INTERSECTION OF POLYHEDRONS AND A PLANE WITH GEOGEBRA

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Abstract

Using GeoGebra, we present an innovative method for teaching of the intersection of polyhedrons with a plane using infinite points and the swap of finite and infinite points. The method presented is efficient and powerful, allowing one to generate solutions of a whole set of problems by solving one instance and using a pre-made applet at any stage of the solution process.

Keywords: Dynamic geometry software; GeoGebra; Infinite point; Intersection of a plane and a polyhedron.

1 INTRODUCTION

The concept of mathematical competence now includes facility with modern software systems. Using dynamic geometry software (DGS) such as GeoGebra, students develop programming skills and algorithmic thinking, expanding their digital competence beyond that of traditionally passive technologies. Use of interactive software also deepens students' mathematical knowledge and conceptual understanding of mathematics [1, 2, 6, 12, 17, 22].

1.1 Sam software

In Bulgaria, preservice teachers use a specialized dynamic software, *Sam*, in their preparation to work with young students [7, 8, 21] and to deepen understanding of topics associated with the teaching of mathematics. A new function, *Swap finite and infinite points*, is included in the geometry menu of *Sam* which greatly simplifies sketch work while supporting creative problem solving in the teaching and learning of geometry. The tool challenges teachers and students to discover new connections between the geometric objects, to summarize certain tasks (both elementary and from international competitions), and to create new tasks [8, 16, 19]. Research involving the usage of *Sam* in the teaching of geometry across various grades [14, 15, 21], cultures [5], and abilities show that the tool encourages increased student interest and motivation in mathematics.

1.2 GeoGebra

While *Sam* is a specialized DGS, *GeoGebra* is a universal, freely-available DGS with a large library of functions. A particularly useful feature of *GeoGebra* is the software's capacity to preserve attributes of objects and save a sequence of construction steps using scripting tools. The rich possibilities of *GeoGebra* can be combined with the function *Swap finite and infinite points*, using checkboxes [18]. This idea is has been explored with students from fifth grade in [14].

Coxeter notes that two lines intersect at an *infinite point* if they are parallel ([3], p. 98). This notion is based on an idea originally developed by Kepler. Specifically, an infinite point U_∞ is associated with every set of parallel lines, indicating the direction of the lines. For this reason, it is helpful to present an infinite point with a vector in GeoGebra.

In the following discussion, we present the notion of infinite points with the help of *GeoGebra* in the context of an exploration of the intersection of polyhedra with a plane. Our method fosters creativity while helping make student and teacher sketchwork more efficient. Specifically, we consider three basic tasks and their solutions and with a pre-fabricated applet, demonstrating the possibility of generating new tasks. All figures in the paper can be downloaded from: `http://fmi-plovdiv.org/GetResource?id=2068`.

2 Preliminaries

The function *Swap finite and infinite points*, defined in DGS *Sam* [8], allows the visualization of the wide variety of quadrangular and triangular prisms, pyramids, and truncated pyramids with a single button click [8, 20, 21]. The construction of a universal parallelepiped highlights the functionality of *Swap finite and infinite points*. This construction is completely described in [8]. For completeness sake, we describe it here again.

2.1 Universal Parallelepiped

A sketch of a completed universal parallelepiped is shown in (refer to Figure 1). The free objects in its construction are the points `A, U, V, W,` U_∞, V_∞, W_∞. After constructing these points, we define two Boolean variables, `a` and `b` with labels “`Swap U and` U_∞” and “`Swap V and` V_∞”, respectively. Then we construct the two lines:

`c=If[a,Line[A,U],Line[A,`U_∞`]]` and `d=If[b,Line[A,V],Line[A,`V_∞`]]`.

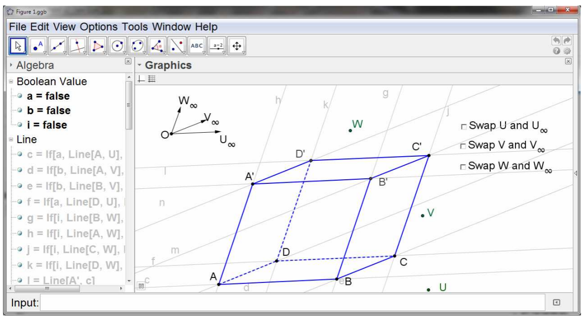


Figure 1. Universal Parallelepiped

If the checkbox “`Swap U and` U_∞” is not checked, `Line[A,`U_∞`]` is visible, otherwise `Line[A,U]` is shown. The same holds for the other checkbox with Boolean variable `b`. With the commands `B=Point[c]` and `D=Point[d]` we choose arbitrary points `B` and `D` on lines `c` and `d`, respectively.

Next, we define the lines `e=If[b,Line[B,V],Line[B,V`$_\infty$`]]` and `f=If[a,Line[D,U],Line[D,U`$_\infty$`]]`. The vertex `C` is the intersection point of the lines `e` and `f` and is defined by `C=Intersect[e,f]`. Next, the edges of the parallelepiped are constructed as segments. The figure `ABCD` is constructed in this way. We consider it as a lower base of the parallelepiped `ABCDA'B'C'D'`, and we call it universal parallelogram [8]. Next, we add the third Boolean variable `i` with labels "`Swap W and W`$_\infty$" to construct the surrounding edges of the prism, which lay on the lines

- `h=If[i,Line[A,W],Line[A,W`$_\infty$`]]`,
- `g=If[i,Line[B,W],Line[B,W`$_\infty$`]]`,
- `j=If[i,Line[C,W],Line[C,W`$_\infty$`]]` and
- `k=If[i,Line[D,W],Line[D,W`$_\infty$`]]`.

The choice of the point `A'` on the line `h` defines the length of the surrounding edges. The edges of the upper base `A'B'C'D'` and the remaining vertices are constructed with the help of the tools *parallel line* and *intersection of two lines*.

