GEOMETRIC TRANSLATIONS: AN INTERACTIVE APPROACH BASED ON STUDENTS' CONCEPT IMAGES

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Abstract

The authors present research-based activities that engage middle school students in knowledge-building of geometric translation. In the activities, students use interactive GeoGebra sketches and confront numerous examples and nonexamples as they develop and refine their understanding of translation, first as a motion then as a mapping on the plane.

Keywords: translation, transformation, vectors, examples, non-examples

1 INTRODUCTION

1.1 Overview

For quite some time, mathematicians and mathematics educators have emphasized the importance of geometric transformations. For example, the National Council of Teachers of Mathematics states that, "Instructional programs from pre-kindergarten through grade 12 should enable all students to apply transformations and use symmetry to analyze mathematical situations" [7, p. 41]. In the approach advocated by the Common Core State Standards for Mathematics (i.e., Content Standard 8.G) [8, p. 55], geometric transformations, including translations, play a central role in understanding congruence and similarity. However, recent research in the learning of geometry has shown that students need more systematic opportunities to develop their understanding of geometric transformations and in particular geometric translations [5, 14].

In this article we present activities for middle school students that are based on the nature of students' concept images of geometric translations. In the activities, we propose the use of interactive GeoGebra files and systematic use of examples and nonexamples. The examples and activities are carefully chosen and sequenced to help students develop their understanding through a research-based, translation-as-motion approach. The activities are sequenced to help students develop understanding of translation as motion, making both the direction and the distance of the motion explicit. Some address common misconceptions students have about translations. The activities are thus aimed to develop consistency between students' concept images of translation and the definition of translation. The activities and choice of examples are based on research of student growth in understanding of transformations in technological and non-technological environments [5, 14].

Before presenting the activities, we discuss the central role of examples and nonexamples in the learning of mathematical concepts. Next, we briefly describe three stages in middle-school students' thinking about geometric translation as a motion. Ten activities for middle school students constitute the main body of the article. In most cases, each activity is preceded by a brief discussion that provides a rationale or a description of students' approaches or difficulties addressed by the activity. We conclude with a discussion of more advanced stages of understanding that should be developed beyond middle school.

1.1.1 Role of examples and non examples in the learning of concepts

Students rely on individual mental representations as they engage in mathematical activities and problem solving tasks. These representations might be influenced by previous experiences [1]. Psychologists point out that most natural human concept formation is based on examples, and sometimes on a specific example [6]. Students generally rely on special examples of mathematical concepts which are called prototypes [9]. Hershkowitz [3] points out that every concept has a set of examples and a set of critical attributes, with prototypes playing a special role. One way to define prototype is as "those category members to which subjects compare items when judging category membership" [10, p. 575]. Usually textbooks do not have a large variety of examples, and some students develop limited conceptions and use prototypes that are too restricted. For instance, a student may conclude that "you cannot translate diagonally" [14, p. 42] because all the examples of translations he has seen in the book are horizontal or vertical translations. Students also appreciate examples, either as a means of understanding the concept or as a means of checking their understanding.

When learning a new concept, students go through different levels of concept learning [11]. In the first four levels, examples (rather than a definition) play a crucial role. It is only at the last level that definitions play a role in developing the understanding of a concept. First, students recognize examples they have experienced earlier. Next, students recognize examples encountered earlier even though the examples may be observed from a different perspective. At the next level, students can distinguish examples from nonexamples. Next, students produce their own examples. Finally, at the fourth level, students state definitions of the concept.

Using nonexamples as well as examples enables the student to focus on concept-relevant properties by noting their absences. Nonexamples enable students to recognize the insufficiency of an incomplete concept and helps them develop their concepts. In our activities, we use principles advanced by [4], introducing nonexamples after students have experienced a handful of examples first. We choose nonexamples which closely resemble examples. To use a transformation that completely distorts figures as a nonexample of translations would not add much precision to the student's concept of a translations, whereas a reflection, a dilation, or a rotation combined with a translation might. We focus the attention of the students to nonexamples (why isn't this an example?) to help students process the information in nonexamples more completely.

1.1.2 Geometric translations as motion translations

Hollebrands [5] and Yanik [14] found that students' initial thinking about translations is based on the perceived motion of an object. There are three stages of thinking about translation as translation motion. Initially students think about translations as undefined translation motion. At this stage, students think about translation in terms of motion of physical objects, like moving a pencil or a table. They incorporate physical concepts as part of their thinking about translations, such as needing a force to move an object to overcome friction. Students describe translations without specifying the direction of the motion, or the distance that the object was translated. Students may or may not be aware that translations do not involve rotations. At an intermediate stage, students think about motion and specify one of the parameters, either the direction or the distance, but not both. A third stage is marked by students thinking about motion in terms of both a direction and a distance. At this most advanced level, translations as motions may be described in terms of vectors although such explanations may remain challenging [5, 13, 14]. To help students reach the stage where translations can be thought in terms of direction, distance, and motion of mathematical objects (rather than physical objects), and develop a more complete understanding of the role of the vector that determines the translation, we propose several activities with GeoGebra that include examples and nonexamples. In each, it is important for students to focus on the result of the translation.

2 ACTIVITIES FOR STUDENTS

2.1 Free exploration with software (embedded motion with constraints)

The purpose of the free explorations is, first, to help students focus on motions that are solely translations and do not involve other types of motions such translations combined with rotations. When students move physical objects with their hands or bodies they often move the object not as a pure translation but rotate the object along the way, too. Free exploration with software addresses a misconception common among students first learning about translation, namely, that translations are combined with a rotation. Yanik [14] found that over half of the students in his study (grades 6-8) thought of translation as a motion that also incorporates some rotation. Some students identified transformations that not only moved a figure but that changed orientation of the figure as translations. For instance, in one case, a student identified the example in Figure 1 as a translation, despite the fact that the marked point revealed that point A had been rotated [14, p. 41].

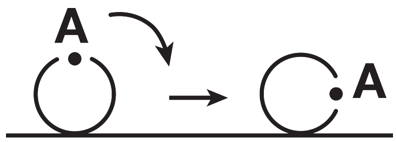


Figure 1. A circle and transformed image with a marked point.

As students drag the figures on the computer screen, they need to notice that the orientation of the shape does not change and that the shape does not turn. On the other hand, the computer environment allows translations in any direction, thus addressing the misconception that translations are always horizontal or vertical. Second, the language in our activities focus on direction and distance, providing students with a basis for understanding that translations are motions of shapes that are determined by two parameters - namely, direction and distance. As mentioned earlier, many students consider translations as motions without explicitly considering these parameters. Providing initial verbal descriptions that include direction and distance lay the foundation for stronger understanding of vectors.

