BUILDING DYNAMIC FRACTION BAR MODELS WITH GEOGEBRA

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Abstract

In this article, the author demonstrates how GeoGebra can be used to build fraction bar models for conceptualizing equivalent fractions as well as addition, subtraction, multiplication, and division of fractions. Each construction uses check boxes and input boxes to allow users to dynamically interact with the constructions and sequences of polygons to create the fraction bars.

Keywords: fractions, preservice elementary teachers, bar models

1 INTRODUCTION

Fractions are arguably the most challenging topic covered in elementary school mathematics. In fact, studies ([1, 2]) have shown that many elementary school teachers lack a deep conceptual understanding of fractions and their operations. To combat these difficulties, most current textbooks for preservice elementary school teachers (e.g., [3–5]) incorporate the use of some form of bar models or “fraction bars.” In these models, a rectangular region represents the whole unit, and fractions are represented by dividing the unit into equal parts. With such models, we can give visual representations to fractions and their operations. Consider for example, the following task. “James went to the market and bought a half gallon of milk. He used one-fourth of this in a recipe. How much milk was in the recipe?” This problem can be modeled using fraction bars in the manner shown in Figure 1. If one divides a unit rectangle in half and then divides each half into fourths, it becomes clear that one fourth of one half is one eighth of the original unit.

Figure 1. A fraction bar model showing $\frac{1}{4}$ x $\frac{1}{2}$

In the remainder of this article, I describe how GeoGebra can be used to create dynamic constructions in which the end user can quickly explore different examples of equivalent fractions and the four basic arithmetic operations.

2 EQUIVALENT FRACTIONS

A particular fraction, say $\frac{n}{d}$, can be shown with a bar model by slicing the rectangle representing the unit into d equally sized pieces and shading or coloring n of these pieces. An equivalent fraction, say $\frac{kn}{kd}$, can then be formed by subdividing each of the n pieces into k equally sized pieces. For example, to show that $\frac{2}{6}$ is equivalent to $\frac{1}{3}$, one could start by shading one of three equal parts from a fraction bar and follow this by slicing each of the three pieces into two equal parts. As shown in Figure 2, this leads to the same area being shaded out of the whole for each fraction.

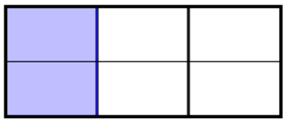

Figure 2. A fraction bar model showing $\frac{1}{3} = \frac{2}{6}$

I created a GeoGebra applet (shown in Figures 3 and 4) that allows students and teachers to interact with this concept. The example shown in Figure 3 demonstrates that $\frac{1}{3}$ is equivalent to $\frac{2}{6}$. In the next section, I share the steps I used to construct the applet. (Editors' note: A completed sketch is available at GeoGebraTube `http://www.geogebratube.org/student/m138997`).

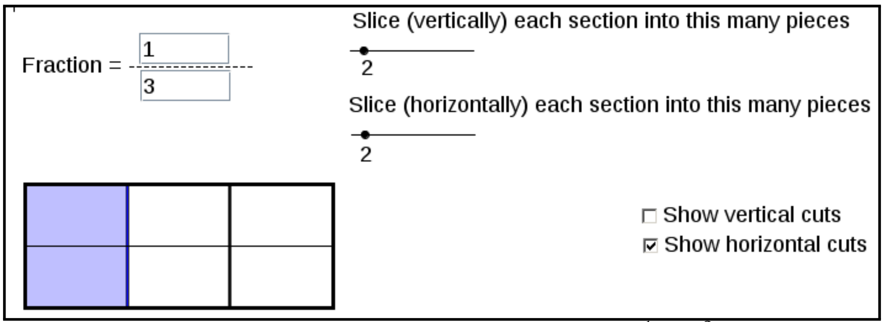

Figure 3. GeoGebra applet showing $\frac{1}{3} = \frac{2}{6}$

When users ignore the horizontal slices or deselect the checkbox, $\frac{1}{3}$ of the whole unit is shaded blue. If the check box is selected and users focus on the smaller rectangles, they see that the same area is $\frac{2}{6}$ of the whole. Similarly, Figure 4 shows that the improper fraction $\frac{6}{5}$ is equivalent to $\frac{12}{10}$, using the vertical slices. Since $\frac{6}{5}$ is more than one whole unit, the shaded region extends beyond the original unit rectangle.

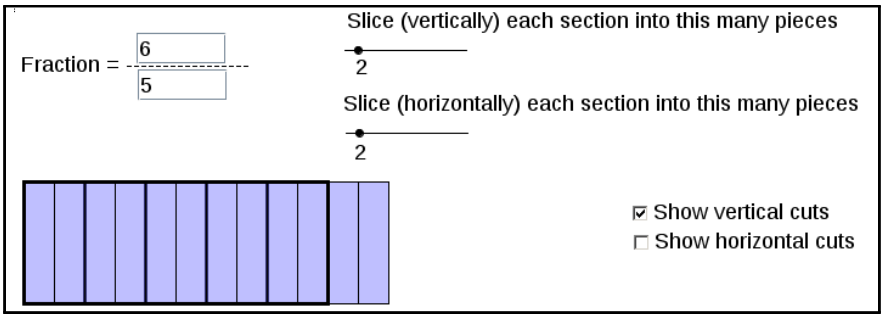

Figure 4. A screen capture of a fraction bar model demonstrating $\frac{6}{5} = \frac{12}{10}$

2.1 Instructions for Making the Equivalent Fractions Applet

Before constructing the applet, I recommend that you hide the axes in your sketch. To do this, right click within the **Geometry View** window and click on the word "axes." The directions in this article use particular colors and line thicknesses. Of course, you are welcome to change these properties to suit your personal tastes. The following is a step-by-step list of procedures to construct the applet.

1. **Build the unit.** Type `Polygon[(0,0),(5,0),(5,2),(0,2)]` into the GeoGebra **Input Bar**. This constructs a rectangle, as shown in Figure 5. The rectangle can have any positive dimensions, but the choice of coordinates will affect other parts of the construction.

