocity for an inelastic collision. Physically the two objects each have different initial velocities and "stick" together with one final velocity:

$$\mathbf{v}_f = \frac{m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i}}{m_1 + m_2} \quad (1)$$

The momentum of each object is represented both through individual and total momentum vectors and equivalently the area between the mass and velocity plots shown in Figure 2.

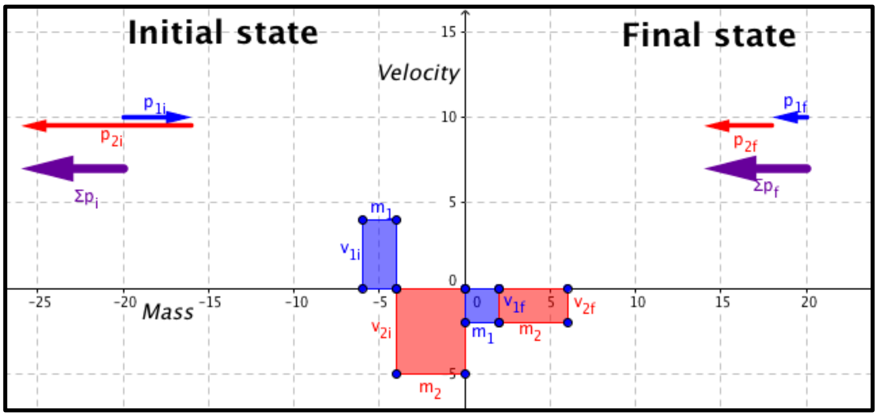


**Figure 2.** Inelastic collision example with initial conditions of $m_1 = 2, \mathbf{v}_{1i} = +4, m_2 = 4, \mathbf{v}_{2i} = -5$ and solution $\mathbf{v}_f = -2$. The final velocity can either be estimated from the grid or numerically from the Algebra panel. With the right units this could represent a head on collision between a sports car going in the opposite direction of a Sport Utility Vehicle (SUV) both traveling at the same speed.

### 2.2 Explosions

Since explosions represent a "time reversal" of an inelastic collision, the two objects of total mass equal to the sum of $m_1$ and $m_2$ are initially stuck together and could be moving at a velocity $\mathbf{v}_i$. After the explosion they separate into two separate masses and velocities $\mathbf{v}_{1f}$ and $\mathbf{v}_{2f}$. Because mechanical energy is not conserved during explosions the problem can only be solved if one of the final velocities (e.g. $\mathbf{v}_{1f}$) is known along with the initial velocity:

$$\mathbf{v}_{2f} = \frac{(m_1 + m_2)\mathbf{v}_i - m_1\mathbf{v}_{1f}}{m_2}$$

An example of an explosion is provided in Figure 3.

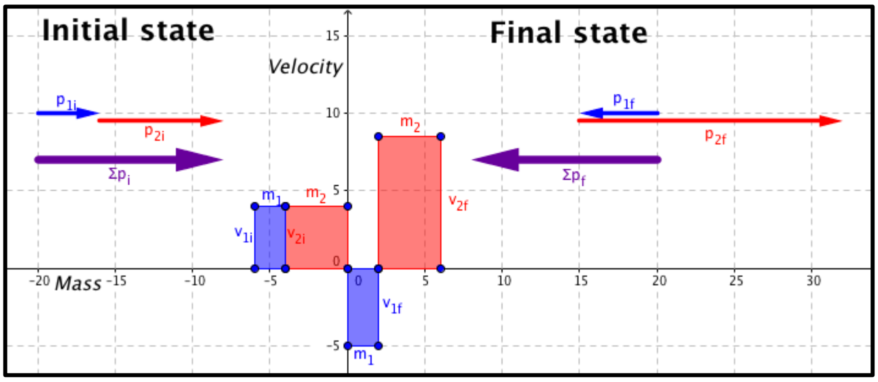


**Figure 3.** Explosion example with same initial conditions of $m_1 = 2, \mathbf{v}_{1i} = +4, m_2 = 4, \mathbf{v}_{2i} = -5$ but with solution $\mathbf{v}_{1f} = -5$ and $\mathbf{v}_{2f} = +8$. With the appropriate units this could represent a drifting astronaut pushing off a large piece of space debris in an attempt to get back to the mother ship.

