EXPLORING CYCLIC QUADRILATERALS WITH PERPENDICULAR DIAGONALS

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Abstract

After a brief discussion of two preliminary activities, the author presents three student explorations about cyclic quadrilaterals with perpendicular diagonals and an extension to cyclical quadrilaterals. In the first exploration, students explore the relationships among the intersection of the segments joining midpoints of opposite sides, the intersection of the diagonals, and the center of the circle. In the second, the relationship between the sums of squares of opposite sides and the square of the radius is examined. In the third, students explore properties of a line perpendicular to a side through the intersection of the diagonals. Lastly, students are introduced to the concept of anticenter as they extend one of the results to cyclic quadrilaterals in general.

Keywords: Cyclic Quadrilaterals

1 Introduction

1.1 Overview

Students explore quadrilaterals that satisfy two conditions - namely, 1) they are cyclical, that is, their four vertices lie on a circle, and 2) their diagonals are perpendicular to each other. With GeoGebra [2] it is straightforward to construct such quadrilaterals using the coordinate axes and the circle tool, as suggested in Figure 1). In two preliminary activities, students explore Varignon's theorem for quadrilaterals and a result about the intersection of lines that connect midpoints of opposite sides of a quadrilateral. After three follow-up explorations, students extend one of the results to a general cyclical quadrilateral.

1.2 Background knowledge

Students only need basic knowledge of GeoGebra to explore and discover the results discussed in this paper. Basic results from high school geometry and algebra may be used to prove hypotheses students generate from these explorations. Our students have used the following to construct proofs: a) the Pythagorean theorem; b) angles inscribed in a circle that subtend congruent chords are congruent, and conversely; c) the altitude on the hypotenuse of a right triangle forms two right triangles similar to the original triangle; d) the diagonals of a parallelogram bisect each other; e) a triangle with two congruent angles is isosceles; f) the line connecting the midpoints of two sides of a triangle is parallel to the other side; g) side-angle-side principle for congruency of triangles; h) the equation for a circle in coordinate geometry; i) finding the coordinates of the midpoint of two given points; i) solving a quadratic equation.

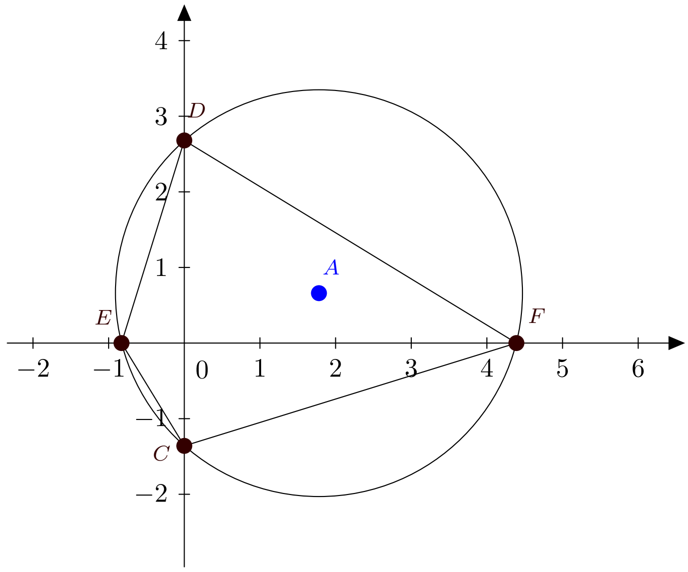


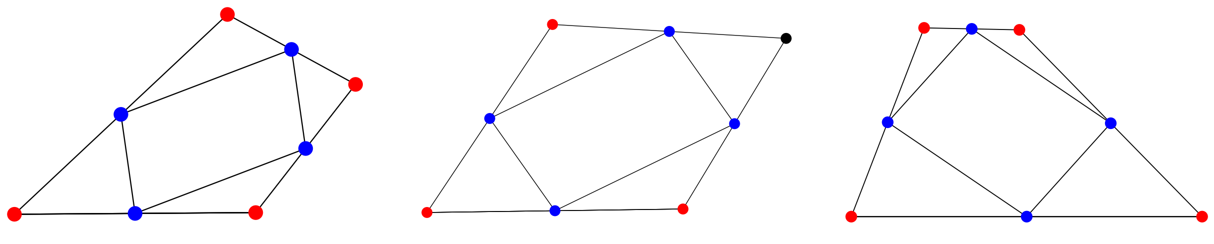
Figure 1. Intersection of a circle with the coordinate axes.