The above described construction of the universal parallelepiped creates an obligated status of its faces `ABCD` and `A'B'C'D'` to be its bases. The rotation of the infinite points `U`$_\infty$ and `V`$_\infty$ changes the angle of the parallelogram and the rotation of `W`$_\infty$ transforms the parallelepiped from sloping to a right-angled parallelepiped. The limited freedom of the points $B \in AU_\infty, D \in AV_\infty$ and $A' \in AW_\infty$ allows us to change the length of the edges of the parallelepiped.

2.2 Connected Figures

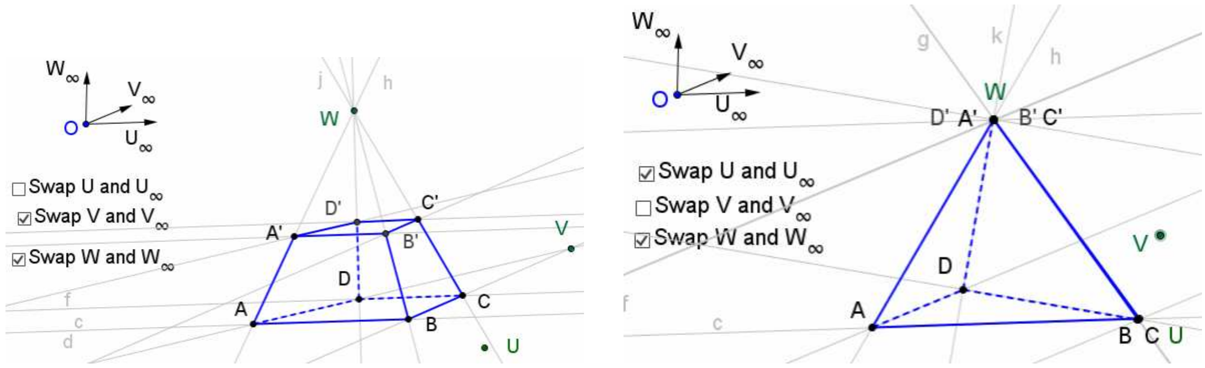
We refer to figures that are obtained from one another with the help of the function *Swap finite and infinite points* as *connected figures* [8]. For example the Figures 1, 2a, and 2b are connected.
A truncated pyramid with base `ABCD` is obtained from the universal parallelepiped by swapping `V`$_\infty$ and `V`; `W`$_\infty$ and `W` (refer to Figure 2a). A triangular pyramid is obtained from the universal parallelepiped by swapping `U` and `U`$_\infty$ as well as `W` and `W`$_\infty$, positioning the point `B` (and `C`) to coincide with `U` and positioning the points `A'`, (`B'`, `C'` and `D'`) to coincide with `W`.

Remark 1 *A new sketch is displayed after clicking the checkbox "`Swap V and V`$_\infty$." One may need to move the point `D` to carry out the relation `D/AV` (point `D` is between the points `A` and `V`). After clicking the checkbox "`Swap W and W`$_\infty$" one may need to move the point `A'` to carry out the relation `A'/AW` (refer to Figure 2a). After clicking the checkbox "`Swap U and U`$_\infty$" one may need to move the point `B` to carry out the relation `B/AU`.*

2.3 Basic Constructions for finding the intersection of polyhedrons with a plane

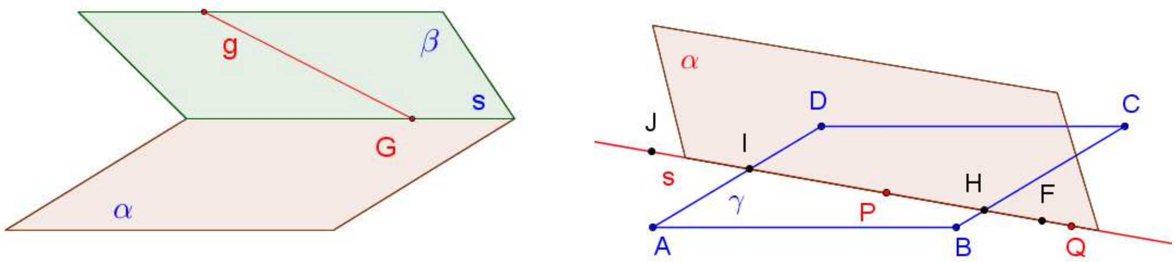
For sake of clarity, we will denote the plane of the intersection by α.The basic constructions used to find the intersection of α and a polyhedron are:

1. Finding the piercing point `G` of a line `g`, generated by two vertices of the polyhedron and the plane α. To find the point `G=g`$\cap\alpha$ we perform the following constructions:
   (a) Construct an auxiliary plane β, which is incident with the line `g`;



(a) Truncated Pyramid **(b)** Triangular Pyramid

Figure 2. Main figure caption



(a) Construction A **(b)** Construction C

Figure 3. Main figure caption

   (b) Construct the line $\mathtt{s}=\alpha\cap\beta$;

   (c) Locate the point $\mathtt{G}=\mathtt{g}\cap\mathtt{s}=\mathtt{g}\cap\alpha$ (refer to Figure 3a).

2. Locate the intersection line $\mathtt{s}$ of the plane α with the plane γ, containing a face of the polyhedron. To find $\mathtt{s}$ it is enough to locate two common points $\mathtt{P}$ and $\mathtt{Q}$ for the planes α and γ. Then the intersection line is $\mathtt{s}=\alpha\cap\gamma=\mathtt{PQ}$.

3. Finding the piercing points of the edges of the polyhedron, located in one of the faces with α, is simplified because these edges have a common auxiliary plane $\beta=\gamma$ – the plane of the face (refer to Figure 3b). Then the intersection points of the line $\mathtt{s}$ with all the lines incident with the edges that belong to the face γ, are the piercing points of these lines with α. From Figure 3b, we see that the edges $\mathtt{BC}$ and $\mathtt{AD}$ have common points $\mathtt{H}$ and $\mathtt{I}$ with α, respectively, while there will be no vertices from the polygon, the intersection of the polyhedron with α, that lay on edges $\mathtt{AB}$ and $\mathtt{CD}$.