After interacting with the computer, translating shapes on a paper grid helps students to further internalize translations in terms of units moved in different directions while encouraging students to think about translations of mathematical objects rather than physical objects.

Activity 1a

Open the file at https://www.geogebratube.org/student/m396571. Drag point H. By moving point H you will be able to translate the L shape (see Figure 2).

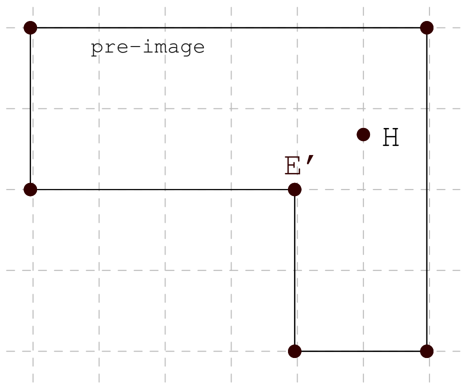


Figure 2. Translating a shape by dragging point H.

When you release, the final position of the L shape will be the result of the translation. We will call the original shape the pre-image, and the result of the translation the image. Translate the L shape on the grid as indicated.

1. Move the L shape horizontally
2. Move the L shape vertically
3. Move the L shape in any other direction

Student questions

What point on the image corresponds to point A on the pre-image? Denote the corresponding point with A′ . Describe the direction you need to move A to arrive at A′ .

Possible answers

Different students may choose different distances. Figures 3-5 illustrate some possibilities.

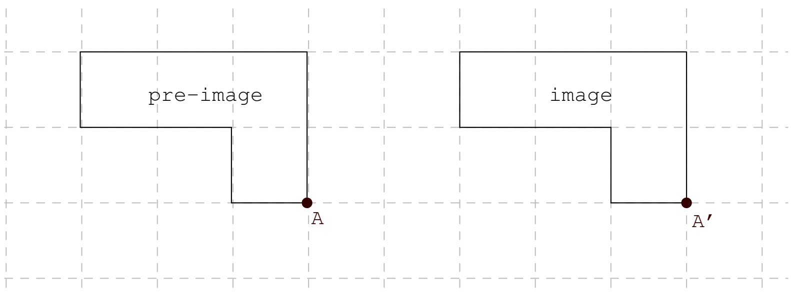


Figure 3. Horizontal translation 5 units to the right.

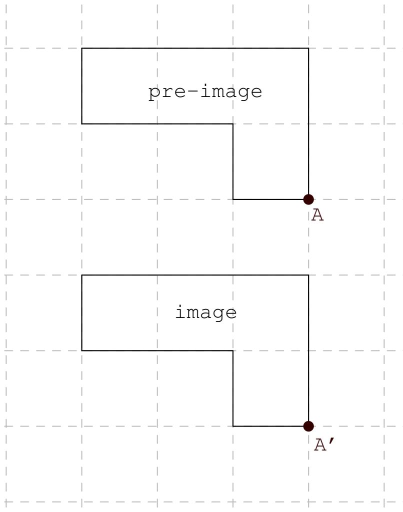


Figure 4. Vertical translation 3 units down.

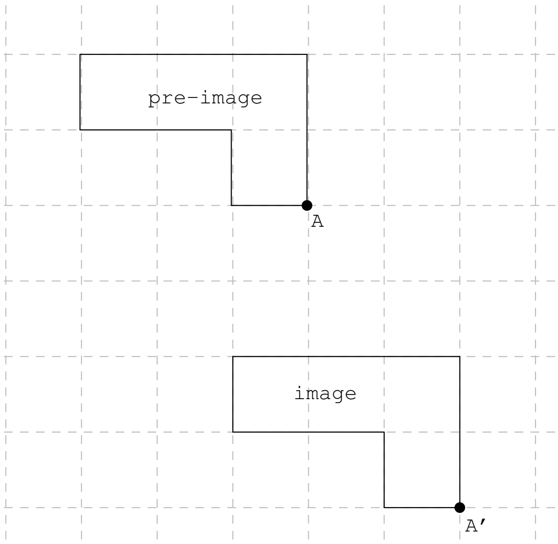


Figure 5. Translation two units to the right and four units down.

Activity 1b

Use Figure 6 to sketch one of the translations you did above. Choose another point `B` on the pre-image and identify the corresponding point `B′` on the image.

Student question

In what direction do you need to move to go from `B` to `B′`?

2.2 *Examples of translations*

Providing a variety of examples and opportunities for students to translate different shapes may expand their understanding of geometric translations. Yanik [14] found that some students focus on corners and edges to determine the translation and struggle with the concept of translating circular shapes. One student said, "You cannot translate a circle. It does not have a corner. How am I going