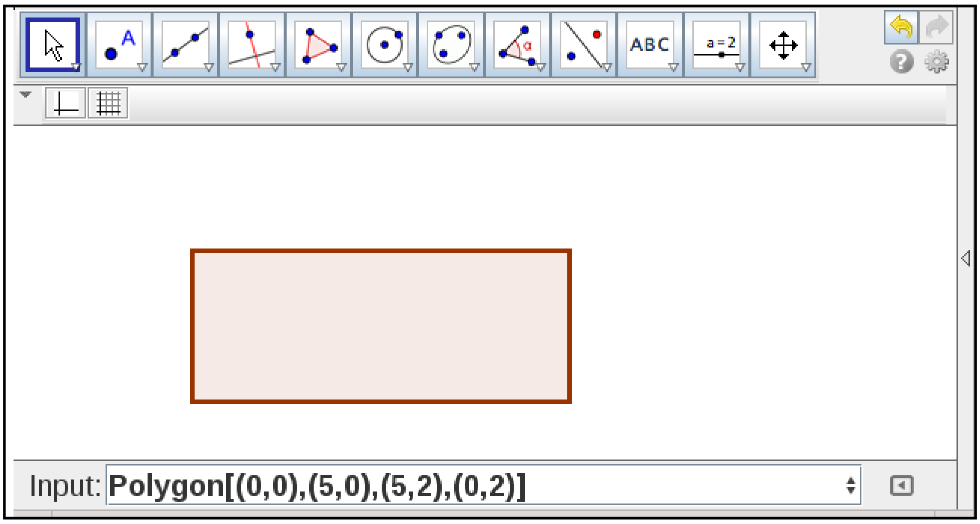

Figure 5. Unit rectangle generated from `Polygon[(0,0),(5,0),(5,2),(0,2)]`

2. **Change appearance.** Right click on the polygon and select **Object Properties**. Click on the **Style** tab and change the line thickness to 6. Next, click on the **Color** tab and set the color to black with opacity 0.

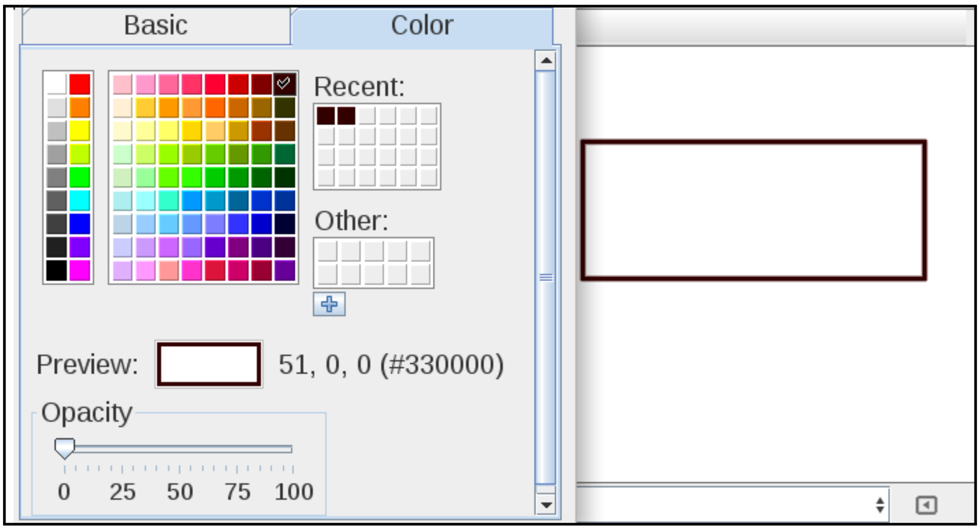

Figure 6. The unit rectangle with adjusted properties.

3. Use the **Move Graphics View** tool to place the rectangle in the desired location on the screen. That is, you will want the origin to be near the lower middle left side of the screen. Note that a user may have to zoom out as well if he or she enters a large improper fraction.

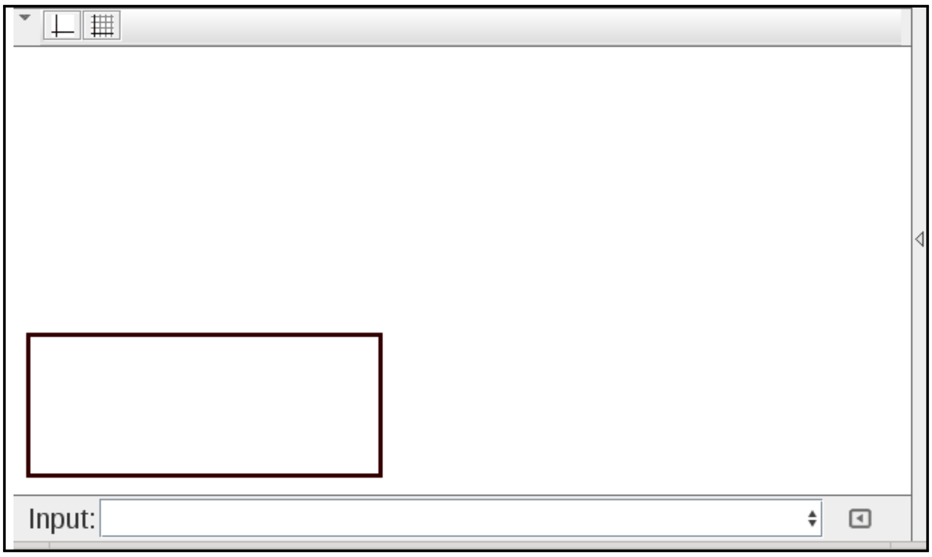

Figure 7. The unit rectangle relocated to the lower left corner

1. **Initialize the fraction.** The sketch needs an initial fraction to display when it is first opened by the user. The numerator and denominator need to be controlled separately, so type `N = 3` and `D = 5` into the GeoGebra **Input Bar** to define the numerator and denominator respectively for an initial fraction of $\frac{3}{5}$.

2. **Set up the input boxes.** In order for the user to easily change the values of the numerator and denominator, input boxes can to be created.

   (a) In the **Input Bar**, type "Fraction = ——————" in quotes to create the text string, and drag it to the upper left corner.

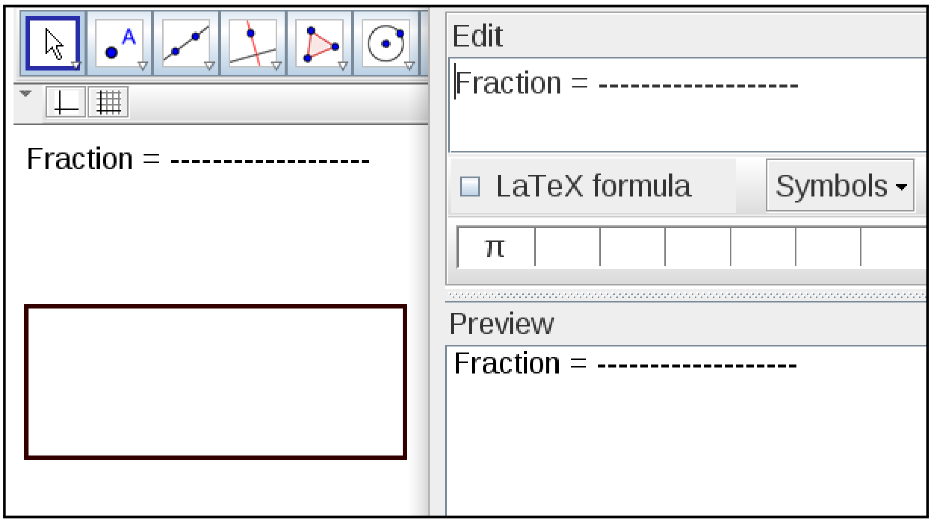

Figure 8. The sketch with the fraction label added

(b) Use the **Insert Input Box** tool to create an input box (see Appendix A for more details on this step) linked to N. Leave the caption blank. The software will label the box with the name `inputBox1`, which can be hidden. Once the label is hidden, the box cannot be easily moved. So first right click and drag the box into the position shown in Figure 9.

