FOSTERING UNDERSTANDING OF MONTE CARLO SIMULATIONS FOR ESTIMATING π USING DYNAMIC GEOGEBRA APPLETS

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Abstract

Simulations of events, using Monte Carlo methods in Geogebra, provide students with opportunities to actively participate in statistical events. Simulations can be used to derive statistical information about underlying distributions. In the following paper, we explore dart throwing and needle drop simulations as vehicles for estimating π. Connecting abstract concepts to visually compelling media provides teachers with innovative approaches for teaching probability.

Keywords: statistics, estimation, geometry package, applet, simulation

1 INTRODUCTION

In the following paper, we explore the use of GeoGebra as a visualization and simulation tool to improve the teaching of undergraduate mathematics and engineering education. Chaturvedi and Akan [1] describe how simulations - particularly those that allow the visualization of otherwise abstract concepts - provide students with opportunities to clarify their understanding of content in the engineering sciences. Simulation provides similar opportunities in mathematics and statistics since both are generally regarded as abstract subjects. Crowe [2] provides an extensive survey of different mathematics applications in teaching and learning, with geometric visualisation one of the most powerful features of such software. Numerous articles have noted the power of technology to foster more positive attitudes of students towards mathematics [8].

The Monte Carlo simulation method [6] is a computerized mathematical technique based on repeated random sampling to uncover probability distributions. Monte Carlo methods provide users with methods to account for risk in quantitative analysis and decision making. The method was named after Monte Carlo, the Monaco resort town renowned for its casinos.

The geometry package GeoGebra [5] [3] [4] is a freely available tool that provides teachers and students with the ability to make mathematics and related fields more interactive. In particular, the software provides tools for conducting Monte Carlo simulations and enables users to visualize various influencing parameters of estimations.

In the pages that follow, we describe two activities that we've used successfully with students to motivate estimations of π - namely, an experiment with throwing darts and another involving needle drops. As the value of π is known, students can evaluate the quality of the obtained estimator. We performed the experiments as part of a basic statistics course in the second year of industrial engineering program. The activities provide students with examples of unbiased estimators. When we engage students in the simulations, we assume that they have a solid background with integration since such knowledge provides them with the understanding necessary to justify estimation with the Monte Carlo method.

2 Estimating π with the Monte Carlo method

2.1 Estimating π by throwing darts

Dart throwing provides a real-world context for considering Monte Carlo methods, where the position of the thrown dart on the target is arbitrary. We assume the random variable of the position within the chosen square target is uniformly distributed across the interior of the square with side R in which a disk with radius $R/2$ resides. Such a scenario is modeled in the GeoGebra applet shown in Figure 1.

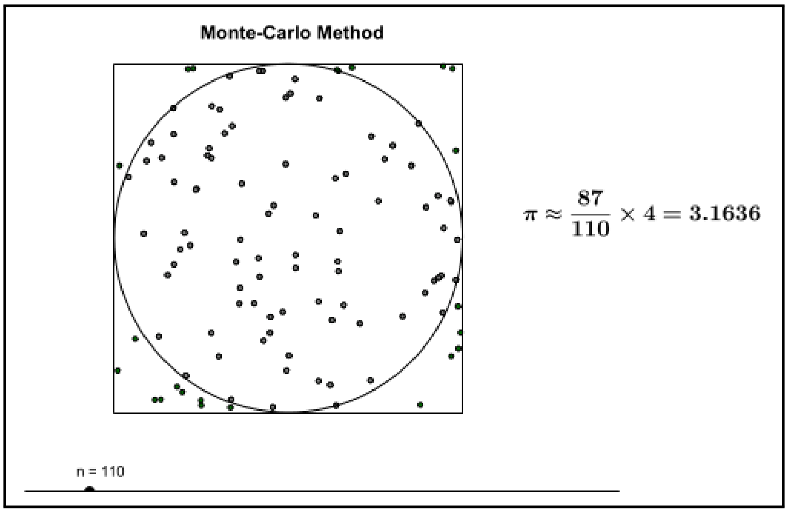

Figure 1. Dart throwing applet (available at `tube.geogebra.org/material/show/id/112416`).

The ratio of successful throws (i.e., those that hit the disk) out of total throws is denoted as $\hat{p}$. Note that this ratio may be thought of as the area of the disk divided by the area of the square as expressed in (1). From (1), it follows that $4\,\hat{p}$ is an estimation of π.

$$E[\hat{p}] = E\left[\frac{\#\text{hits}}{\#\text{throws}}\right] = \frac{\pi\, R^2/4}{R^2} \tag{1}$$

Figure 1, illustrates the estimation of π as an experiment in GeoGebra. Using the software, students can vary the value of n by manipulating the slider at the bottom of the sketch, providing them with an opportunity to discover the relationship between sample size and accuracy of estimates.