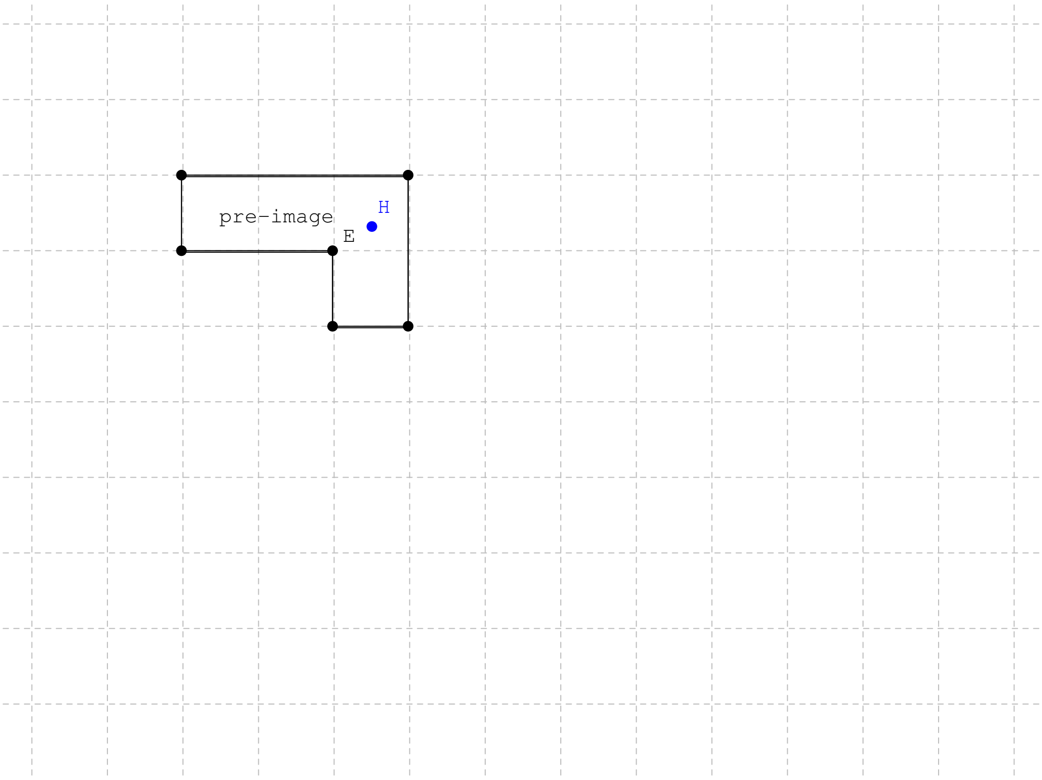


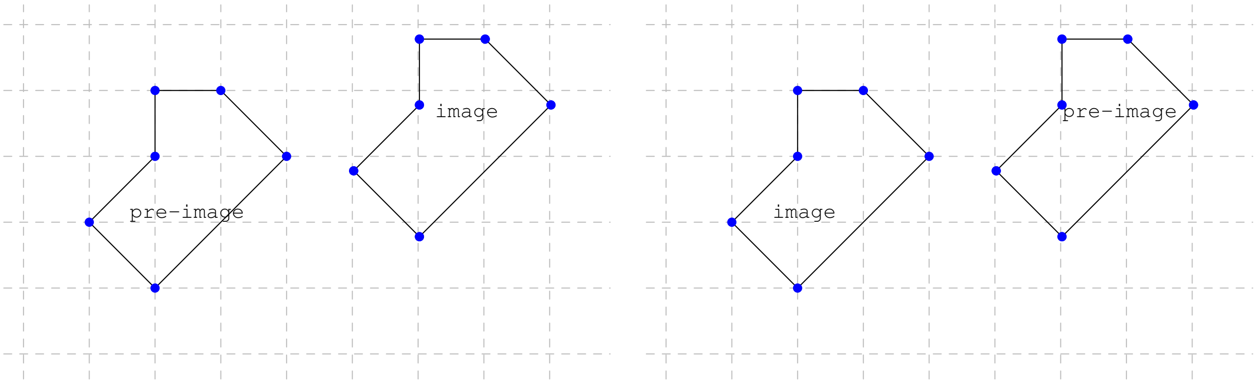
Figure 6. Grid for sketching the image of a figure and two points.

to know where I can hold it and translate it?" [14, p. 41]. Another student stated, "In translation... we measure the distance between the corners of the pre-image and image. Since circle does not have a corner we cannot measure that distance" [14, p. 41]. Along with translating polygonal shapes, translating circular shapes might also be helpful for students to comprehend one-to-one mapping of points on pre-image and image figures. In the following section, we provide various translation examples, including polygonal and circular shapes. For the circle, several points of reference are identified, the center and a few arbitrary points on the circumference. These points are meant to help students measure the distance between pairs of pre-image and image points. The use of different colors for the points on the circle help students identify what pairs correspond to each other.

Activity 2a

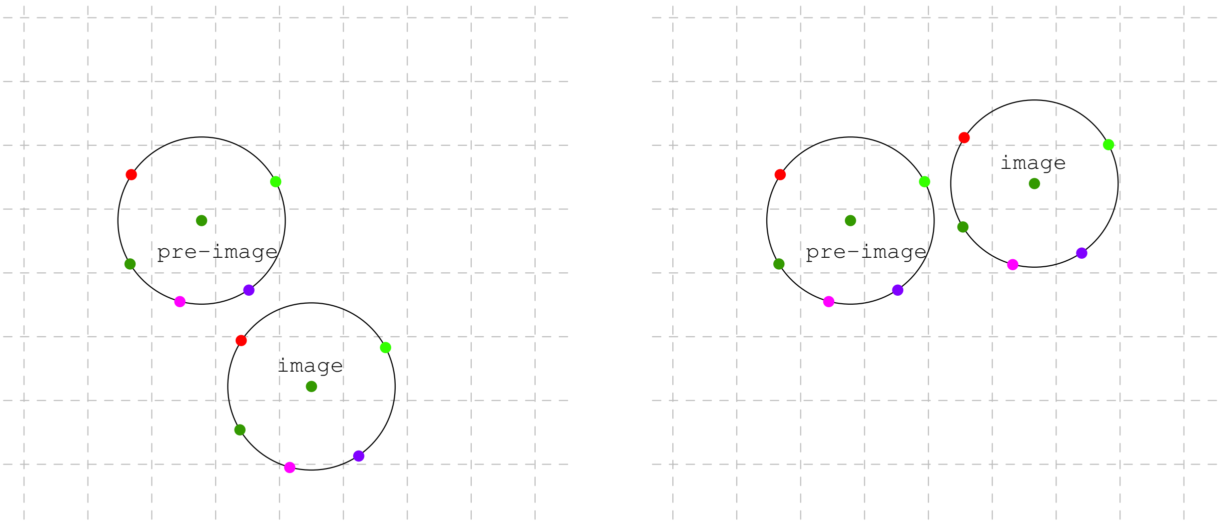
Open the file at http://tube.geogebra.org/m/1336109. Hide and show shapes with checkboxes within the sketch. Hide the square and the circle and focus on the polygon. Then focus on the square and finally on the circle.

The pairs of shapes in Figures 7 - 9 represent different translations. In each pair we show the initial position of the geometrical shape (the pre-image), and the result of the translation (the image). In your own words, describe all the characteristics that the pre-image and image have in common. Address issues such as size, shape, orientation, direction, etc. For each translation make explicit the direction of the translation as well as the distance(s) that various parts of the shape were moved.



(a) **(b)**

Figure 7. Translation of Polygons.