3 Intersection of polyhedrons with a plane

As one might imagine, intersections of polyhedra with planes is one of the most difficult concepts for Bulgarian school students to grasp. For this reason, the concept is not included in the course of study of all secondary schools. In addition, different presentations of the topic appear in various textbooks in Bulgaria [4, 9–11, 13].

In the paragraphs that follow, we present a new strategy for the construction of intersections of polyhedra with planes with *GeoGebra* and the help of a teacher-constructed applet, namely *Swap finite and infinite points*. This approach introduces a creative component in the study of the concept. Our approach is based on the projective property – a preservation of the incidence of the basic objects (points, lines and planes) in the central projection. We avoid using the affine property that the parallel faces of the parallelepiped intersect the plane of the section in parallel lines because after the transition into prisms with bases different from parallelograms or into pyramids (truncated or not), this property is not preserved (except for the bases).

Finding the lines of intersection of the planes of two faces of the polyhedron with α ensures the availability of two common intersection points. Thereafter we can determine all the piercing points of the edges of the polyhedron with α. Without loss of generality, we choose one of the first intersection lines to lie on a base of the polyhedron.

Let the plane of the intersection α be determined by the non-collinear points P, Q, R. We use the notations $(A_1A_2 \ldots\ A_n)$ for the face $A_1A_2 \ldots A_n$ of the polyhedron and $\gamma(A_1\ A_2 \ldots\ A_n)$ for the plane of this face. We introduce our method for determining the intersection of a polyhedron and a plane with the following classical problem:

Problem 1 *Find the intersection of the parallelepiped $ABCDA'B'C'D'$ with the plane $\alpha(P,Q,R)$, where $P \in BC$, $Q \in AA'$, $R \in C'D'$.*

Solution: Initially, we find the intersection line s of the plane α and the plane $\gamma(ABCD)$ (refer to Figure 4). Since we have one point P belonging to s, it will suffice to find the piercing point F of the line RQ with the plane $\gamma(ABCD)$. We consider the auxiliary plane $\beta(Q, R, W_\infty)$.

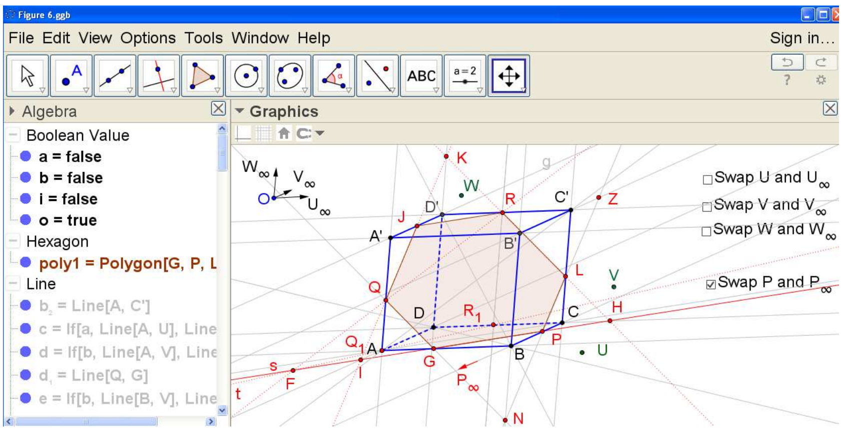


Figure 4. Problem 1

To find the intersection line $t=\beta \cap \gamma$, we determine two points incident with $R_1=RW_\infty \cap \gamma=RW_\infty \cap CD$ and $Q_1=QW_\infty \cap \gamma=A$. They are the projections of R and Q on the plane $\gamma(ABCD)$ along W_∞. This step is often referred to as method of projections. Consequently there holds the relation $F=RQ \cap \gamma=RQ \cap t=RQ \cap R_1Q_1$ and $s=\alpha \cap \gamma=PF$. From C), the points $G=AB \cap s$, $I=AD \cap s$ and $H=CD \cap s$ are the piercing points of the lines AB, AD and CD with α, respectively. With the help of the lines $IQ=\alpha \cap \delta(ADD'A')$ and $HR=\alpha \cap \epsilon(CDD'C')$, we determine the vertices of the intersection $J=IQ \cap A'D'$ and $L=HR \cap CC'$.

We will illustrate how, with the applet *Swap finite and infinite points*, we can generate a set of new problems with their solutions in *GeoGebra*. Specifically, we will swap the points P and P_∞. For the correct functioning of the above construction it is necessary for the point P to be swapped with an arbitrary infinite point P_∞, which is incident with the plane $\gamma(ABCD)$. The position of the point P_∞ can be changed in GeoGebra's Object Properties window. To illustrate this, we will present several intersections generated from Figure 4 by changing the point P_∞. When the points Q and R are not fixed, for each new drawing we adjust the type of polygon depending on our preference and manually outline it. If $P_\infty \in BC$, then the plane α is parallel to the line BC (refer to Figure 5). If $P_\infty \in BD$, then the plane α is parallel to the line BD (refer to Figure 6); If $P_\infty \in AC$, the plane α is parallel to the line AC (refer to Figure 7). Besides, we have swapped U with U_∞ and W with W_∞ in Figure 7 and thus we have produced an intersection of α with a truncated pyramid with bases, which are trapeziums.