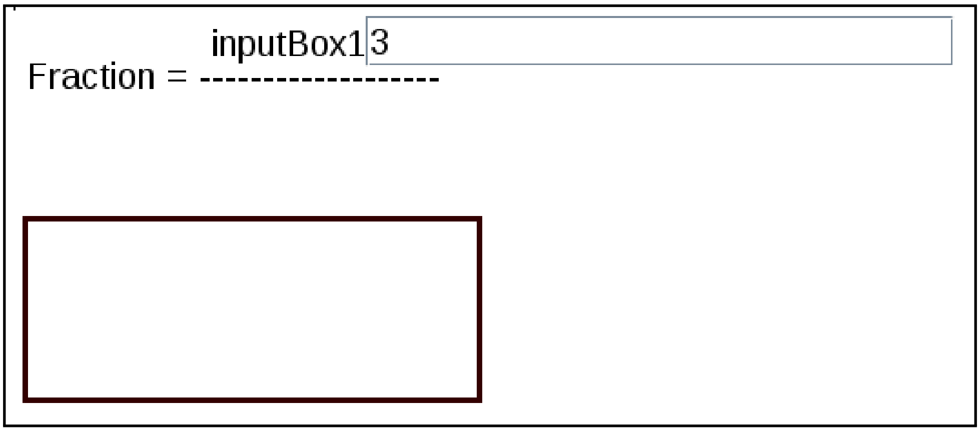

Figure 9. The input box linked to the numerator

(c) Right click again on the input box label and select **Object Properties** to edit the properties. Set the Input Box Length to 5 under the style tab, and deselect the **Show Label** check box under the basic tab.

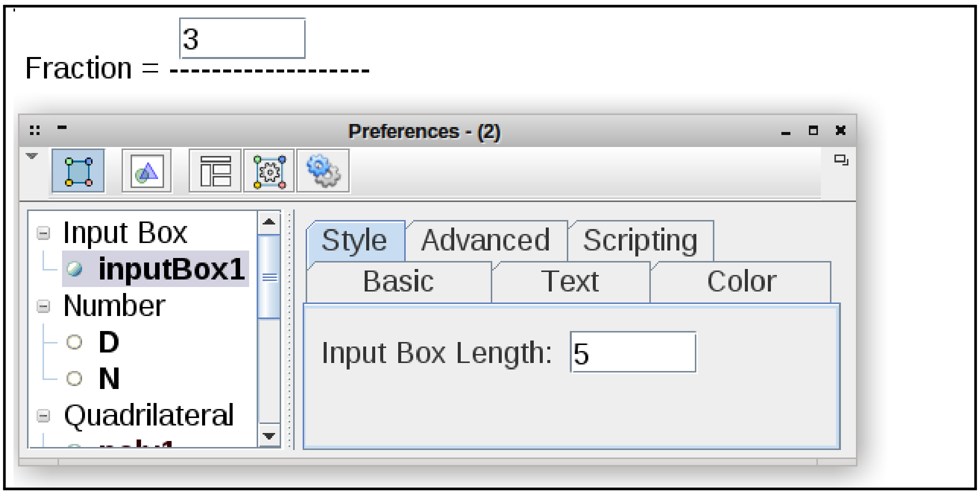

Figure 10. The input box after formatting

(d) Repeat the process from step (c) to make another input box linked to the variable D.

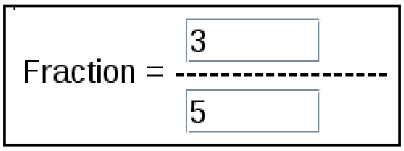

Figure 11. The completed fraction input boxes

3. **Construct a visual representation of the fraction.** Create the sequences of polygons to divide the unit and show the fraction.

   (a) Since the unit rectangle has been constructed with a length of 5 in GeoGebra's coordinates, each section when divided into the given fraction will have width $\frac{5}{D}$. In the **Input Bar**, type `S = 5/D`.

   (b) Next the equal sections determined by the numerator can be constructed as D polygons using the command
   `UnitCut = Sequence[Polygon[(i,-3),(i+S,-3),(i+S,-1),(i,-1)], i,0,5-S,S]`.
   This sequence command will plot D polygons, incrementing the variable i from 0 to 5 by steps of size S, cutting the unit into D equal rectangles of width S.

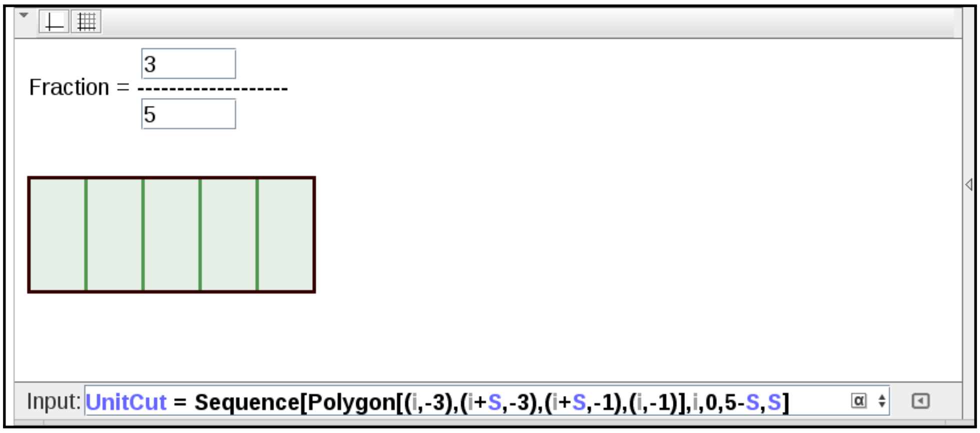

Figure 12. The unit fraction cut into equal pieces based on the denominator D

   (c) Right click on `UnitCut` in the algebra view to access the **Object Properties** and set the line thickness to 4. Then go to the color tab and set the color to black and the opacity to 0.

   (d) Next, N of the rectangles are shaded to show the numerator. Use the command `TheFrac = Sequence[Polygon[(i,0),(i+S,0),(i+S,2),(i,2)],i, 0,S(N-1),S]`. This creates a sequence of N rectangles of width S by incrementing i from 0 to $S(N-1)$ by step size S. Due to overlapping borders, these rectangles will not be clearly visible until the completion of the next step. Note the size of this sequence is controlled by N allowing improper fractions when $N > D$.