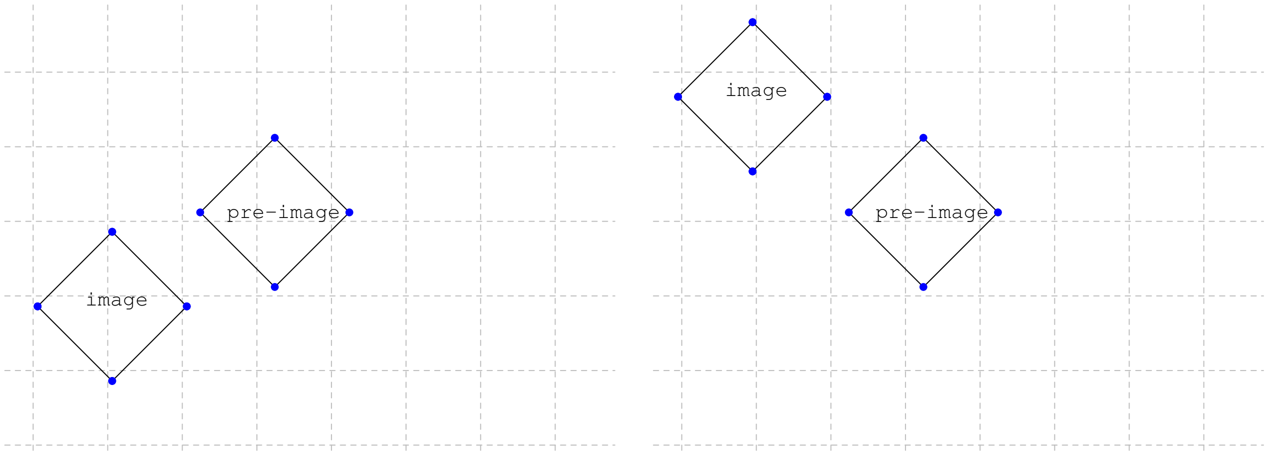


(a) **(b)**

Figure 8. Two Translations of a Circle

Activity 2b

Summarizing characteristics of translations. After looking at a rich variety of examples, students should make a list of what properties are preserved by translations. Often students use their own informal language; the teacher can provide more precise mathematical terms. At this point the purpose is not to define translations but to list their properties. Some middle school students will be able to make explicit that translations preserve shape, size and the direction of a figure being translated [14]. That is, the original shape and the image are congruent and have the same direction. One student referred to these two properties by saying, "when you put the final triangle on top of the initial triangle they would match exactly" (p. 41). A second student stated, "translation does not include rotational motion" (p. 40). Another student also made explicit that rotations are not part of translations: "Translation is like sliding a figure or an object. You don't rotate it. You just slide it" (p. 39). Some students will only refer to what happens to the figure as a whole, whereas other will make reference to corners (vertices) and sides of the shapes. As one student noted, "we measure the distance between the corners of the pre-image and image" [14, p. 41]. Furthermore, some students will realize that each of the points of the original shape, not just the corners, is displaced in the same



(a) **(b)**

Figure 9. Two Translations of Squares

direction and by the same distance. Some students will think that the shape of the pre-image figure is not important for translations, while some others may think that the shape is crucial [14, p. 40].

2.3 Examples and nonexamples of translations

After sharing a variety of examples of translation with our students, we introduce non-examples into our instruction. Comparing examples and nonexamples of translations help students further understand the effect of translation on pre-image figures. Past research (e.g., [12–14] suggested that students often confuse translation with reflection and change the orientation of shapes as a result of translation. Students may also alter the direction of figures being translated or change the original size of the pre-image figure. Such nonexamples can be provided for students to discuss whether or not these transformations are examples of translations, and why or why not. Below, we provide various non-examples of translations. In the activities, students are asked to use informal language to explain whether or not the pairs of shapes illustrate an example of a translation. Based on their descriptions, the teacher may help students use more precise mathematical language. For instance, when discussing whether the pair of triangles shown in Figure 10 could represent a translation, one student stated, "at the end it turned to a different triangle, it doesn't look like the initial triangle" [14, p. 41]. Here the student was referring to the direction of the triangle. The teacher may focus the attention of the student on parts of the figure, not just the figure as a whole. For instance, the smallest angle in the original triangle is "pointing up" whereas in the transformed triangle it is "pointing down."

Activity 3a

Identifying examples and non examples. Consider the pairs of shapes in Figures 11 - 13.

Student questions

Determine whether or not each pair illustrates a translation. In your own words, explain why or why not. Additionally, if the figures do represent a translation, state the direction that pre-image is moved

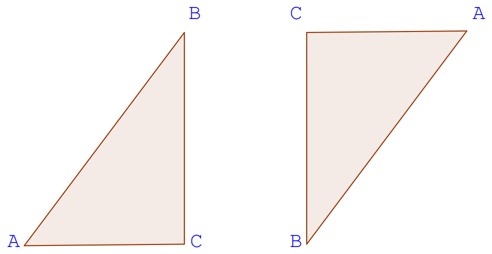


Figure 10. Two triangles that appear "different" to students.

and the distance. If not, determine whether the pairs represent another type of transformation, perhaps a combination of a translation with another transformation.

Hints

Notice that the orientation of corresponding points changed. Moving from A to B to C in the preimage results in a counterclockwise motion, but moving from A′ to B′ to C′ in the image results in a clockwise motion. Connect some of the points in the pre-image with their corresponding images. Do all of the points move the same distance?

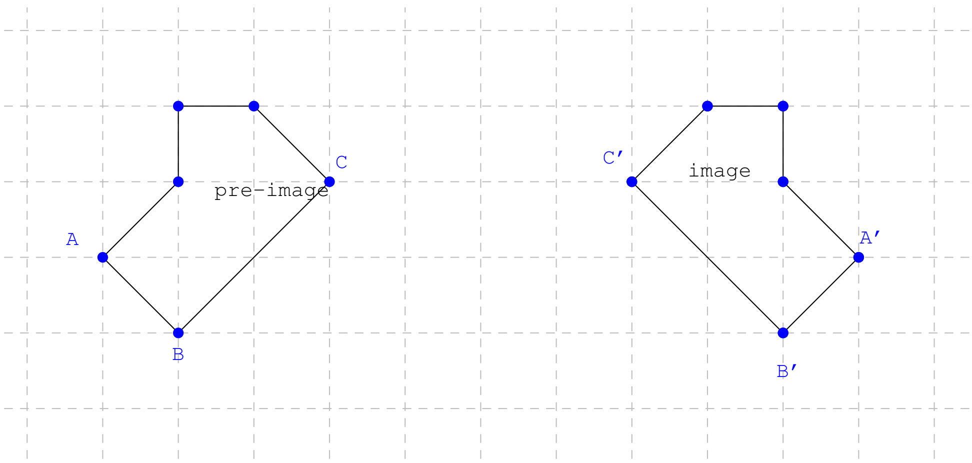


Figure 11. Preimage and image pair. Does this illustrate a translation?