2 PRELIMINARY ACTIVITIES FOR STUDENTS

2.1 Connecting midpoints of consecutive sides of the quadrilateral

Varignon's theorem states that the inscribed quadrilateral formed by connecting the midpoints of consecutive sides of any quadrilateral is a special quadrilateral. Students generate sketches similar to those shown in Figure 2. As they explore these shapes, we ask them to consider the following prompts.

- What is the shape of the inscribed quadrilateral in general?
- What is the shape of the inscribed quadrilateral when the diagonals of the original quadrilateral are perpendicular? (Figure 3)
- State your conjecture and provide a convincing argument. Hint: look at the diagonals of the original quadrilateral.



(a) **(b)** **(c)**

Figure 2. Varignon's theorem for quadrilaterals.

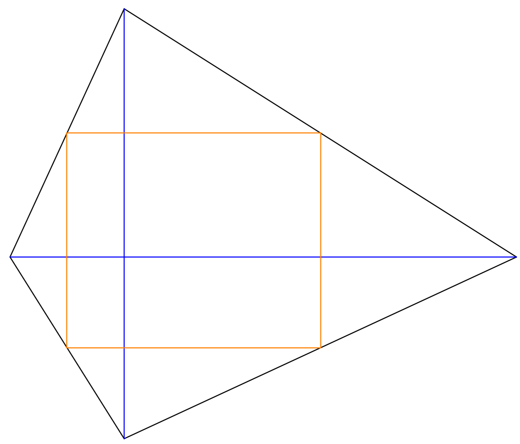


Figure 3. Special case of Varignon's theorem.

2.2 Connecting the midpoints of opposite sides of a quadrilateral

Next, we ask students to connect the midpoints of *opposite* sides of an arbitrary quadrilateral in GeoGebra. We ask them to consider the following prompts.

- Observe the location of the intersection point with respect to the whole segment, measure if necessary and state a conjecture.
- Give a convincing argument that the lines connecting the midpoints of opposite sides bisect each other. Hint: What can you say about the intersection of the diagonals of a parallelogram?

3 EXPLORATIONS FOR STUDENTS

3.1 Three interesting points in the cyclical quadrilateral with perpendicular diagonals

Students begin by constructing a cyclic quadrilateral with perpendicular diagonals. Students can use the *x*- and *y*- axes as diagonals, creating a quadrilateral such as *DECF* depicted in Figure 4. Once they have completed this construction, we ask students to consider the following prompts.

- Construct the line segments connecting the midpoints of opposite sides of the quadrilateral.
- Observe, conjecture and verify the relationship of the intersection of the segments joining midpoints of opposite sides, *O* (called the centroid), with the intersection of the diagonals, *P*, and the center of the circle, *A*.

Students can use coordinates to verify the conjecture that the centroid O is the midpoint of segment $\overline{PA}$ (see Figure 4).