   (e) Right click on $TheFrac$ in the algebra view to access the **Object Properties**. Change the line thickness to 4. Set the color to your favorite choice. Set the opacity to a desired darkness. The author has used a dark blue and an opacity of 25%.

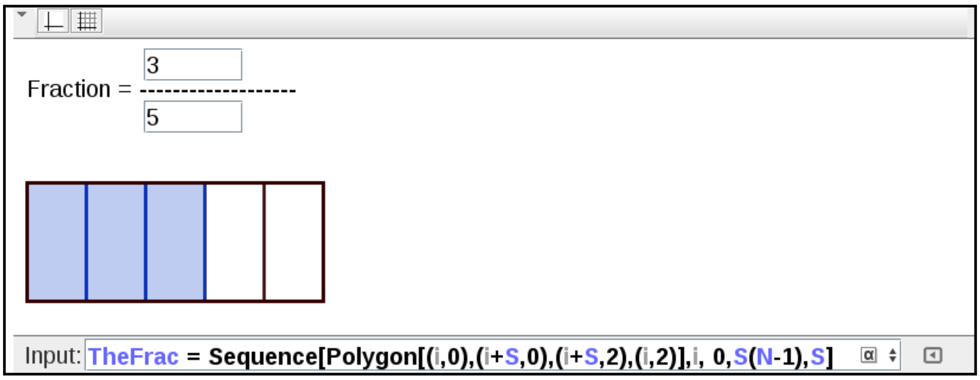

Figure 13. The sketch after adding and formatting the fractional pieces

3 ADDING HORIZONTAL AND VERTICAL PARTITIONS

In the following section, I discuss how to add vertical and horizontal partitions to our fraction bar. These steps allow students to visualize equivalent fractions within the sketch.

(a) **Insert a slider.** Use the slider tool to create a slider named v with a minimum of 1, maximum of 10, and an increment of 1. This will allow each piece of the original fraction to be divided into as many as 10 equal pieces each. Of course, any positive whole number could be used for the maximum.

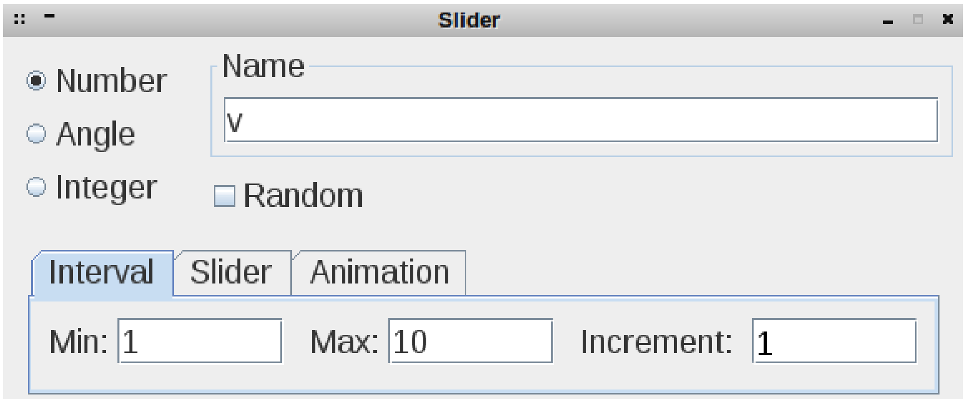

Figure 14. Adding a slider.

(b) **Insert text.** Use the mouse to drag the slider label to a position below the slider bar, and add the text "Slice vertically (into v pieces)."

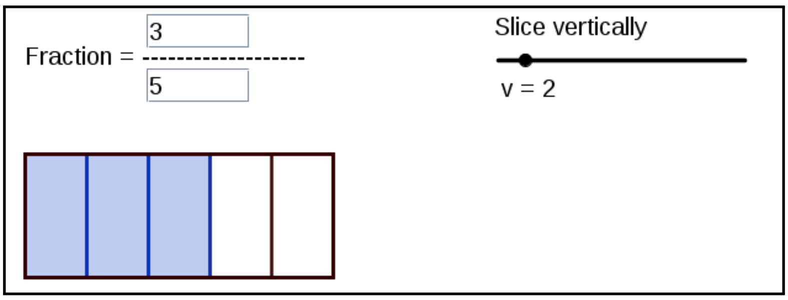

Figure 15. Additional text added to clarify the use of the slider.

(c) **Create vertical partitions.** A sequence of polygons can be used to create the vertical partitioning. Use the command: `VCut = Sequence[Polygon[(i,0),(i+S/v,0),(i+S/v,2),(i,2)],i,0,Max[S(N-1/v), 5-S/v],S/v]`.

This creates a sequence of rectangles of width $\frac{S}{v}$ to divide each of the rectangles of width S into v equal pieces. The `max` command is necessary to ensure that the entire unit rectangle is subdivided for proper fractions and that all N pieces are subdivided for improper fractions.

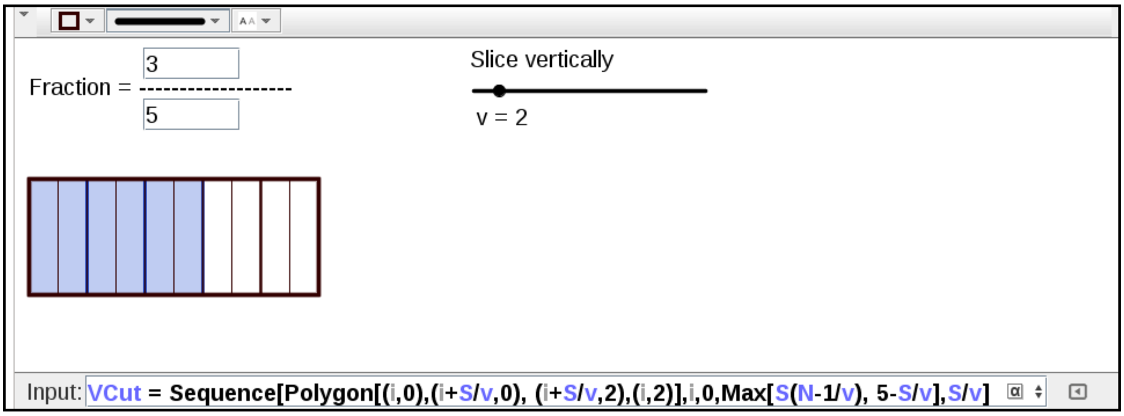

Figure 16. The sketch with the addition of the vertical partitioning

(d) **Change color.** Adjust the color to black and the opacity to 0 for the $VCut$ polygons to create a more aesthetically pleasing image.

(e) **Add check box.** In order to make it easy to switch between views of different equivalent fractions, add a check box to hide and unhide $VCut$ (see the Appendix A for more details on check boxes).