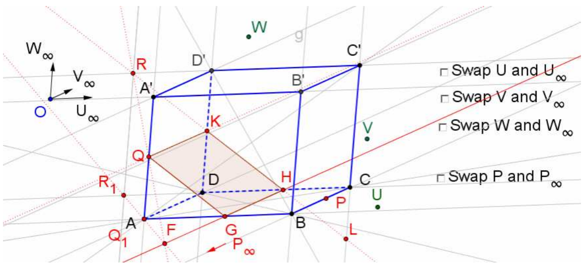


Figure 5. $P_\infty \in BC$

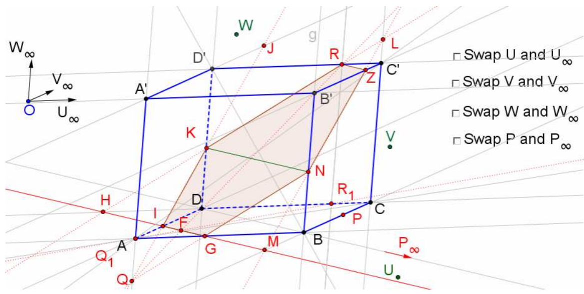


Figure 6. $P_\infty \in BD$

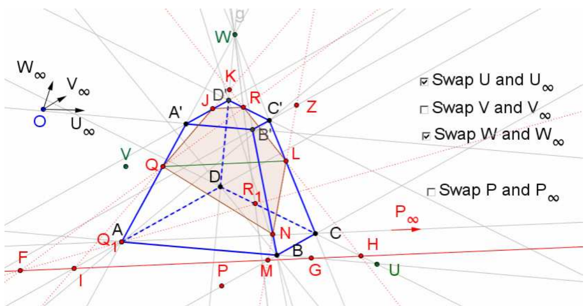


Figure 7. $P_\infty \in AC$

The figures in Figures 6, 7, 8 and 9 are connected. Using GeoGebra, teachers and their students can generate a basic problem of their own and produce a set of new problems with their solutions. The construction of the solutions can be presented step by step with the help of a slider. For example let us consider the following problem.

Problem 2 *Find the intersection of the parallelepiped $ABCDA'B'C'D'$ with the plane $\alpha(P,Q,R)$ provided that two of the points P, Q, R lay on the plane of one and the same face of the parallelepiped $ABCDA'B'C'D'$ and the third point is arbitrary.*

Solution: Without loss of generality, we assume that the points $P, Q \in \gamma(ABCD)$ and R are arbitrary (refer to Figure 8). Although R is an arbitrary point, we need to determine its position in the space because the universal parallelepiped $ABCDA'B'C'D'$ can be considered a main axonometric image of an arbitrary parallelepiped in general axonometric view, where the three groups of parallel lines are parallel to the virtual axonometric axes. Note that the projections of the vertices A and A' on the plane $\gamma(ABCD)$ along W_∞ are $A_1=A_1'=A$, on the plane $\delta(ADD'A')$ along V_∞ are $A_2=D$, $A_2'=D'$. The projections of the other vertices are readily seen. For this reason, denote it by R_1, the projection of R on the plane $\gamma(ABCD)$ along W_∞. We see that $s=PQ=\alpha \cap \gamma$.

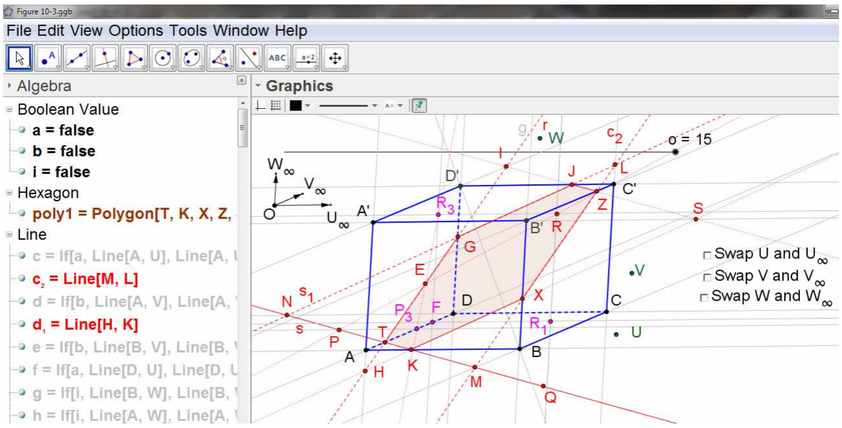


Figure 8. Problem 2

According to Remark 1 the points $K=s\cap AB$, $T=s\cap AD$, $M=s\cap BC$ and $N=s\cap CD$ are the piercing points of α with the lines AB, AD, BC and CD, respectively. Each of these points belong to the plane $\gamma(ABCD)$ and to the plane of one more of the faces of the parallelepiped.

Now we find the intersection line of α with the plane of another face of the parallelepiped. Without loss of generality, we may search for the intersection line with the plane $\delta(ADD'A')$. We need a second point from this line, because already $T\in AD$ is a point incident with it. It is enough to find the piercing point of the line PR (or the line QR) with the plane δ. For this purpose we use the auxiliary plane $\lambda(P,R,U_\infty)$. Projecting the points P and R on the plane $\delta(ADD'A')$ along U_∞ yields projections $P_3=AD\cap PU_\infty$ and $R_3=RU_\infty\cap FW_\infty$, where $F=R_1U_\infty\cap AD$. Consequently $\delta \cap \lambda=P_3R_3$ and the point $E=PR\cap P_3R_3$ is the piercing point of the line PR with the plane $\delta(ADD'A')$. Then the line $r=TE$ is the intersection line of α with the plane $\delta(ADD'A')$.

According to C from Section 2, we determine the piercing points $H=r\cap AA'$, $G=r\cap DD'$, $I=r\cap A'D'$. We construct the line NG, which is the intersection line of the plane α with the plane $\epsilon(CDD'C')$ and we find the piercing points $L=CC'\cap\alpha=CC'\cap NG$, $J=C'D'\cap NG$. We construct the line ML, which is the intersection line of the plane α with the plane $\rho(BCC'B')$ and thus find the piercing points $X=BB'\cap\alpha=BB'\cap ML$ and $Z=B'C'\cap\alpha=B'C'\cap ML$. Note the piercing point $S=A'B'\cap\alpha=d_1\cap A'B'$, where $d_1=HK$. There hold the relations $TK\|JZ$, $TG\|XZ$, $GJ\|KX$, because there hold $\gamma(ABCD)\|\mu(A'B'C'D')$, $\delta(ADD'A')\|\rho(BCC'B')$, $\epsilon(CDD'C')\|\sigma(ABB'A')$, respectively.

Definition 2 *Following [8] we will accept the definition of satellite problems as follows: Problems, which are generated from a problem A with the help of the applet "Swap finite and infinite points" will be called* ***satellites*** *of Problem A and are denoted by A.1, A.2,. . . etc.*

The applet *Swap finite and infinite points* in *GeoGebra* allows teachers and students to produce a set of satellites of Problem 2 with their solutions. We will illustrate some of these possibilities, and we will comment the creative stages of their realization.