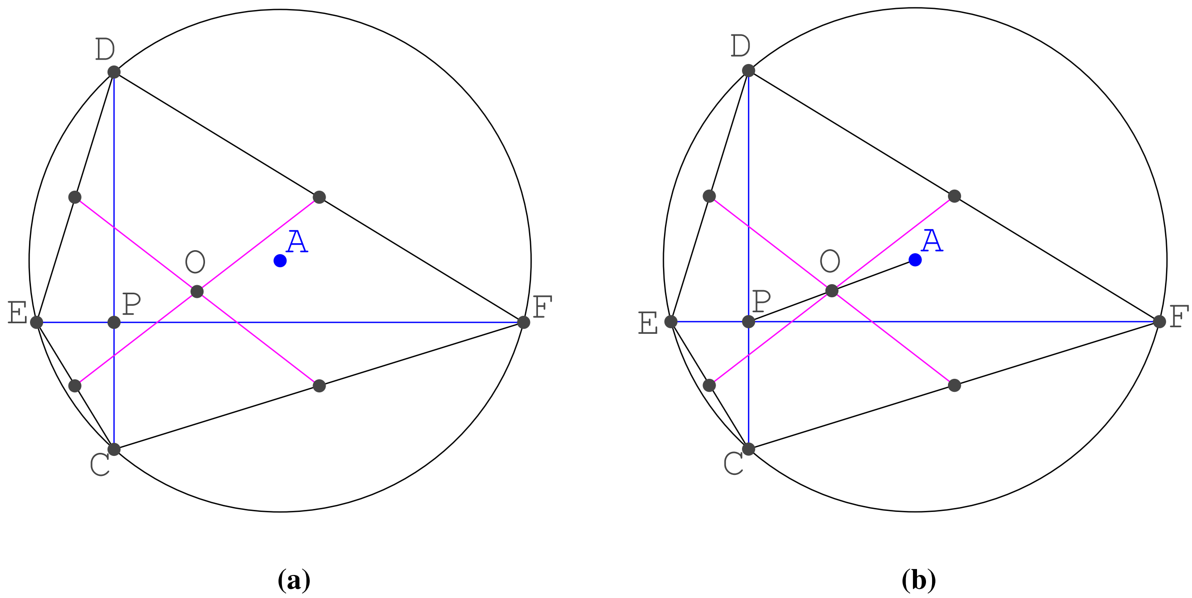


Figure 4. Three interesting points are collinear

Proof: Let P be at the origin, and A at the point (a, b). The equation of the circle is given by $(x-a)^2 + (y-b)^2 = r^2$. The x coordinates of the intersections of the circle with the $x-$axis are the solutions of the equation $(x-a)^2 + b^2 = r^2$ or $x^2 - 2ax + a^2 + b^2 - r^2 = 0$. Thus,

$$x_{1,2} = \frac{2a \pm \sqrt{4a^2 - 4(a^2 + b^2 - r^2)}}{2} = a \pm \sqrt{r^2 - b^2}$$

The x coordinate of the midpoint of one side is the arithmetic average of the x coordinates of its end-points. The x coordinate of the intersection point is the average of the x coordinates of the midpoints of two opposite sides. Thus the x coordinate of point O is the weighted average of the x coordinates of the quadrilateral, $0, 0, a + \sqrt{r^2 - b^2}$ and $a - \sqrt{r^2 - b^2}$, that is, $\frac{2a}{4} = \frac{a}{2}$. In the same way we can see that the y coordinates of the intersections with the vertical axis are $b + \sqrt{r^2 - c^2}$ and $b - \sqrt{r^2 - c^2}$ and that the weighted average of the four y coordinates of the intersection points with the axes is $\frac{b}{2}$. Thus the coordinates of O are $(\frac{a}{2}, \frac{b}{2})$, which correspond to the midpoint between $(0, 0)$ and (a, b).

3.2 Sums of squares of opposite sides and the square of the radius

In the second exploration, we ask our students to begin by considering the squares of the distances of P to the vertices of the quadrilateral, that is the values PE^2, PD^2, PF^2, PC^2. Students determine these values through quick use of built-in GeoGebra measurement tools. Next, we ask students to consider the following prompts.

- Combine the squares in pairs to form the squares of the sides of the quadrilateral.
- State your observations about the sums of the squares of opposite sides.

Students can use the coordinates of the vertices of the quadrilateral to prove that $PE^2 + PD^2 + PF^2 + PC^2 = 4r^2$. An alternative argument may be constructed by showing that $DF^2 + EC^2 = (2r)^2$ and recognizing that the indicated angles marked with the same letter in Figure 5 are congruent, then showing that $\overline{EC}$ is congruent to $\overline{FS}$.

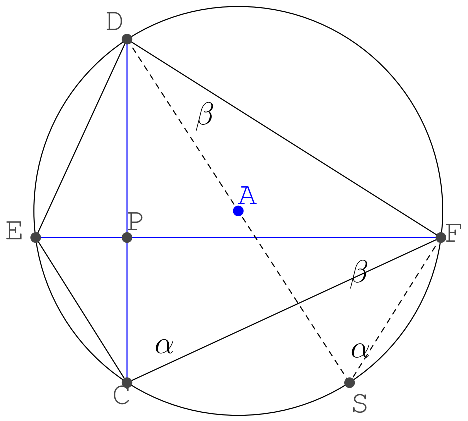
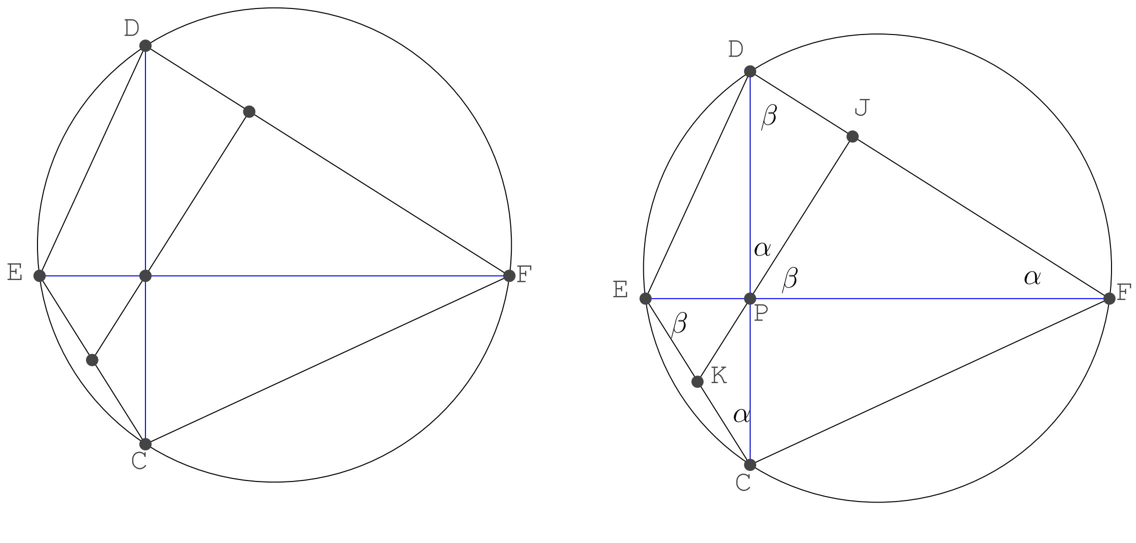