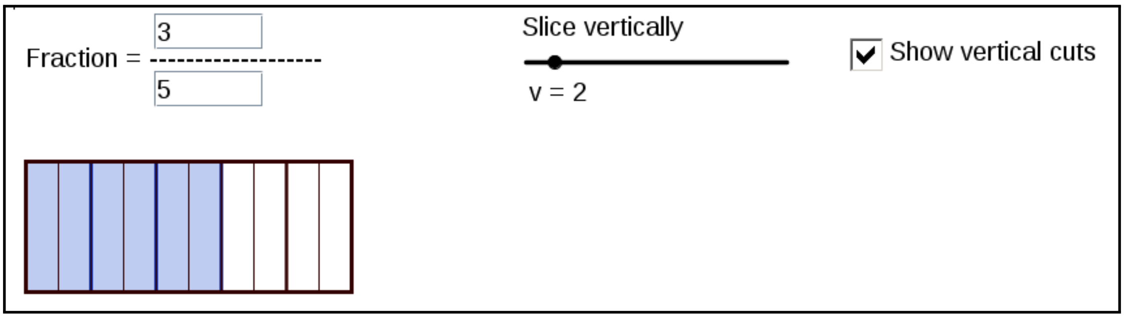

Figure 17. Sketch after the addition of a check box to hide and unhide the vertical partitioning

(f) **Add the ability to subdivide the fraction horizontally**. Similar to the vertical partitioning, the fraction bar can be partitioned horizontally to show equivalent fractions.

i. Use the slider tool to create a slider named h with a minimum of 1, maximum of 10, and an increment of 1.

ii. Add the text "Slice horizontally (into this many pieces)" to explain the slider to the users.

(g) Because the unit rectangle has a height of 2 as constructed in GeoGebra, it will be divided into $2/h$ pieces. Define this by typing `T = 2/h` in the **Input Bar**.

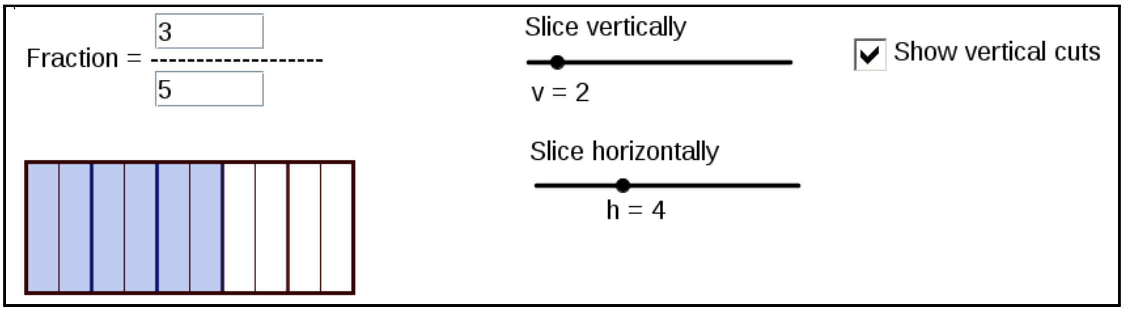

Figure 18. The sketch after adding the slider for horizontal partitioning

(h) Construct the T rectangles defining the horizontal partitioning with the commands `M=Max[5,S*N]` and `HCut = Sequence[Polygon[(0,i),(0,i+T),(M,i+T),(M,i)],i,0},2-T,T]`. This will create a sequence of T rectangles with width 5 and height T. The max command ensures that the horizontal segments run across the entire unit rectangle for proper fractions and across the entire fraction for improper fractions.

(i) Format the $HCut$ list of polygons to have the color black and opacity 0%.

(j) Finally, add a check box to allow the user to hide and unhide $HCut$, and close the algebra view as shown in Figure 19.

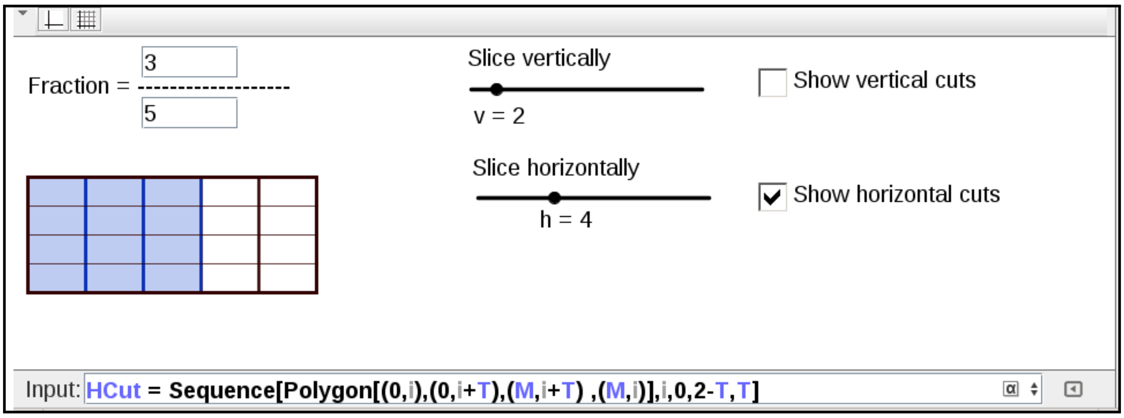

Figure 19. The sketch with horizontal partitioning

4 USING MULTIPLE BARS FOR ADDITION, SUBTRACTION, AND DIVISION

By using multiple fraction bars in the same window, we illustrate that addition and subtraction only make sense when the two fractions are converted to equivalent fractions with common denominators. For example, Figure 20 shows that $\frac{1}{4}$ and $\frac{2}{3}$ can be converted to equivalent fractions $\frac{3}{12}$ and $\frac{8}{12}$ respectively. Once the common denominator is established, we can do computations such as $\frac{1}{4}+\frac{2}{3}=\frac{11}{12}$. Interestingly, common denominators also provide a powerful conceptual model for division of fractions. I'll provide an example of division after presenting steps for constructing such sketches.