2.2 Estimating π by Buffon's needle

2.2.1 The experiment of dropping needles

Buffon's needle [9] is a well-known probability experiment involving dropping needles between a pair of parallel lines.

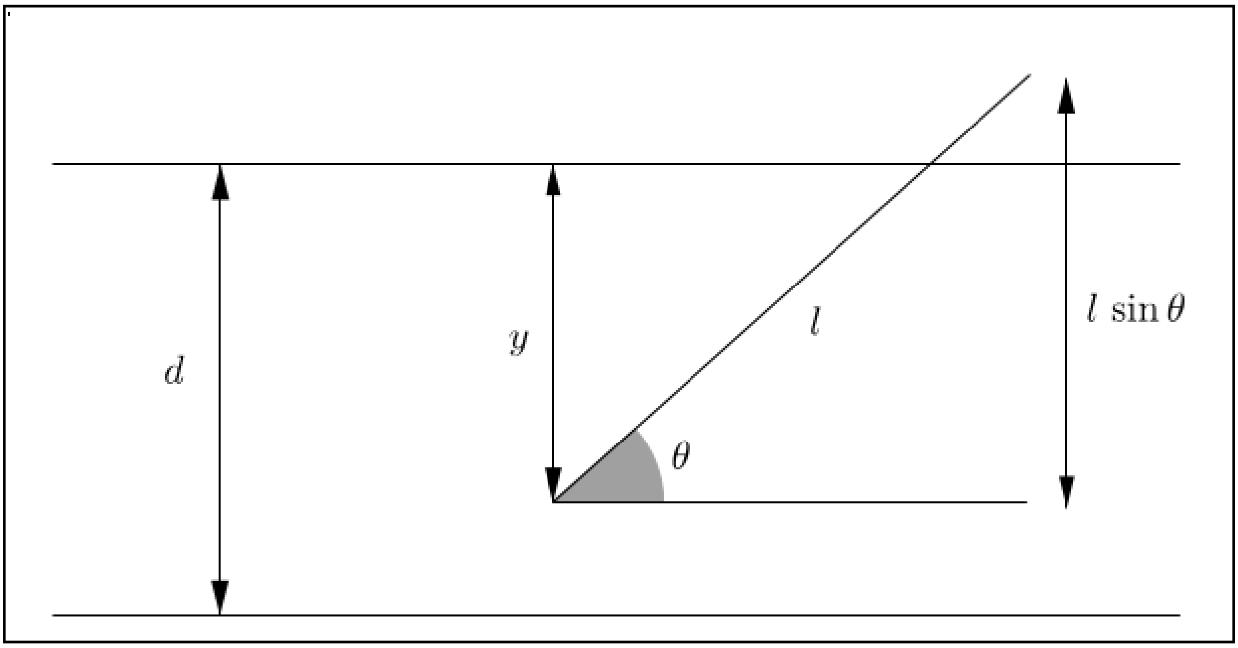

Figure 2. Mathematical symbols for the Buffon's needle experiment.

Figure 2 illustrates several parameters involved in the experiment: l is the length of the needle, d is the distance between the parallel lines, and $\theta\ (\in [0,\ \pi])$ is the amplitude of the angle of inclination of the needle.