Comments about transformations

In Figure 12 the placement of the image as a whole changed, although orientations of both shapes are the same. Students can determine whether the two shapes have the same direction by looking at corresponding segments. For Figure 13 we can imagine two transformations. First we translate four units horizontally to the right. Then the figure is rotated counterclockwise around the lowest vertex, A. For Figure 14 the image is not congruent to the preimage.

Activity 3b

Consider the following pairs of shapes of the same color.

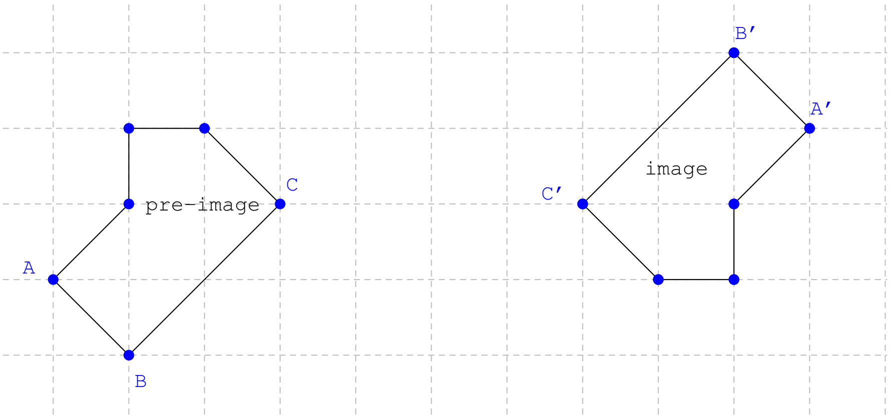


Figure 12. A pair of corresponding shapes.

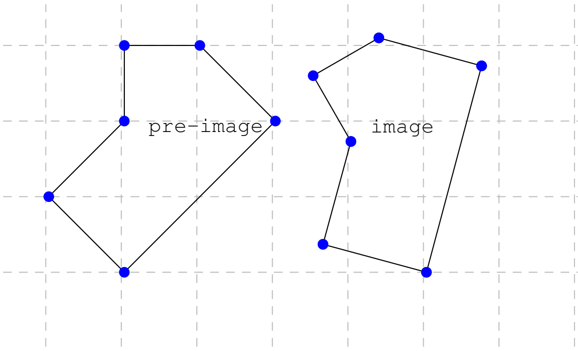


Figure 13. Is this a translation?

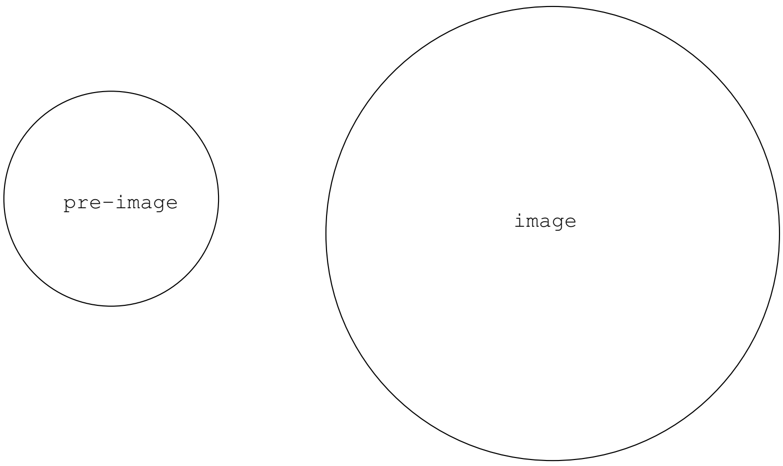


Figure 14. A preimage circle and its scaled image.

Student questions

Determine whether or not each pair illustrates a translation. In your own words, explain why or why not.

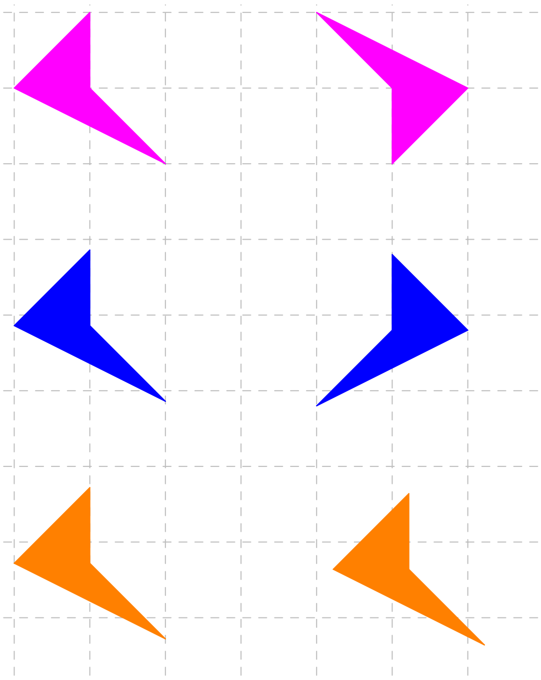


Figure 15. Which pairs of shapes represent a translation? Which do not?

2.4 Representing both direction and distance with a vector

Understanding the role of vectors in translating figures is often challenging for students. For instance, students often have misconceptions with respect to the vector that determines the translation [14]. Some use the given vector as a reference line, for example to mark the middle of the translation. Other students think of the vector as part of a line of symmetry. Others use the endpoint of the vector to locate the translated figure, rather than the direction and length of the vector.

The initial activities with vectors focus on sketches with complete translations that include pre-image, image and translation vector. Students may alter the direction and the magnitude of the vector to see the effect of this change on the image. While students see that altering the magnitude and the direction of the vector changes the location of the image, students may have difficulties making a direct connection between the magnitude of the vector and the distance between corresponding pre-image and image points.

Activity 4a

Open the file at http://www.geogebra.org/m/1336071. Drag `K` to change the vector and observe how the pre-image is translated. What can you say about the colored arrows connecting corresponding points on the pre-image and the image?

Activity 4b

Open the file at http://www.geogebra.org/m/1336079. Drag `H` to translate the circle. Notice that although the circle does not have corners, we can still mark points on the original circle and find the corresponding points on the image.

It is crucial for students to understand that connecting points in the pre-image with corresponding image points determines the same direction and the same distance for all points. Furthermore, the distance and direction are determined by the vector.