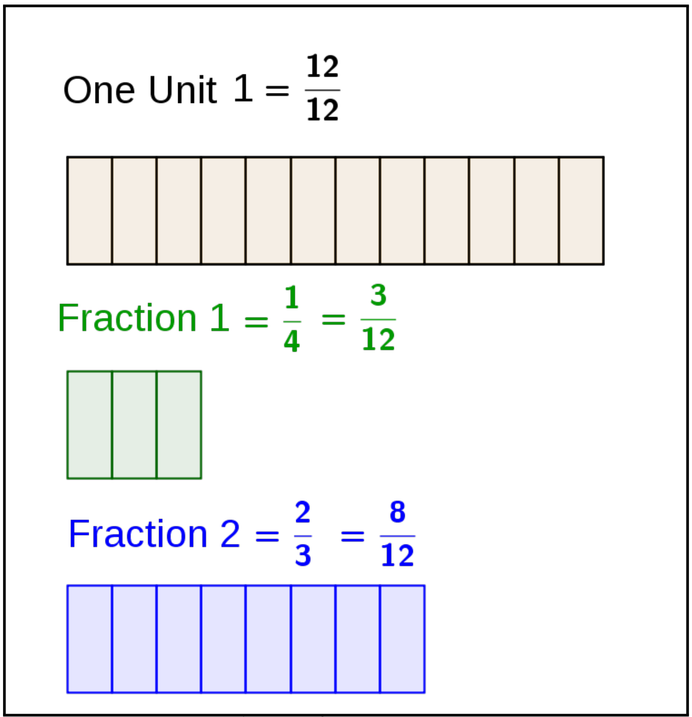

Figure 20. Representing $\frac{1}{4}$ and $\frac{2}{3}$ with a common denominator of 12

4.1 Instructions for Making the Construction with Two Fraction Bars and a Common Denominator Feature

The only modification that we need to make to the previous construction is to add an additional fraction bar that we can subdivide, but a few extra features will allow users to easily convert to common denominators and make comparisons with the unit.

1. Make the unit rectangle using the command `Polygon[(0,0),(5,0),(5,1),(0,1)]` to make a rectangle with width 5 and height 1.

2. Change the color of this polygon to the color of your choice.

3. Use the **Move Graphics Tool** to reposition the origin as shown in Figure 21. The axes are shown for clarity, but these should be hidden after this step. The label "One Unit" has been added.

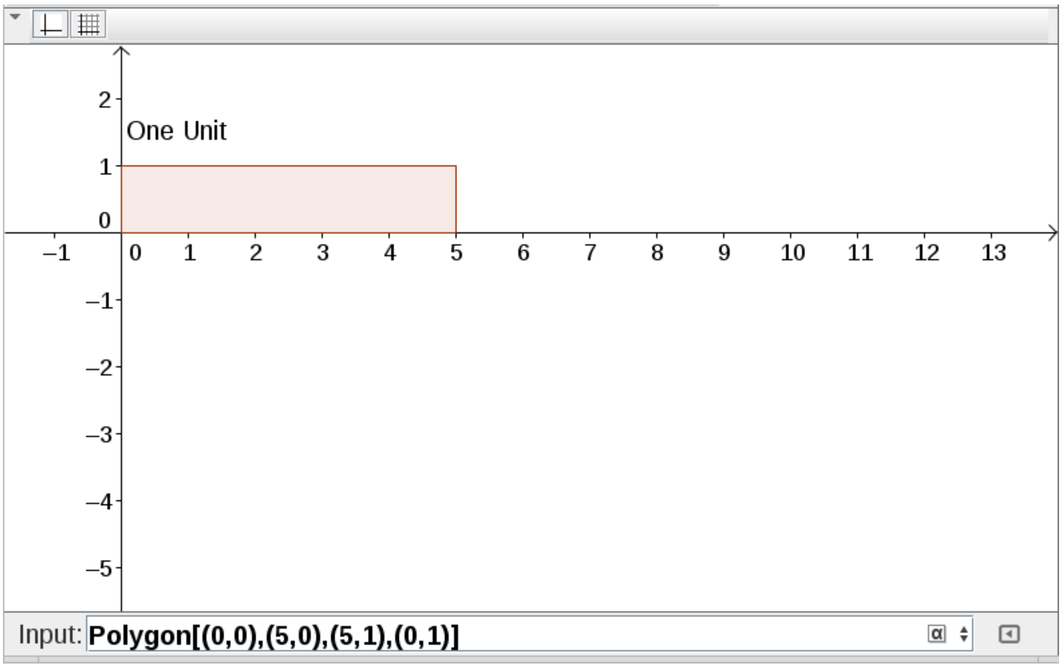

Figure 21. Sketch with the unit fraction bar

4. **Create the fraction bars.** We've constructed the numerator of fractions 1 and 2 as **Input Boxes** linked to variables $n1$ and $n2$, respectively. Similarly, denominators are constructed as **Input Boxes** linked to variables $d1$ and $d2$. This is suggested in Figure 22.

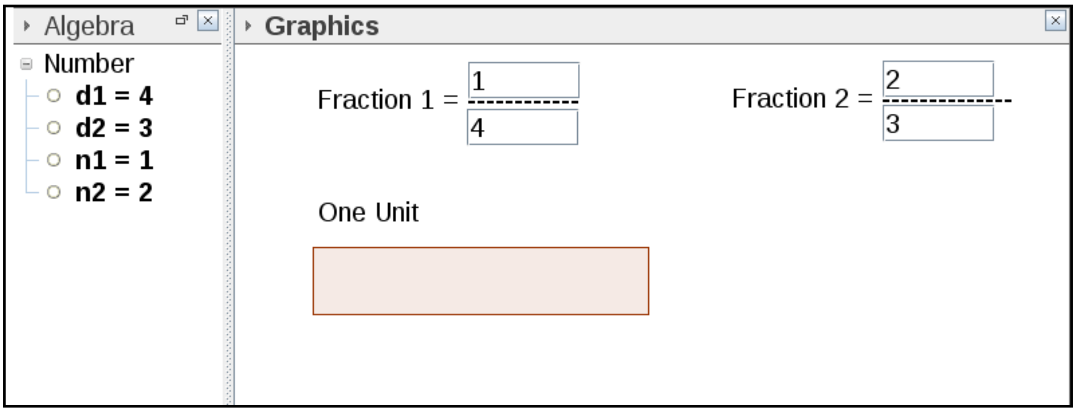

Figure 22. The sketch after the addition of input boxes for the fractions

5. The bar for Fraction 1 will be divided based on its denominator using width `w1 = 5/d1`. The bar for Fraction 2 will use width `w2 = 5/d2`. Each width is entered in the **Input Bar**.

6. Next, divide the two fraction bars using sequences of polygons:

```
Frac1 = Sequence[Polygon[(i,-2),(i+w1,-2),(i+w1,-1),(i,-1)],i
,0,5*n1/d1-w1,w1]
Frac2 = Sequence[Polygon[(i,-4),(i+w2,-4),(i+w2,-3),(i,-3)],i
,0,5*n2/d2-w2,w2]
```

These sequences construct a connected set of rectangles with the number of pieces equal to the numerator and the widths determined by the denominator as shown in Figure 23.