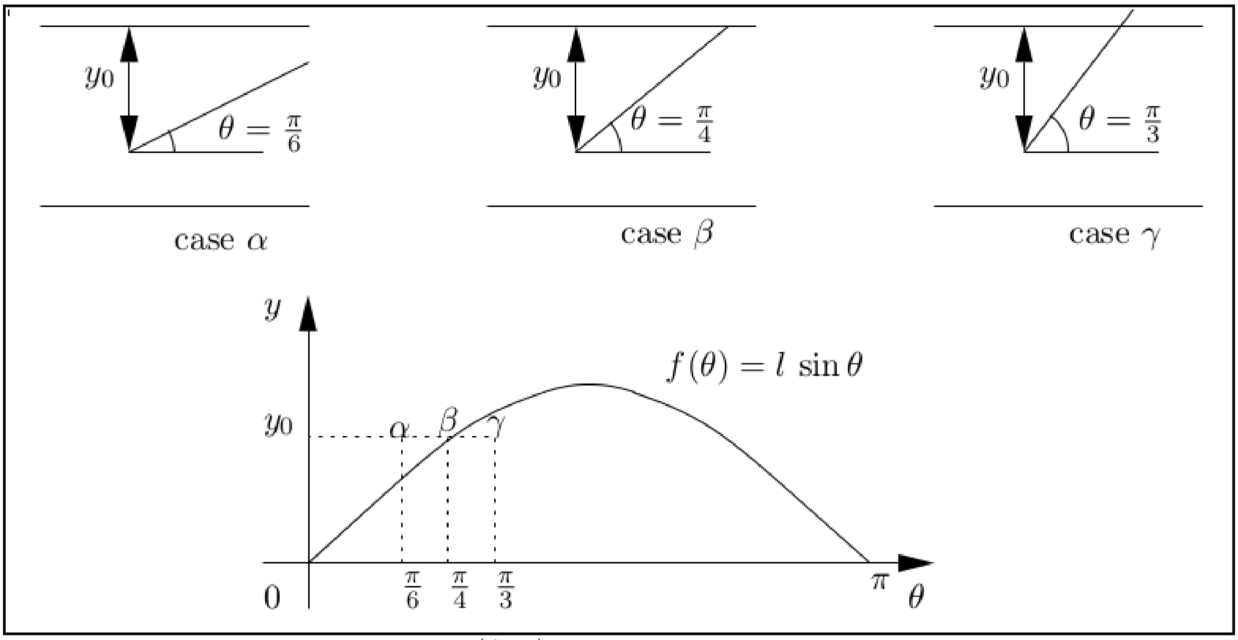

Figure 3. Position in the $(\theta,\ y)$-plane for the Buffon's needle experiment.

The position of the needle is determined by the ordered pair $(\theta,\ y)$ with $0 \leq \theta \leq \pi$ and $0 \leq y \leq d$. Note that the needle will cross a horizontal line if $(y \leq l\ \sin\theta)$, as shown in Figure 3 for the cases β and γ. When the ordered pair $(\theta,\ y)$ is plotted in the plane, the point $(\theta,\ y)$ falls on or below the curve $y = l\ \sin\theta$ when the needle crosses a horizontal line.

2.3 Mathematical derivation of the estimator for π with Buffon's needle

Mathai [7] explained the relation between π and the needle drop. Because the position of each dropped needle is random, we can define the probability p_C for the needle to cross a horizontal line as

$$p_C = \frac{\text{area A}}{\text{area B}},$$

with A the crossing region of the $(\theta,\, y)$-plane and B the "accessible" region, i.e., the rectangle in the $(\theta,\, y)$-plane with $0 \leq \theta \leq \pi$ and $0 \leq y \leq d$. These regions are depicted visually in Figure 4 for the two cases: $0 < l \leq d$ and $l \geq d$.

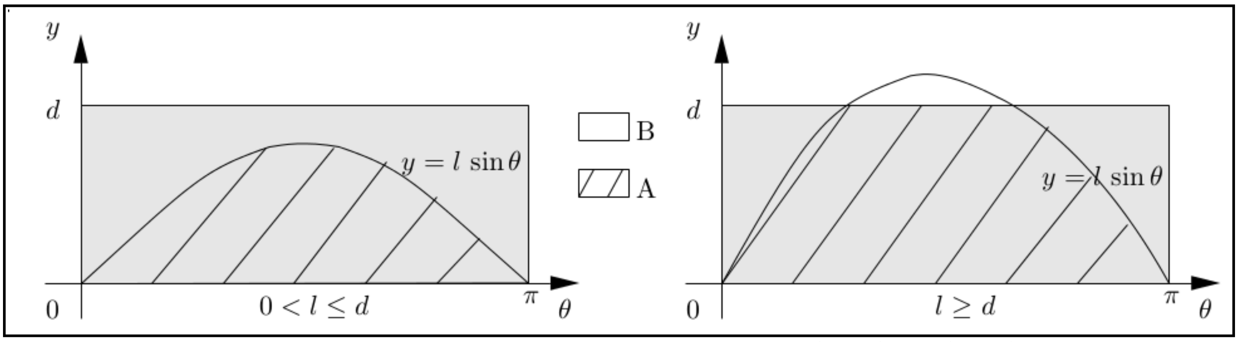

Figure 4. Regions A and B for the cases: $0 < l \leq d$ and $l \geq d$.