Figure 5. Congruent angles

3.3 A line perpendicular to a side through the intersection of the diagonals

In a third exploration, students again explore a cyclic quadrilateral whose diagonals are perpendicular (see Figure 6a). We ask them to consider the following prompts.

- From the intersection point of the diagonals, P, construct a perpendicular to side $\overline{DF}$.
- Observe the intersection of this line with the opposite side $\overline{EC}$.
- Make a conjecture about the position of this point K in the segment $\overline{EC}$. Measure to informally verify your conjecture.
- Provide a convincing argument that confirms your conjecture. Hint: Show that triangles $\triangle KPC$ and $\triangle KPE$ are isosceles triangles, using the fact that the marked angles are congruent.



(a) **(b)**

Figure 6. Segment perpendicular to a side through intersection of diagonals

3.4 Extensions to other cyclical quadrilaterals

As an extension, we ask students to explore cyclical quadrilaterals whose diagonals are *not* perpendicular. After constructing such a quadrilateral, our students consider the following prompts.

- Examine the point determined by the intersection of the diagonals, the centroid, and the circumcenter. Are they collinear? Why or why not?
- Examine the line connecting the point of intersection of the diagonals to the midpoint of one side. Is the line perpendicular to the opposite side? Why or why not?

Through exploration, students note that for general cyclical quadrilaterals the point determined by the intersection of the diagonals, the centroid, and the circumcenter are not necessarily collinear. We illustrate this in Figure 7. Likewise, the line connecting the point of intersection of the diagonals to the midpoint of one side is not necessarily perpendicular to the opposite side, as shown in Figure 8.

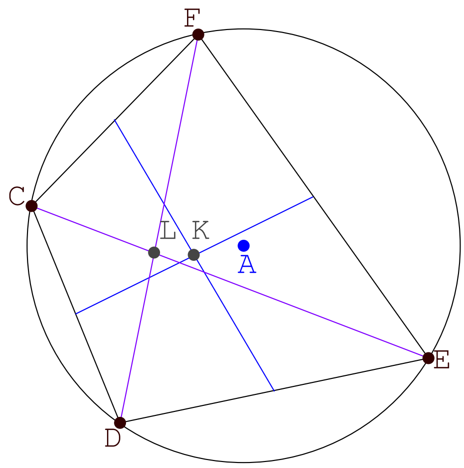


Figure 7. Points not collinear in general.

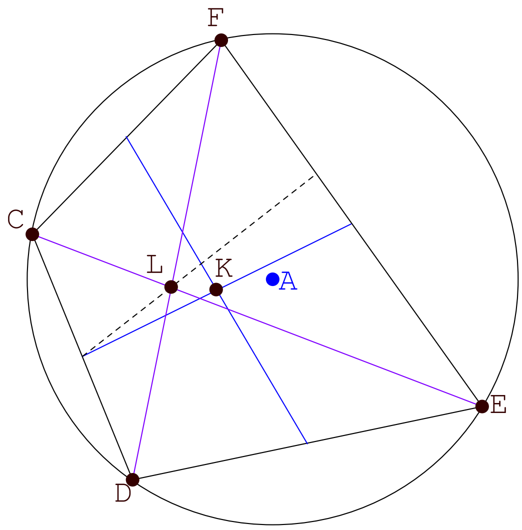


Figure 8. Segment through intersection of diagonals doesn't have the same property.

Lastly, we remind students that, in the special case of perpendicular diagonals, the intersection of the diagonals is the reflection of the center of the circle across the intersection of the medians. Moreover, this point of reflection has the special property that the line connecting the midpoint of one side to this reflected point is perpendicular to the opposite side (truly remarkable results!). After we provide students with some time to consider these observations, we ask them the following question.