Problem 2.1 *Find the intersection of a truncated pyramid* $ABCDA'B'C'D'$ *with parallelogram bases and surrounding edge* AA' *orthogonal to the bases with the plane* $\alpha\,(P,Q,R)$*, where* $P,Q\in \gamma\,(ABCD)$ *and* $R\in \rho\,(BCC'B')$*.*

Solution: It is enough to change the definition of the point R_1 from `Point[i1]` to `Intersect[i1,e]`, where i_1=RW_∞ and e=BV_∞ in the window "Object properties" (refer to Figure 9). This ensures that $R\in \rho\,(BCB'C')$. With the help of the checkbox "Swap W and W_∞," we change the Boolean value of i from `false` to `true`. After applying the dynamics on the point A', we realize the relation A'/AW.

The truncated pyramid, with its intersection with the plane α, is displayed within the sketch. By using dynamics of the point W, we provide the condition $AA'\perp AB$, which means $AA'\perp \gamma\,(ABCD)$ in the parameters of the cabinet projection (usually used in the secondary school).

Problem 2.2 *Find the intersection of the oblique prism* $ABCDA'B'C'D'$ *with trapezum bases* $(AB \parallel CD)$ *with the plane* $\alpha\,(P,Q,R)$*, where the points* P *and* Q *are the midpoints of the segments* BC *and* AB*, respectively, and* R *is the intersection point of the diagonals* AC' *and* BD'*.*

Solution: It is enough to change in the window "Object properties" the definitions of the points: `R=Intersect[q1,b2]`, where q_1=BD', b_2=AC', `P=Midpoint[B,C]` and `Q=Midpoint[A,B]` (refer to Figure 10). Using the checkbox "Swap V and V_∞," we change the Boolean value of b from `false` to `true`, and thus the lines BC and AD transform from parallel lines to intersecting into the finite point V. After rotation of W_∞, we determine the direction of the surrounding edges of the prism.

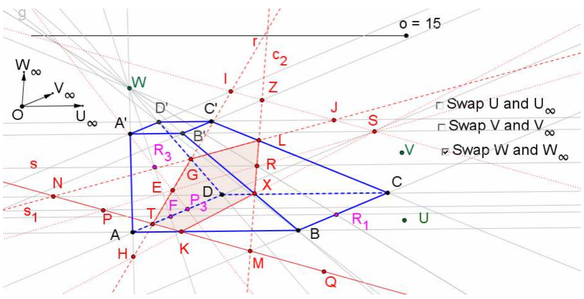


Figure 9. Problem

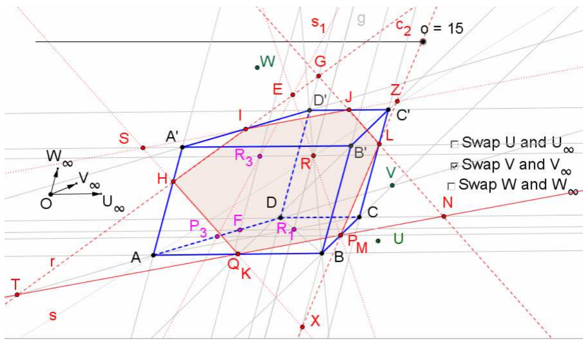


Figure 10. Problem

Remark 3 *The conditions* $R\in BD'$ *and* $R_3\in AD'$ *are equivalent because the projection of the line* BD' *on the plane* $\delta\,(ADD'A')$ *along the point* U *is the line* AD' *and also the central projecting (regardless from a final or an infinite center) preserves the incidence.*

The last Remark simplifies the solutions of the following two satellite problems.

Problem 2.3 *Find the intersection of a truncated quadrangular pyramid* $ABCDA'B'C'D'$ *with a plane* α*, which passes through the midpoints of the edges* AB *and* BC *and* α *is parallel to the diagonal* BD'*.*

Solution: We use the dynamic sketch Figure 10, but now P=Midpoint[A,B], Q=Midpoint[B,C] (refer to Figure 11). With the help of the checkboxes "Swap U and U_∞" and "Swap W and W_∞," we transform the trapezium ABCD into an arbitrary quadrangular and the parallelepiped into a truncated pyramid. We define a new Boolean variable o with label "Swap R and R_∞," with $R_\infty \in BD'$. According to Remark 3, points $R_3\ (R_\infty)_3$ belong to AD' and $E=P_3(R_\infty)_3 \cap PR_\infty = AD' \cap PR_\infty$.

After these modifications to the sketch, we draw the segment SG to emphasize the presence of the relation $SG \parallel BD'$ following from $\alpha \parallel BD'$ and $SG=\alpha \cap \pi(BDD'B')$.

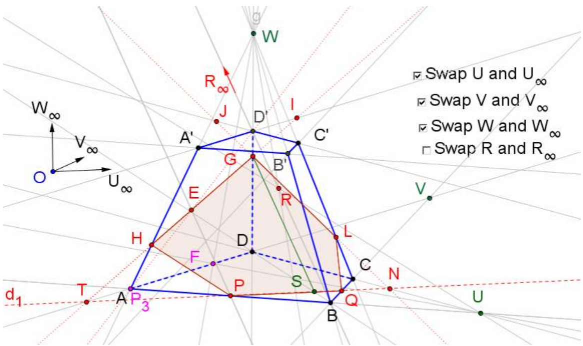


Figure 11. Problem 2.3

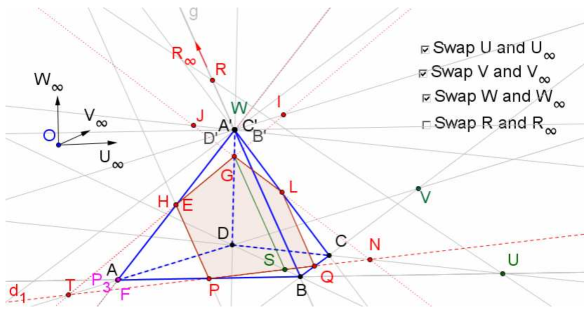


Figure 12. Problem 2.4

Problem 2.4 *Find the intersection of a quadrangular pyramid ABCDW with a plane α passing through the midpoints of the edges AB and BC with α parallel to the surrounding edge WB.*

Solution: The solution is generated as suggested in Figure 11 by moving the point A' to coincide with W (refer to Figure 12).