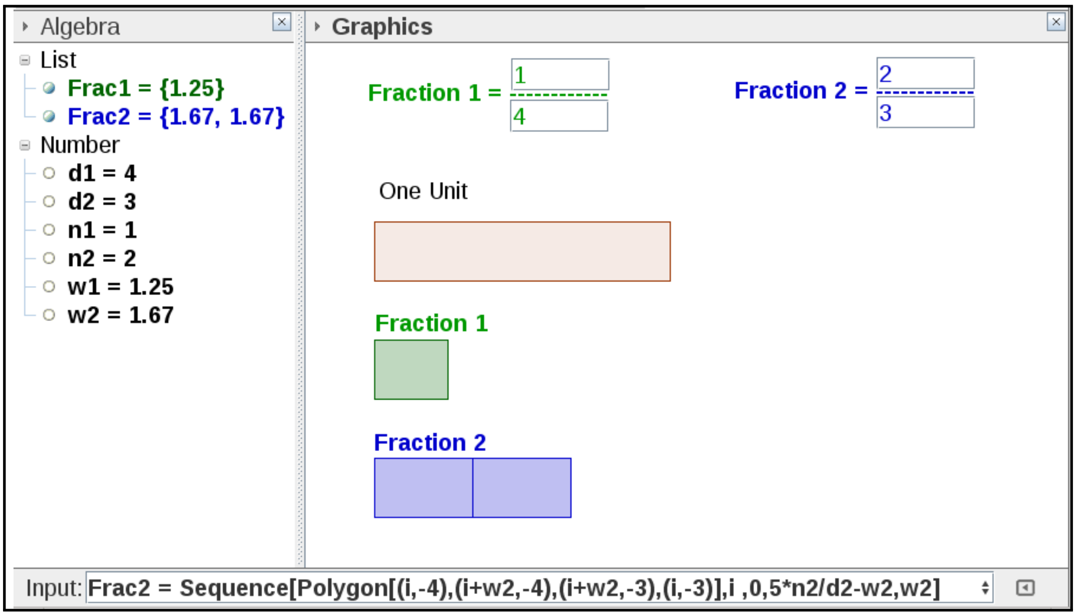

Figure 23. The sketch with the fraction bars added

At this point, we have the unit fraction bar and two different fractions that we can alter with the input boxes. Next we can create scale factors to subdivide the two fraction bars and a feature to subdivide each fraction bar using the common denominator.

7. Initialize the scale factors and common denominator.
   (a) Type `s1 = 1` and `s2 = 1` in the **Input Bar** to initialize scale factors for each fraction.
   (b) For a common denominator, enter the least common multiple command with the two denominators in the **Input Bar**: `cd = LCM[d1,d2]`.
8. Create input boxes for $s1$ and $s2$ with the captions "scale factor 1" and "scale factor 2."

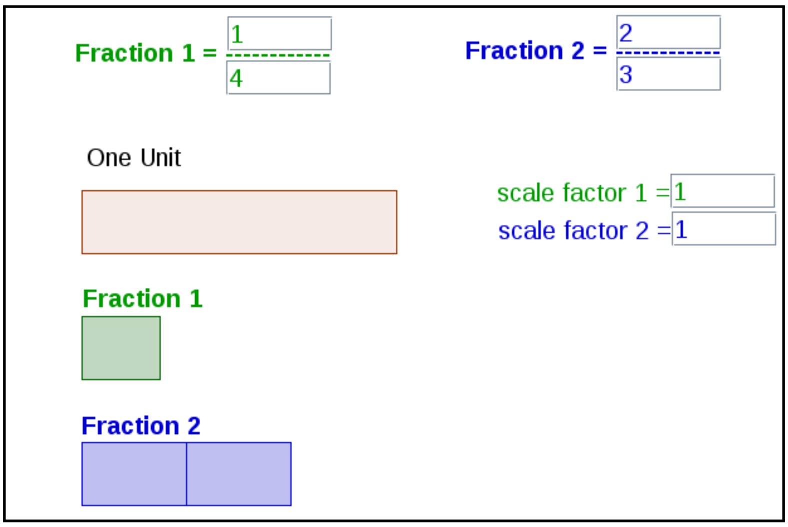

Figure 24. Input boxes added for scale factors

9. Create a check box to select whether or not to use a common denominator. Note that we only need the check box to represent a yes or no decision, so it does not need to be linked to anything at this point. If you have followed the order in the instructions, you should see a Boolean value `e=true` in the **Algebra View**.

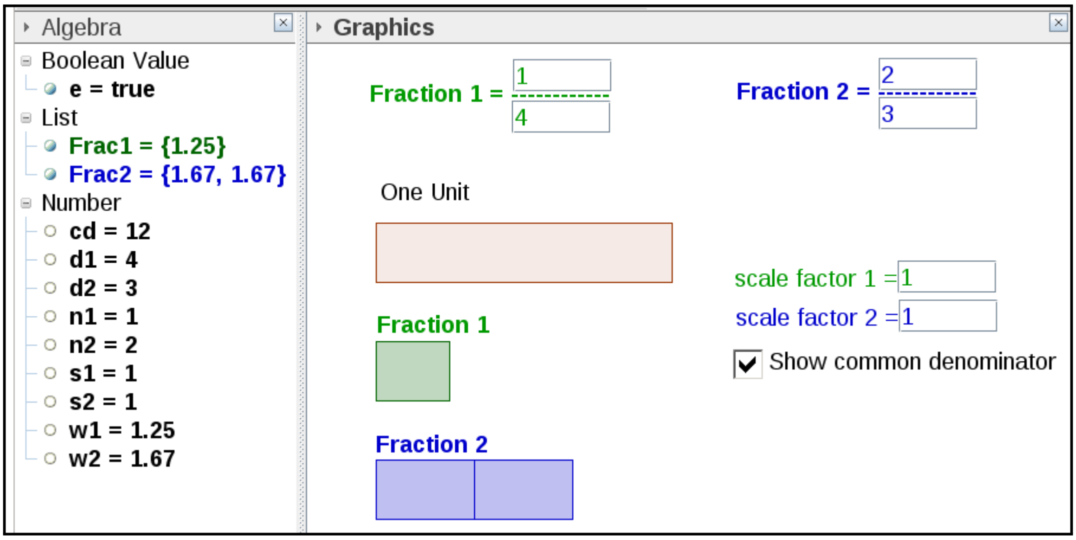

Figure 25. A Boolean check box added to control use of a common denominator

10. Create scale factors for use with the common denominator when it is selected.

    (a) The sketch needs to use scale factors based on the common denominator whenever a user clicks on the checkbox. When the checkbox is not clicked, individual scale factors $s1$ and $s2$ are used. To do this, we define new scale factors in the **Input Bar** with the following commands: `ns1=If[e==0,s1,cd/d1]` and `ns2=If[e==0,s2,cd/d2]`.

    (b) After defining `ns1` and `ns2`, click on $w1$ in the **Algebra View** window and change the definition to `5/(d1*ns1)`. This will divide each section of the Fraction 1 bar into $ss1$ pieces. Similarly, redefine $w2$ as `5/(d2*ns2)`. The redefinition is entered in a dialog box similar to the one depicted in Figure 26.