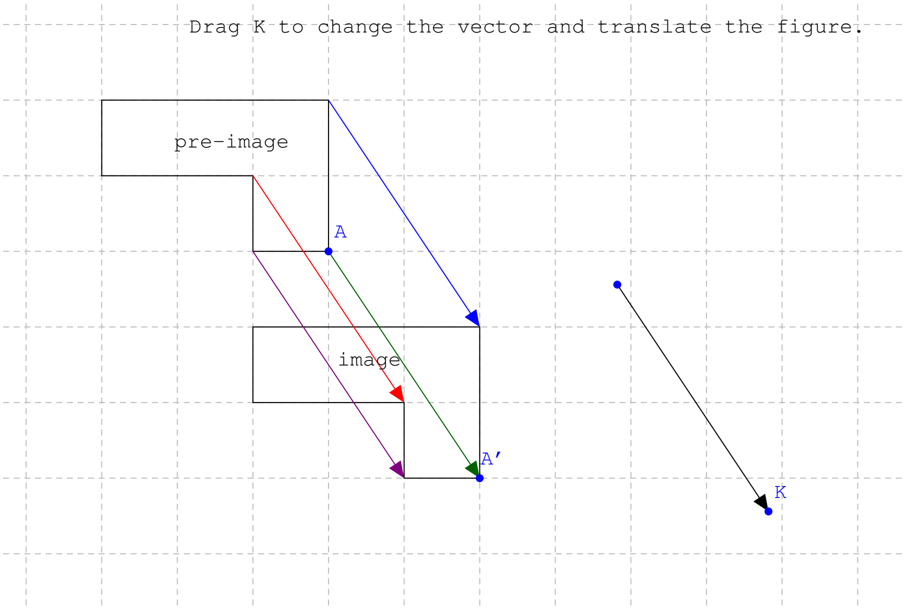


Figure 16. Translation by given vector.

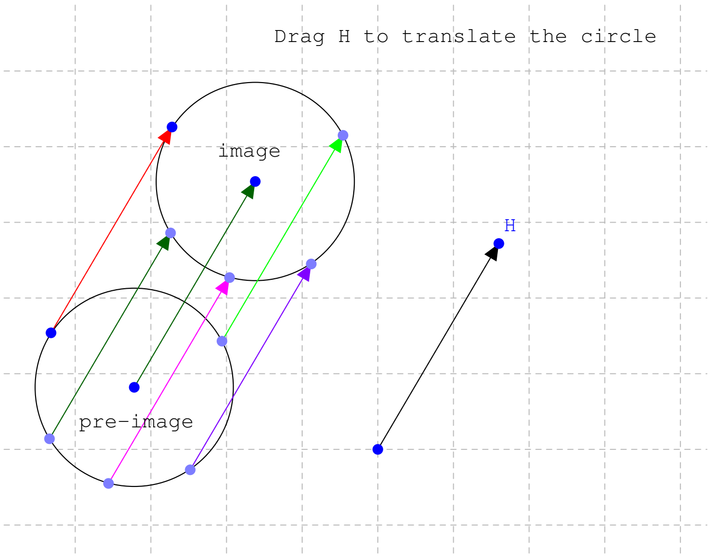


Figure 17. Corresponding points in the translation of a circle

Translations with vectors with the same direction but different distances

The following activities focus on changing only one aspect of a vector, namely the magnitude, while keeping direction the same. We ask students to alter the magnitude of the vector without changing its direction and see how this affects the result of the translation.

Activity 5

Open the file at http://tube.geogebra.org/student/m398973. Drag `K` to translate the L shape. `Image` and `image2` are translations of the pre-image in the same direction with different distances. What can you say about the distance between `A` and `A''` compared to the distance between `A` and `A'`? How do you know that the direction from `A` to `A'` is the same as the direction from `A'` to `A''`?

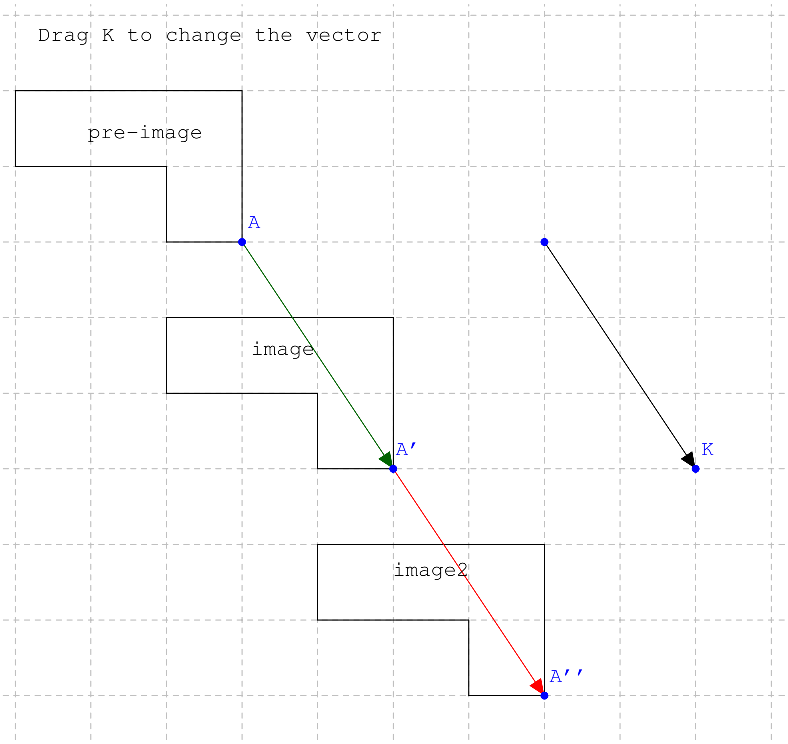
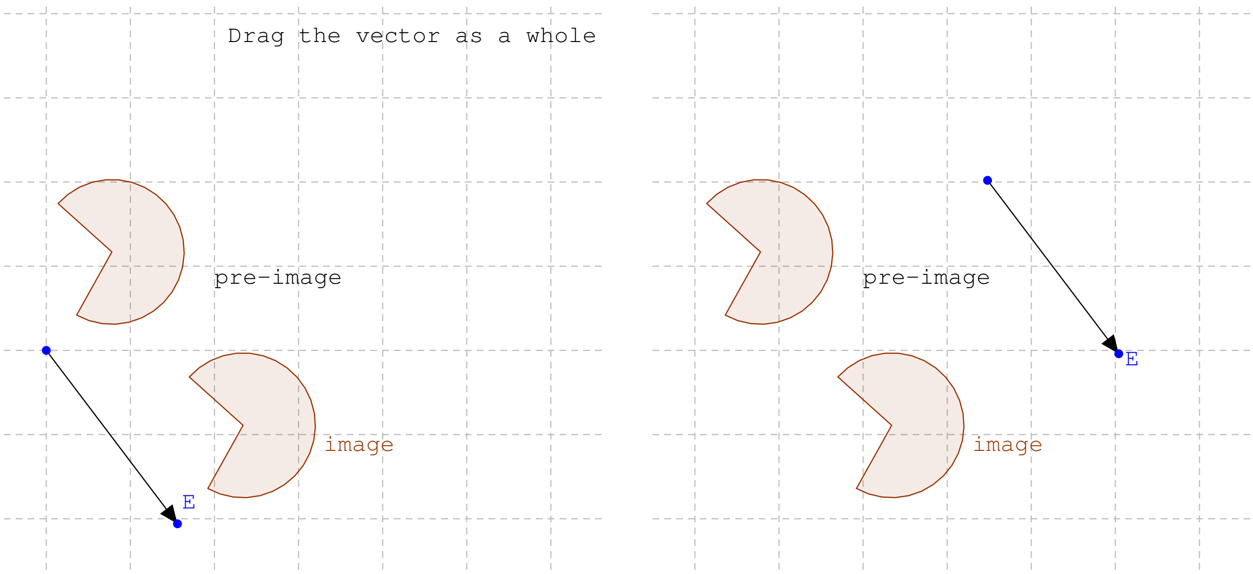