Problem 2.5 *Find the intersection of a parallelepiped $ABCDA'B'C'D'$ with a plane $\alpha(P,Q,R)$, where the point P coincides with a vertex of a base, Q lays in the plane, $\gamma(ABCD)$ and R is an arbitrary point. Consider the following cases:*

1. *The plane α is parallel to a diagonal of the base, and*
   (a) *$ABCDA'B'C'D'$ is a prism with base an arbitrary quadrangle;*
   (b) *$ABCDA'B'C'D'$ is a pyramid;*
2. *The point R lays on a diagonal that connects two vertices which are not from the same face of the parallelepiped $ABCDA'B'C'D'$.*

Solution: We use the sketch of the Problem 2 (refer to Figure 8). We chose the point P to coincide with the vertex B. Thus the definition of the point P must be changed from a free point to Intersect[c,e], where $c=AU_\infty$ and $e=BV_\infty$.

a) From the choice of the point P it follows that it is possible to consider α to be parallel to the diagonal AC. Therefore, the lines AC and PQ will be parallel, i.e., $Q_\infty \in AC$. We define a new Boolean variable o with label "Swap Q and Q_∞" (refer to Figure 13).
a_1) It is enough to change the Boolean values of a and b to true and the values of i, o to false to generate the solution (refer to Figure 14).

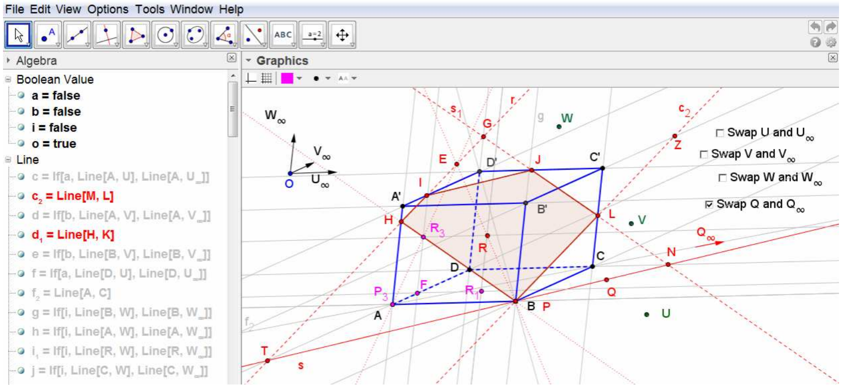


Figure 13. Problem 2.5 a

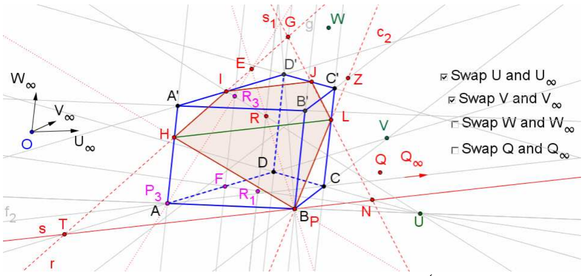


Figure 14. Problem 2.5 (a_1)

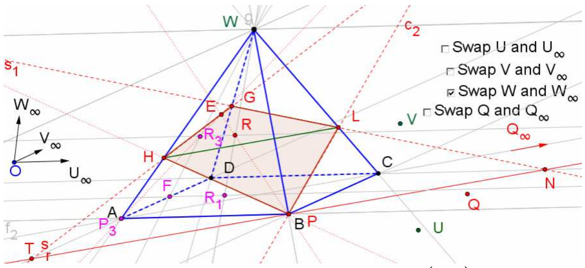


Figure 15. Problem 2.5 (a_2)

a$_2$) We set the value $\mathtt{false}$ for $\mathtt{a,\ b,\ o}$ and $\mathtt{true}$ for $\mathtt{i}$ to generate Figure 15. We draw the segment $\mathtt{HL}$ to emphasize that $\mathtt{HL} \parallel \mathtt{AC}$.

b) The requirement $\mathtt{R}$ to be incident with a diagonal which is not lying in a face of the parallelepiped generates two cases:

b$_1$) $\mathtt{R}\in\mathtt{BD'}$. We define the points $\mathtt{R}$ and $\mathtt{R}_1$ by $\mathtt{R=Point[q_1]}$ and $\mathtt{R_1=Intersect[i_1,g_2]}$, respectively, where $\mathtt{q_1=BD'}$, $\mathtt{i_1=RW}$, and $\mathtt{g_2=BD}$ (refer to Figure 16). Let $\mathtt{ABCDA'B'C'D'}$ be a truncated pyramid. In this case, the intersection is always a quadrangle with vertex $\mathtt{D'=PR}\cap\mu\mathtt{(A'B'C'D')=E=G}$.

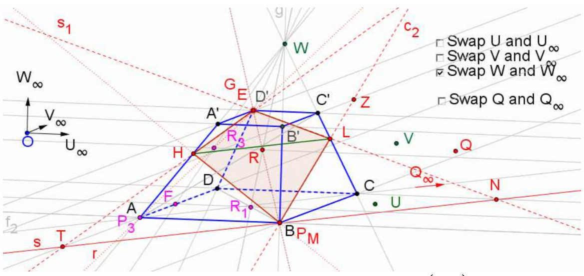


Figure 16. Problem 2.5 (b_1)

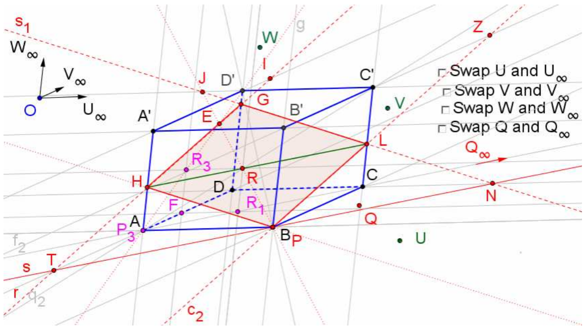


Figure 17. Problem 2.5 (b_2)

b$_2$) $\mathtt{R}\in\mathtt{AC'}$. We define points $\mathtt{R}$ and $\mathtt{R}_1$ by $\mathtt{R=Point[q_2]}$, and $\mathtt{R_1=Intersect[i_1,f_2]}$, respectively, where $\mathtt{q_2=AC'}$, $\mathtt{i_1=RW}$, and $\mathtt{f_2=AC}$ (refer to Figure 17). Let $\mathtt{ABCDA'B'C'D'}$ be a parallelepiped. The intersection could be either pentagon or parallelogram, and the point $\mathtt{G}$ is not necessarily coincident with $\mathtt{D'}$.