### *2.3 Elastic Collisions*

The algebra associated with elastic collisions is slightly more complex involving both conservation of momentum and kinetic energy for motion in the horizontal plane:

$$\Sigma \mathbf{p}_f = m_1\mathbf{v}_{1f} + m_2\mathbf{v}_{2f} = \Sigma \mathbf{p}_i = m_1\mathbf{v}_{1i} + m_2\mathbf{v}_{2i}$$

$$\Sigma E_{kf} = \frac{1}{2}m\mathbf{v}_{1f}^2 + \frac{1}{2}m\mathbf{v}_{2f}^2 = \Sigma E_{ki} = \frac{1}{2}m\mathbf{v}_{1i}^2 + \frac{1}{2}m\mathbf{v}_{2i}^2$$

The final velocities of each object can be solved to yield:

$$\mathbf{v}_{1f} = \frac{(m_1 - m_2)\mathbf{v}_{1i} + 2m_2\mathbf{v}_{2i}}{m_1 + m_2} \tag{2}$$

$$\mathbf{v}_{2f} = \frac{2m_1\mathbf{v}_{1i} - (m_1 - m_2)\mathbf{v}_{2i}}{m_1 + m_2} \tag{3}$$

Figure 1 is an example of elastic collisions using the above solutions. With the appropriate units this could represent the head on collision between a proton and an alpha particle. The same unitless mass and initial velocities are those used in Figures 2 and 3. Importantly this applet assumes constant masses for the objects before and after the interaction.

## 3 Classroom Uses: Applications in Introductory Physics Courses

This applet has previously been used in introductory physics courses at a small private liberal arts college in New England and is designed for investigating momentum conservation in depth at the high school and college level. In this setting, the applet has been used to supplement pen and paper solutions of conservation of momentum in order for students to check their answers. The purpose is to provide an alternative diagrammatic tool to the traditional vectorial momentum representation. The areal approach helps students disentangle the momentum concept, which is not usually transparent in numerical solutions developed in either lecture or lab. The applet provides a better conceptual understanding of what conservation of momentum means in terms of final speeds and directions under different collision conditions. In the case of elastic collisions, the applet is particularly helpful in both confirmation of results and examination of problems that might be mathematically taxing for some audiences because the applet automatically provides solutions to simultaneous linear equations for multiple unknown final velocities.

### 3.1 *Inelastic Freight Car Collision*

Figure 4 is a simpler version of the inelastic collision shown in Figure 2. An empty freight car of mass 2 and velocity $+5$ collides with a stationary loaded freight car of mass $8$. Students can determine the final velocity of the two cars stuck together. The graphical plot indicates the final velocity (read directly from the grid) is about $+1$. The exact solution to Equation 1, also found in the Algebra panel is $+1$. The graphs and vectors clearly indicate that the majority of the momentum has been transferred to the more massive freight car after the collision, but only the graph proportions clearly show how much velocity is changing.

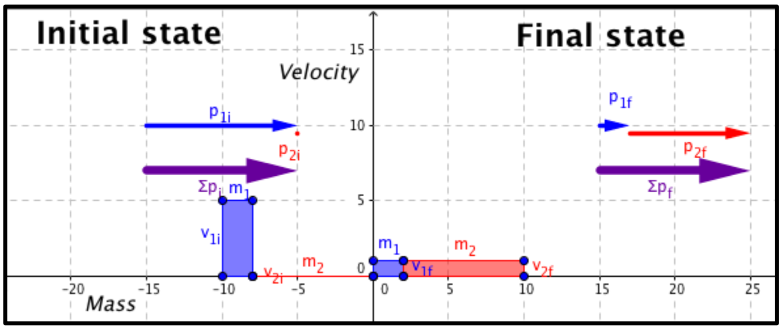


**Figure 4.** A simulation of the inelastic collision of two freight cars where the mass could be in metric tons and speed in meters/second.

### 3.2 *A Moving Explosion*

Figure 5 is a simplified version of Figure 2 in which one of the objects is at rest after an "explosion". For example this could be a child of mass $25$ kg rolling on a skateboard of mass $5$ kg, both at velocity $+2$ m/s. Suddenly the child launches the skateboard from under them and she lands motionless. At what velocity does the skateboard take off? By choosing $C = 0$ the applet will calculate the final velocity of the skateboard to be $+12$ m/s which is pretty fast and the reason why such behavior is not recommended. At face value the problem above may not obviously be an explosion, but it does involve the key idea of two objects initially being stuck together and then separating.