Figure 18. Translations in the same direction, different distances.

2.5 The position of the vector on the plane is not relevant, only its direction and magnitude

Students also need to realize that the position of the vector on the plane does not affect the result of translation. For this purpose students drag the whole vector without changing magnitude or direction and observe effects on the image.



(a) Translation by vector. **(b)** Same vector in different position

Figure 19. Same Translations with vector in different position

Activity 6

Open the file at http://tube.geogebra.org/m/1336123. Drag `E` to see how the vector determines the translation. Now, clicking in the center of the vector (not its endpoints) drag the vector as a whole. Notice that the vector is translated parallel to itself and its length stays the same. Notice that the image of translated figure did not change. However, as you saw before, if you drag the endpoint of the arrow to change the direction or length of the vector, the image changes position.

2.6 *Length of vectors and distance of translation*

One way to help students make a connection between the length of a vector and the distance of its corresponding translation is to consider the corresponding points on pre-image and image figures, and encourage students to focus on the relative positions of these points.

Activity 7

Open the file at http://www.geogebra.org/m/1338903. Identify two points, `A` and `B`, on the pre- image and their corresponding points, `A′` and `B′` on the image. Click the checkbox to connect these pairs of points with a segment.

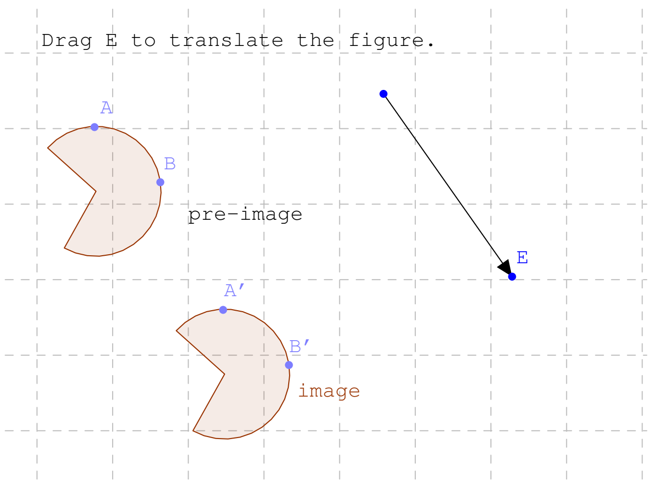


Figure 20. Length of vector and distance between original point and image.

What is the distance from point `A` to `A′`? What is the distance from point `B` to `B′`? What is the length of the vector? You can determine the length of the vector and the distance between two points by measuring segments with GeoGebra (click the checkbox to show measured distances) or by hand using the Pythagorean theorem.

2.7 *Translating several objects simultaneously*

Students can also work on translating several objects simultaneously. The following activity helps students understand that translations preserve the relative distance and position of points both within and among figures.

Activity 8

Open the file at http://www.geogebratube.org/student/m398837. Drag `H` and notice that both the L shape and the circle are translated. If we wanted, we could translate any number of shapes, and we could also translate the entire plane!

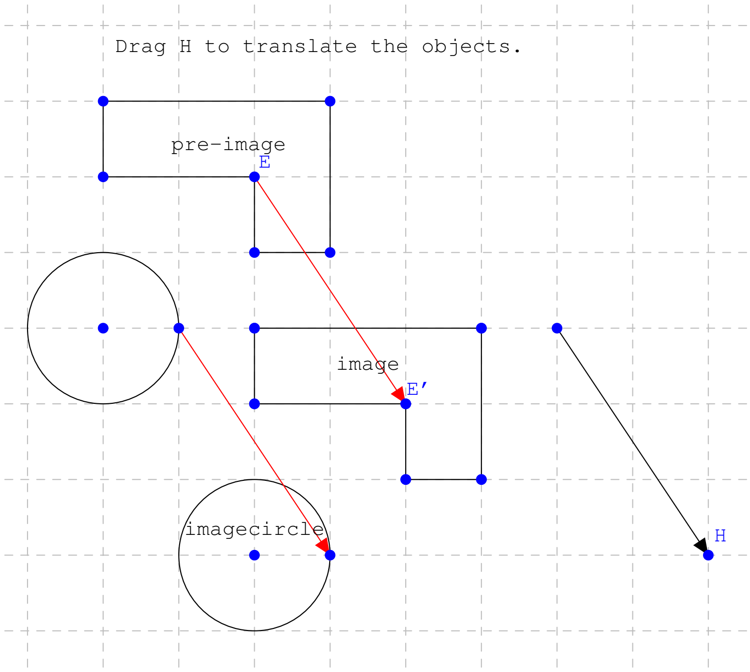


Figure 21. Translating two shapes.

2.8 Determining the vector from a given translation

Activity 9

Open the file at http://www.geogebra.org/m/1338361. A pre-image is shown on the right and the corresponding image on the left. Find a vector that generates this translation. What are its coordinates? Click at the box to show one such vector. Move the vector by dragging it as a whole to verify that it will indeed generate the given translation. That is, put the tail of the vector at a point on the pre-image and verify that the point of the vector is at the corresponding image point.