Remark 4 *We have not included the infinite line in the problems, as it was done in [5], and because the investigation is intended mainly for teachers and students in the secondary schools. This reduces the number of different problems that can be generated from one basic problem.*

In the solutions of the above proposed problems, the preference for the first intersection was for the plane of the lower base of the parallelepiped. While in Problem 1 both bases had equivalent conditions to be selected, in Problem 2 this selection was recommended. The solution of the next problem illustrates the upper base of the parallelepiped as our choice.

Problem 3 *Find the intersection of the parallelepiped $ABCDA'B'C'D'$ with the plane α passing through the point P, belonging to the plane σ $(ABB'A')$ through the point R belonging to the plane δ $(ADD'A')$ with α parallel to the line BD.*

Solution: (refer to Figure 18) First we find the intersection s of α with the plane $\mu(A'B'C'D')$, creating a structure that will allow us to generate decisions and a set of other tasks. For this purpose we find the pierce G of line PR with the plane $\mu(A'B'C'D')$, namely, $G=PR\cap P_4R_4$, where P_4 and R_4 are the projections of P and R, respectively, along W_∞ on the plane $\mu(A'B'C'D')$. Choose an arbitrary point Q on the plane μ and set point $Q_\infty \in B'D'(BD)$. We create Boolean value with label "Swap Q and Q_∞", recording the conditional operator: `If[v,Line[G,Q],Line[G,Q∞]]`.

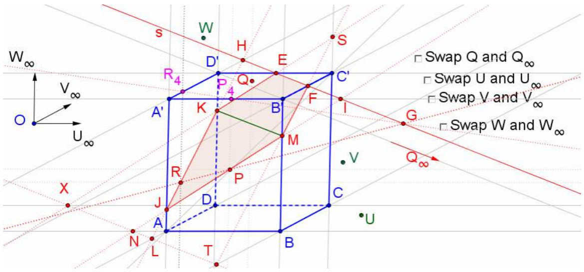


Figure 18. Problem 3

Let us construct the line $s=GQ=\alpha\cap\mu$. By the checkbox "Swap Q and Q_∞," we set the Boolean value to `false` and the line $s=GQ$ is replaced with the line $s=GQ_\infty$. The following steps are similar to Problem 2 and shown in Figure 18. We define the points $E=s\cap C'D'$, $F=s\cap B'C'$, $I=s\cap A'B'$, and $H=s\cap A'D'$ and then the lines $RH=\alpha\cap\delta(ADD'A')$ and $PI=\alpha\cap\sigma(ABB'A')$. The points $M=PI\cap BB'$, $N=PI\cap AB$, $J=PI\cap AA'$, $K=RH\cap DD'$, $L=RH\cap AD$; $S=MF\cap CC'$, $T=MF\cap BC$, $X=KE\cap CD$, E, F, I, and H are potential vertices of the polygon – section of the plane α with the polyhedron.

We suggest the following four satellite problems:

Problem 3.1 *Find the intersection of the truncated quadrangular pyramid $ABCDA'B'C'D'$ with parallelogram bases with the plane α passing through the point P belonging to the plane σ $(ABB'A')$ through the point R belonging to the plane δ $(ADD'A')$ with α parallel to the line BD.*

Problem 3.2 *Find the intersection of the truncated quadrangular pyramid $ABCDA'B'C'D'$ with trapezoid bases $(AD \parallel BC)$ with the plane α passing through the point P belonging to the plane σ $(ABB'A')$ through the point A with α parallel to the line BD.*