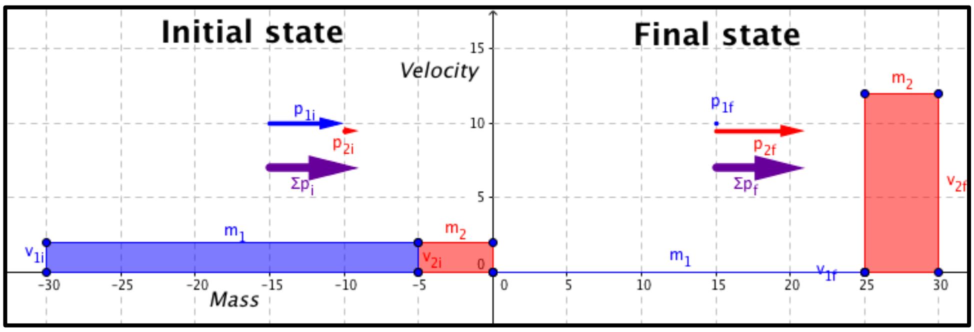


**Figure 5.** A child, launching a skateboard, and landing at rest.

### 3.3 A Stationary Explosion

Figure 6 is a more traditional explosion of a gun fired from a stationary platform. Since the initial velocity of both is zero with respect to the Earth this problem can be misleading in terms of conservation of momentum. Often students simply invoke the solution $m_1\mathbf{v}_1 = m_2\mathbf{v}_2$ without understanding the actual concept of momentum conservation. The applet makes its clear because $\Sigma\mathbf{p}_i = 0 = \Sigma\mathbf{p}_f$ the correct vectorial answer, $m_1\mathbf{v}_{1f} = -m_2\mathbf{v}_{2f}$, is more transparent by viewing the widely differing final velocities. The "Change Vector Shifts" and "Change Measurement Increments" were used to create this figure as can be seen in the Algebra display, along with the positive large final velocity of the bullet and smaller negative final velocity of the gun.

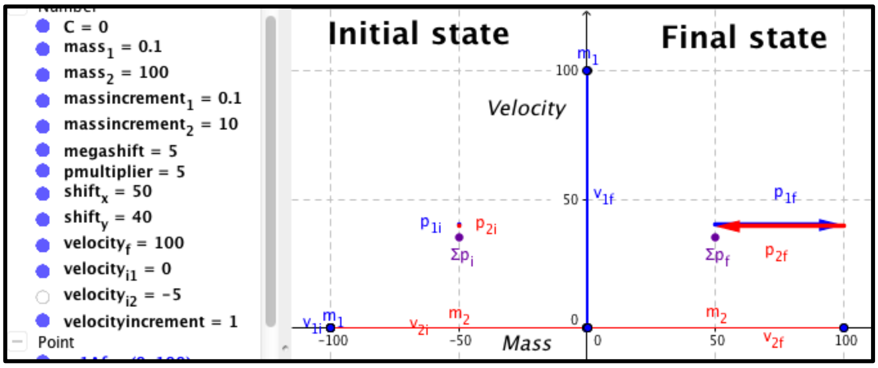


**Figure 6.** Tradition bullet exploding from gun in which total momentum is zero. The light bullet has large positive velocity and heavy gun has small negative velocity.

### 3.4 Molecular Elastic Collisions

Figure 7: A molecule of nitrogen (mass = 28) is traveling at $+4.0$ and collides head on with a molecule of oxygen (mass = 32) traveling at $-2.0$. Students can calculate the final velocities of each molecule. This is a tedious activity to do with Equations 3 and 4. The applet provides the answers quickly, indicating the two molecules' momenta are almost completely transferred with the final velocity of the nitrogen molecule being $-2.4$ and the final velocity of the oxygen being $+3.6$. A physics instructor could also discuss scaling at the point. By increasing the velocities two orders of magnitude and scaling the molecular masses appropriately the answers would not only still be correct but also correspond to realistic range of speeds (in m/s) for gas molecules at room temperature. The labels were turned off in this figure to concentrate on the idea that the momenta are largely transferred between the two nearly identical massed objects.

## 4 Conclusions

This GeoGebra applet can enhance student learning about collisions beyond what is normally described in textbooks, increasing the understanding of momentum and kinetic energy in simple two body one-dimensional collisions. The applet applies a simple slider mechanism and algebraic commands to switch the scenarios shown from inelastic collisions to explosions to elastic collisions and has sliders to change different variables within each scenario. Checkboxes and sliders can change the layout of the program providing variable flexibility for many different collision scenarios to be shown so the program can adapt to a specific classroom activity.

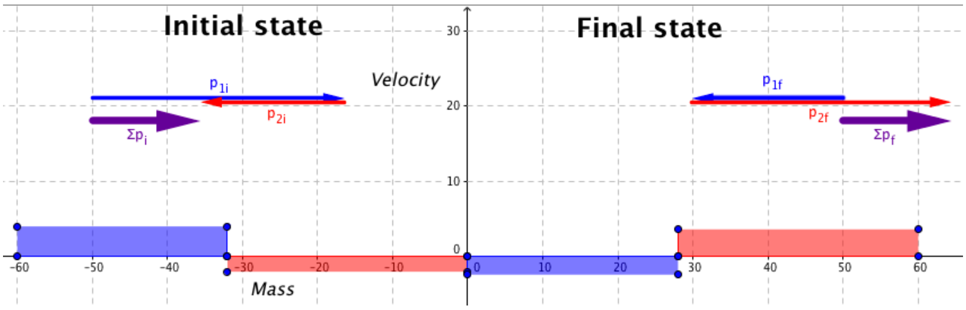


**Figure 7.** Unscaled oxygen and nitrogen molecules colliding together and then bouncing apart.

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