2.9 Determine your own translations with GeoGebra

2.9.1 Activity 10

Construct any shape within a new GeoGebra file. For example, use line segments, circles, polygons, or any other shapes. Next, in the segment menu, choose the *Vector between Two Points* option.

The vector tool, , is now highlighted. Construct a vector on the plane by clicking on two points. This vector will determine how your figure is translated. In the transformations menu, choose the *Translate Object by Vector* option.

The translation tool should be highlighted . First click the object you want to translate, then click the vector. We will illustrate the steps above with a circle, but you can use any shape. Construct a circle.

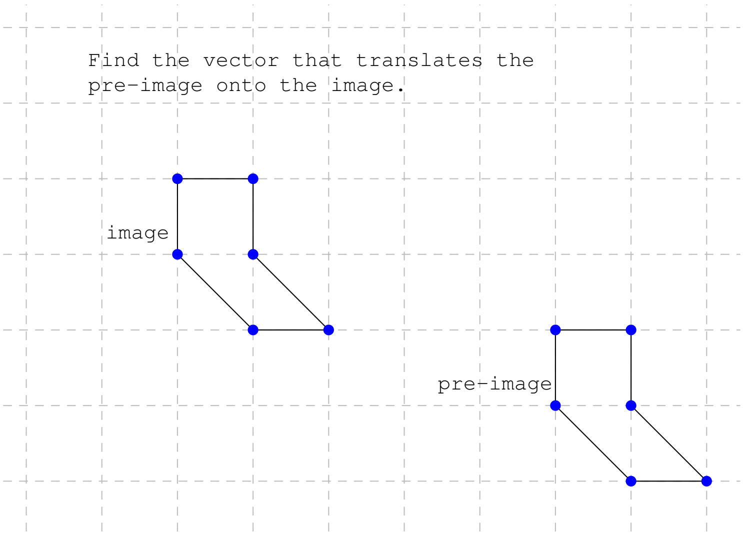


Figure 22. What is the vector for this translation?

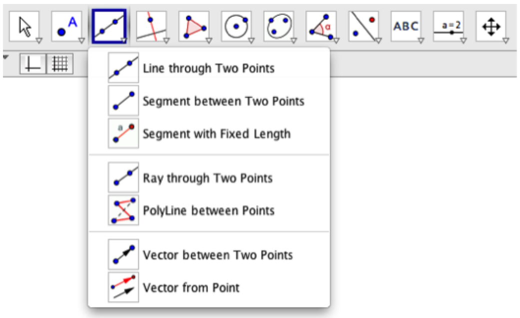


Figure 23. Vector between Two Points in the Line menu.

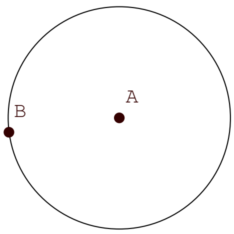


Figure 24. A circle to be translated.

Choose the vector tool  and construct a vector, $\overrightarrow{CD}$.

Choose the translation tool  . In order to translate not only the circle, but also its center A and the point B, we use the translate vector tool to draw a box around the circle (Figure 26) and then click at the vector. The result is a circle with its translated image (Figure 27).

Change the vector by dragging one of its endpoints. See how the translation of circle AB changes.

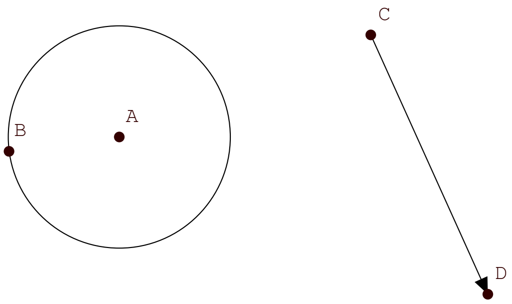


Figure 25. The vector that will determine the translation.

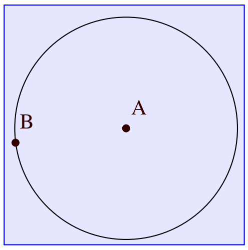


Figure 26. Highlight the object.

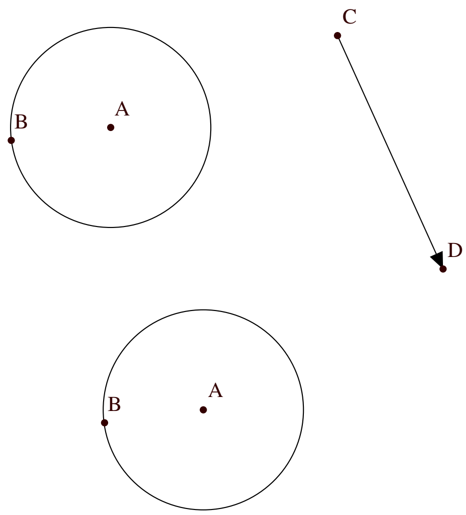


Figure 27. The original object and its translation.

2.10 Definition of geometric translation in terms of a vector

Activity 11

Use your own words and pictures to define a translation of a geometric shape in terms of a given vector. Explain how the direction of the vector and its length determine the distance and direction of the translation.

3 FURTHER STAGES IN THE DEVELOPMENT OF UNDERSTANDING OF TRANSLATIONS

Thinking about geometric translations as motion translations is natural for middle- school students. Thinking about translating a preimage while fixing the plane is also a convenient way of thinking about translations.

However, in high school and later years, geometry instruction needs to address the development of a different way of thinking about translations, namely as mappings of the plane. Activity 8 focuses on more than one shape and is meant provide a first step in extending the domain of the translation. Likewise, the sketch shown at http://tube.geogebra.org/m/1336109 can be used to show three shapes translated at the same time. Extending the domain encourages students to think about translations as mappings, where the domain of the translation is the entire plane. In order to help students transition to thinking about translations as mappings of the plane, rather than just thinking about translations as motions, researchers [5, 14] have found that several concepts are crucial, such as the domain for translations, parameter (vector in the case of translations), translations being one-to-one mappings of points in the plane, and relations and properties of translations. Although activities that address these concepts are beyond the scope of this article, it is important for middle school teachers to have an idea of how thinking about translations evolves. A further step in the development of students thinking about translations is that once they have developed the understanding of translations as mappings of the plane, students will need to have further opportunities to work systematically with translations as objects of mathematical interest in themselves and talk in terms of identities, inverses, and compositions of translations.

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