In case $0 < l \leq d$, the region A is situated in between the curve $y = l\,\sin\theta$, and the θ-axis for $0 \leq \theta \leq \pi$. As $\int_0^\pi \sin\theta\, d\theta = 2$, the first result in (3) is obtained. In case $d \leq l$, the region A has a supplementary border $y = d$. The second result of (3) is found by splitting the integral to calculate the area of A in three parts as in (2) based on the intersection points $(\alpha,\, d)$=(Arcsin $d/l,\, d$) and $(\pi - \alpha,\, d)$ of the curves $y = d$ and $y = l\,\sin\theta$ ($0 \leq \theta \leq \pi$).

$$\begin{aligned} d \leq l:\ \text{Area A} &= \int_0^\alpha l\,\sin\theta\, d\theta + (\pi - 2\,\alpha)\, d + \int_{\pi-\alpha}^{\pi} l\,\sin\theta \\ &= d\,(\pi - 2\,\alpha) + 2\,l\,(1 - \sqrt{1 - \frac{d^2}{l^2}}) \end{aligned} \tag{2}$$

$$p_C = \begin{cases} \frac{2\,l}{\pi\,d} & \text{if } 0 < l \leq d \\ \frac{1}{\pi\,d}\left(d\,(\pi - 2\,\text{Arcsin}\frac{d}{l}) + 2\,l\,(1 - \sqrt{1 - \frac{d^2}{l^2}})\right) & \text{if } d \leq l \end{cases} \tag{3}$$

In the preceding derivation, the experimental proportion $\hat{p}_C$ estimating the number of times the needle crosses a horizontal line out of the total number of fallen needles can be manipulated to estimate π, as illustrated in (4).

$$\pi \approx \begin{cases} \frac{2\,l}{\hat{p}_C\,d} & \text{if } 0 < l \leq d \\ \frac{2}{(\hat{p}_C - 1)\,d}\left(l - d\,\text{Arcsin}\frac{d}{l} - l\,\sqrt{1 - \frac{d^2}{l^2}}\right) & \text{if } d \leq l \end{cases} \tag{4}$$

2.3.1 Integration in the classroom with Geogebra applets

Buffon Needle Applet
In the classroom, we typically explore the Buffon Needle experiment first with pencil and paper. Students draw parallel lines with the distance between the lines, d, based on the length of their pencil. Next, they drop the pencil repeatedly, counting the number of times it crosses one of the parallel lines. Under the condition that the distance d is equal to the length l of the pencil, (3) shows that π can be estimated as $2/p_C$. The hands-on experiment, while time-consuming, provides students with tactile sense of the experiment. After a few trials with pencil and paper, we recommend transitioning students to the simulation since it is much faster and automatically records results for students. This enables students to focus their attention on conceptual understanding that underlies the experiment rather than on implementation details that may distract from the important mathematical ideas of the activity.