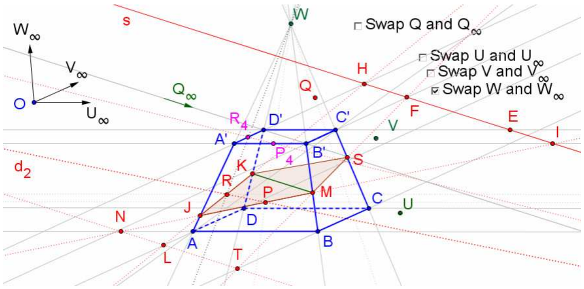


Figure 19. Problem 3.1

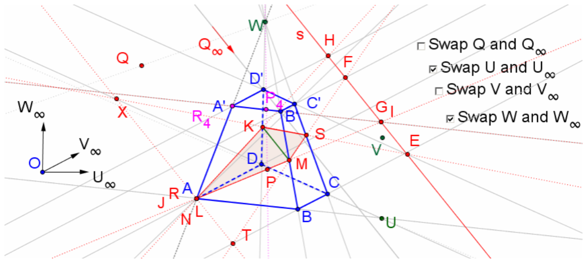


Figure 20. Problem 3.2

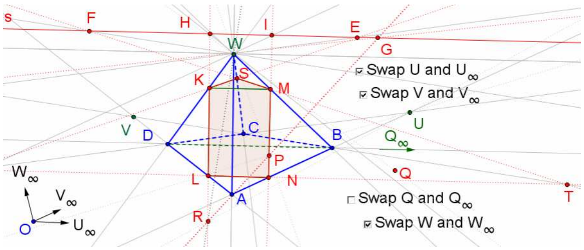


Figure 21. Problem 3.3

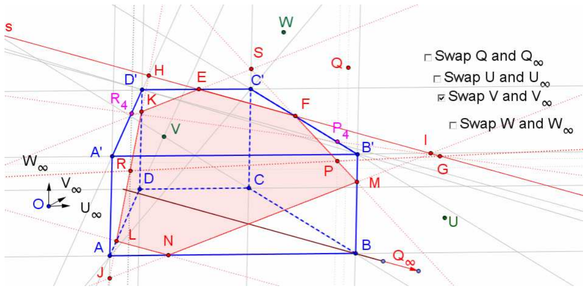


Figure 22. Problem 3.4

Problem 3.3 *Find the intersection of the pyramid* $ABCDA'B'C'D'$ *with trapezoid base* $(AB \parallel CD)$ *with the plane* α *passing through the point* P *belonging to the plane* σ $(ABB'A')$ *through the point* R *belonging to the plane* δ $(ADD'A')$ *with* α *parallel to the line* BD*.*

Problem 3.4 *Find the intersection of the quadrangular prism* $ABCDA'B'C'D'$ *with trapezoid bases* $(AB \parallel CD)$ *with the plane* α *passing through the point* P *belonging to the plane* ρ $(BCC'B')$ *through the point* R *belonging to the plane* δ $(ADD'A')$ *with* α *parallel to the bisector of the angle* $\angle ABC$*.*

4 Conclusion

The exploration of heuristic moments with DGS enhances teaching of school geometry. Building on this idea, we have advanced an innovative method for studying the topic of intersections of polyhedra with a plane within the dynamic environment of GeoGebra. We have shown how the teachers and students can, though the generation of a basic problem of their own construction, produce a set of new problems with solutions using the teacher-generated applet *Swap finite and infinite points*. The function *Swap finite and infinite points* adds powerful functionality to *GeoGebra* for teachers, students, and researchers, alike.

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References

[1] The Fibonacci project, research approach to education in mathematics. `http://www.math.bas.bg/omi/docs/FiboBook-Final-v4.pdf`, 2013.

[2] Francisco Regis Vieira Alves. Visual criterion for understanding the notion of convergence if integrals in one parameter. *Acta Didactica Napocensia*, 7(1):19–30, 2014.

[3] H.S.M. Coxeter. *The Real Projective Plane*. McGRAW-HILL BOOK COMPANY, INC., 1949.

[4] S. Dodunekov, G. Kozhuharova, M. Hristova, D. Kapralova, and S. Doichev. *Mathematics for 11 grade – Specialized training*. Regalia 6 – Sofia, 2015.

[5] S. Grozdev, D. Stefanova, K. Vasileva, S. Koleva, and R. Todorova. Stimulating of bilingual's creativity by a dynamic software. *Mathematics and Informatics*, 57(3):247–273, 2014.

[6] Z. Jiang. The dynamic geometry software as an effective learning and teaching tool. *The Electronic Journal of Mathematics and Technology*, 1(3):245–256, 2007.

[7] S. Karaibryamov, B. Tsareva, and B. Zlatanov. Educational software for interactive training of students on the theme "Mutual intersecting of pyramids and prisms in axonometry". *Acta Didactica Napocensia*, 5(1):29–44, 2012.

[8] S. Karaibryamov, B. Tsareva, and B. Zlatanov. Optimization of the courses in geometry by the usage of dynamic geometry software Sam. *The Research Journal of Mathematics and Technology*, 2(1):1–31, 2013.

[9] G. Kozhuharova. *Intersection of Polyhedrons with Planes*. Daga Plus – S. Zagora, 2014.

[10] C. Lozanov, P. Nedevski, and T. Vitanov. *Mathematics for 11 grade – Specialized training*. Anubis – Sofia, 2015.

[11] G. Paskalev and Z. Paskaleva. *Mathematics for 11 grade – Second level*. Archimed – Sofia, 2015.

[12] P. Pech. How integration of DGS and CAS helps to solve problems in geometry. In *17th Asian Technology Conference in Mathematics, 16–20 December 2012*. Suan Sunandha Rajabhat University, Bangkok, Thailand, 2015.

[13] S. Petkova, P. Petkov, and K. Prodanova. *Mathematics for 11 grade – Specialized training*. Prosveta – Sofia, 2015.

[14] B. Taneva. The information technology as a support to interactive learning on geometry. In *Eighth National Conference on "Education and Research in the Information Society", 28–29 May 2015*, pages 219–228. Plovdiv, Bulgaria: Development of the Information Society, 2015.

[15] P. Trifonova, N. Danova, and R. Todorova. Investigated–oriented training on the topic "Middle section and centroid of a triangle" in a dynamic geometry environment. *Magazine "Academic Forum"*, 4(3):94–105, 2013.

[16] B. Tsareva and R. Todorova. Interactive studying of the circumcribed quadrilaterals in a dynamic environment. *Mathematics and Informatics*, 56(2):142–158, 2013.

[17] D. Yazlik and H. Ardahan. Teaching transformation geometry with cabri geometry plus ii. *Procedia - Social and Behavioral Sciences*, 46(3):5187–5191, 2012.

[18] B. Zlatanov. Some properties of reflection of quadrangle about point. *Annals. Computer Science Series*, 11(1):79–91, 2013.

[19] B. Zlatanov. An etude on one Sharygin's problem. *Global Journal of Advanced Research on Classical and Modern Geometries*, 3(2):50–61, 2014.

[20] B. Zlatanov, S. Karaibryamov, and B. Tsareva. On a new function in the dynamic software. In *The Fibonacci Project, Research Approach to Education in Mathematics*, pages 133–138. Sofia, Bulgaria: Institute of Mathematics and Informatics at Bulgarian Academy of Sciences, 2013.

[21] B. Zlatanov and B. Tsareva. *Textbook for interactive training on Synthetic Geometry in Dynamic environment*. Plovdiv, Bulgaria: Plovdiv Electronic University, 2014.

[22] H. Zulnaidi1 and E. Zakaria. The effect of using GeoGebra on conceptual and procedural knowledge of high school mathematics students. *Asian Social Science*, 8(11):102–106, 2012.

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