> *Is there is a point inside a general quadrilateral that has the same property as the intersection of perpendicular diagonals in this special case?*

For general cyclical quadrilaterals the reflection of the circumcenter across the centroid is called the anticenter [1] (see Figure 9).

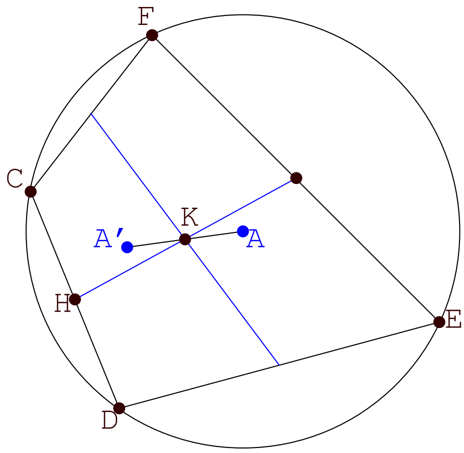


Figure 9. The anticenter.

Students can use GeoGebra to explore, conjecture and verify that the anticenter has the property that a line connecting it to the midpoint of a side is perpendicular to the opposite side (see Figure 10). One proof that our students have found accessible considers triangles $\triangle KA'H$ and $\triangle KAJ$ (see Figure 11).

Proof: $\overline{AJ}$ is perpendicular to $\overline{EF}$ because a radius through the midpoint of a chord is perpendicular to the chord. $\overline{KA'}$ is congruent to $\overline{KA}$ because A' is the reflection of A. $\overline{KH}$ is congruent to $\overline{KJ}$ because the medians bisect each other. Angle $\angle A'KH$ is congruent to $\angle AKJ$ (vertical angles). Therefore (side-angle-side) triangles $\triangle A'KH$ and $\triangle AKJ$ are congruent. Thus $\angle A'HK$ is congruent to $\angle AJK$. These are alternate interior angles, and so $\overline{HN}$ is parallel to $\overline{AJ}$, and thus $\overline{HN}$ is perpendicular to $\overline{EF}$.

4 IN SUMMARY

GeoGebra provides students with opportunities to explore cyclic quadrilaterals with perpendicular diagonals in ways that aren't possible with pencil and paper alone. Dragging vertices of the quadrilaterals or the center of the circle, students can observe quantities that vary and properties that remain invariant. Students form conjectures, using GeoGebra's built-in measurement tools to informally verify their hypotheses. We ask students to follow such verifications with more formal deductive arguments. Starting with the special case of cyclical quadrilaterals that have perpendicular diagonals then examining more general cyclical quadrilaterals, we scaffold students' learning and help them make connections across cases.

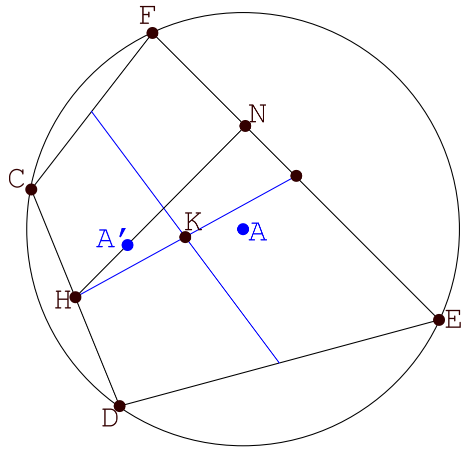


Figure 10. Segment through the anticenter.

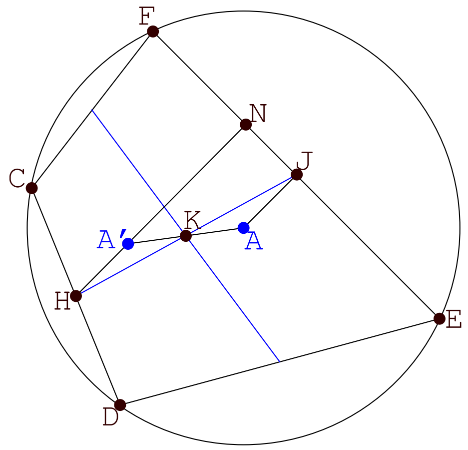


Figure 11. Congruent triangles.

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[2] Hohenwarter, M. (2002). *GeoGebra*. Available on-line at `http://www.geogebra.org/cms/en/`

Alfinio Flores teaches mathematics and mathematics methods courses *con ganas* at the University of Delaware. He uses technology to provide opportunities for students and teachers to explore mathematics on their own. He has conducted activities for students in schools ranging for Kindergarten to 12th grade, and conducted professional development sessions for teachers of grades K-16.