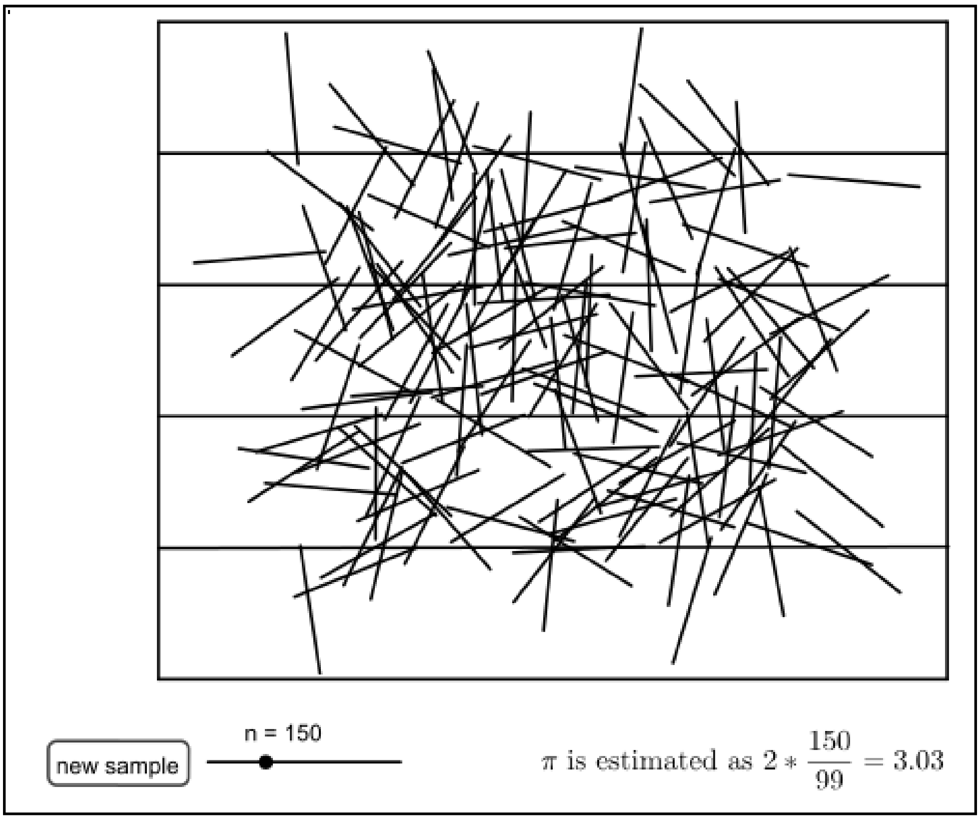

Figure 5. Buffon needle applet (available at tube.geogebra.org/material/show/id/112749)

Figure 5 gives a screenshot of this applet, illustrating the influence of the sample size n (evoked by sliding the dynamic bar for n in the applet) on the accuracy of the estimation of π when $d = l$ is fixed.

Dart Throw Applet
The influence of the length of the needle is an idea that is readily explored with the multi-representational capabilities of GeoGebra. As Figure 6 illustrates, GeoGebra can be used to plot a thrown needle along side a plot of the ordered pair $(\theta,\, y)$ in the plane. Students use this applet to generate conjectures regarding relationships between θ, y, and needle drops that cross parallel lines.

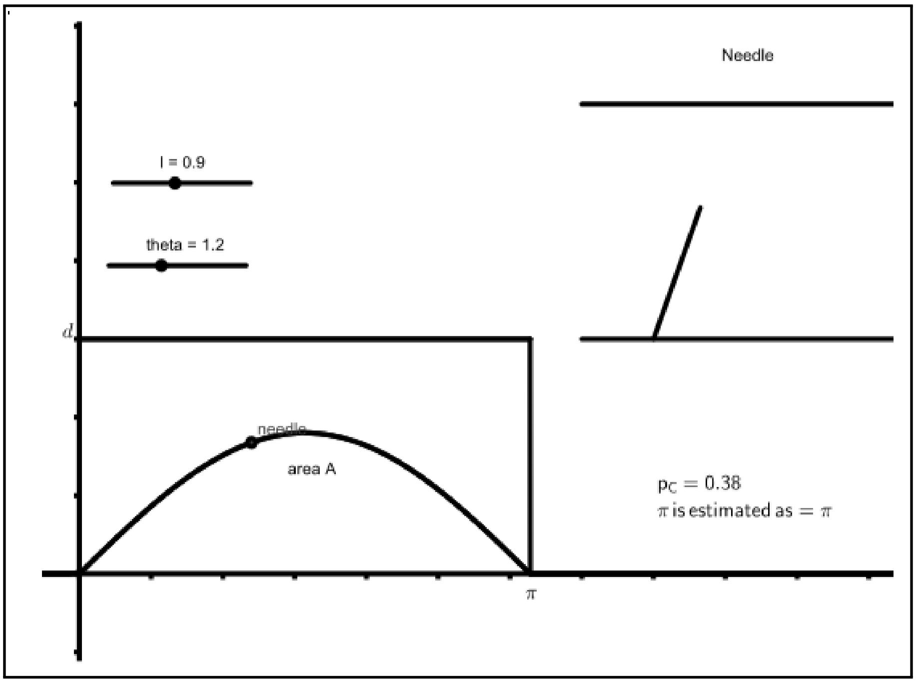

Figure 6. Buffon Needle applet (available at tube.geogebra.org/material/show/id/112419).

Observing the value of p_C when $l \neq d$, helps strengthen student understanding of the formula (3). Students expore the influence of the values of θ and l as they manipulate sliders that control their values in the sketch. By dragging the end of the needle within the applet, students informally confirm the theoretical estimate of π (4) by means of experimentation. If the needle crosses the line, the dot in the $(\theta,\, y)$ plane will fall in the region A (see Figure 4, case γ).

3 Conclusion

Meaningful study of mathematics and statistics involves more than knowing facts. A balanced curriculum provides students with opportunities to build conceptual understanding of various procedures, variables, and formulas. Such content only comes alive when students *experience* variables contained within the formulas they encounter. In this paper, we've shared how technology may be used to help students experience statistical estimations by means of Monte Carlo simulations. A feedback survey from a test group of engineering students ($N = 44$) suggests that student understanding of estimators and Monte Carlo simulations are improved through the use of dynamic GeoGebra simulations. The applets encourage students to ask questions and reflecting on the correctness of associations, valuable activities for any future scientist. Moreover, GeoGebra encouraged students to explain the mathematics illustrated within the sketches.

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