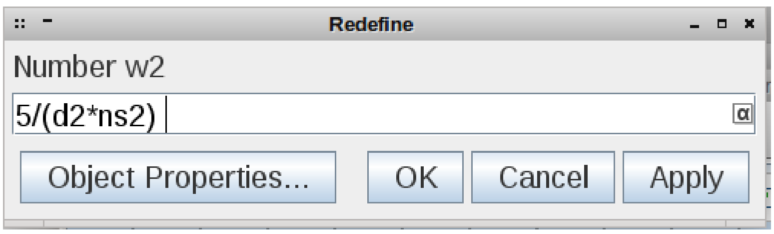

Figure 26. Redefinition of number $w2$

Figure 27 illustrates our sketch after these changes have been made.

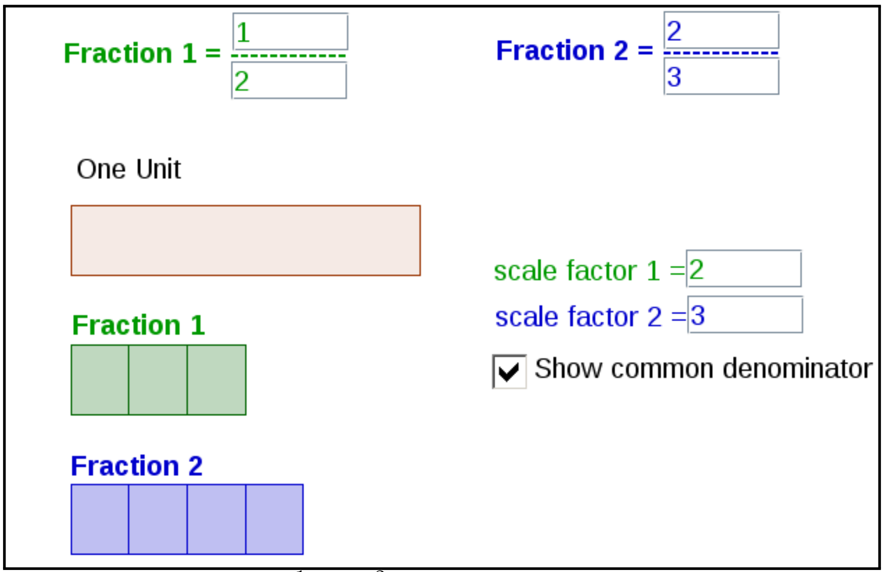

Figure 27. The graphics view of $\frac{1}{2}$ and $\frac{2}{3}$ with common denominator visible (i.e., $\frac{3}{6}$ and $\frac{4}{6}$)

That completes the essential components of the construction, although further work improves the aesthetics of the sketch. For instance, if you wish to subdivide the unit fraction bar using the common denominator, use the following command:

```
Sequence[Polygon[(i,0),(i+5/cd,0),(i+5/cd,1),(i,1)],i,0,5-5/cd,5/cd]
```

Next in the object properties for this list, go to the advanced tab, type `e` in the "Condition to Show Object" box. You may also consider adding text to the sketch to label the fractions or clarify other parts of the sketch. The reader is encouraged to explore these and other features.

4.2 A Division Example

Fraction bars can be used with the Measurement Model of division to conceptualize the division of fractions. The measurement approach to division models the problem $\frac{a}{b}$ by answering the question: How many copies of b are in 1 copy of a? For example, $\frac{12}{2} = 6$ because there are 6 sets of 2 in 1 set of 12. This same model can be used with fractions as long as common denominators are used.

For example, consider a contextual problem. A recipe for a cake calls for $\frac{1}{8}$ gallons of milk, and Sherry has $\frac{1}{2}$ gallons of milk. How many of these cakes can Sherry make, assuming that she is not limited by any other ingredients? The solution to this problem would be the answer to $\frac{1}{2}$ divided by $\frac{1}{8}$. This can be easily answered with the fraction bars as seen in Figure 28. Using a common denominator of 8, it can be clearly seen that there are 4 copies of $\frac{1}{8}$ in $\frac{1}{2} = \frac{4}{8}$.

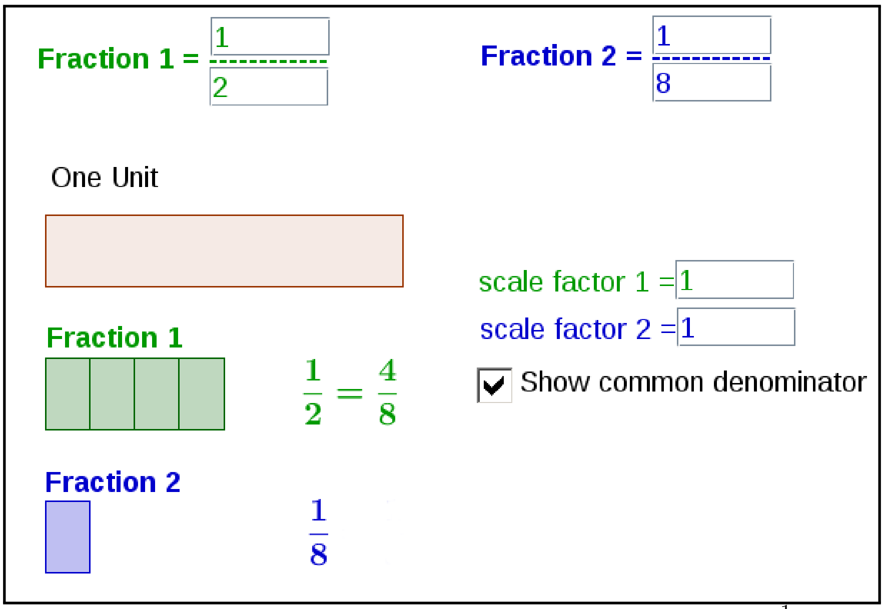

Figure 28. Using the fraction bars to see how many copies of $\frac{1}{8}$ are in $\frac{1}{2}$

This type of problem becomes more complicated, but still manageable, when the there is a remainder. For example, consider a different recipe problem. A cookie recipe calls for $\frac{1}{2}$ a teaspoon of vanilla to make a batch of cookies. Mario has $\frac{3}{4}$ a teaspoon of vanilla. How many batches of cookies can Mario make, assuming that he has all of the other ingredients? The solution to this problem is the answer to $\frac{3}{4}$ divided by $\frac{1}{2}$. Figure 29 shows fraction bars for these two numbers with a common denominator of 4. With the common denominator $\frac{1}{2}$ is two rectangles and $\frac{3}{4}$ is three of the same size rectangles. The solution would be the number of groups of 2 rectangles contained in the three rectangles. This is 1 and $\frac{1}{2}$. That is 3 rectangles contains 1 group of 2 with 1 left over. The remaining 1 is half of another group of 2. So Mario could make 1 and $\frac{1}{2}$ batches of cookies. Another way to see the same answer is to realize that each smaller rectangle in Figure 29 is $\frac{1}{2}$ of Fraction 2, and there are 3 of these in Fraction 1. So Fraction 1 divided by Fraction 2 is $\frac{3}{2}$.

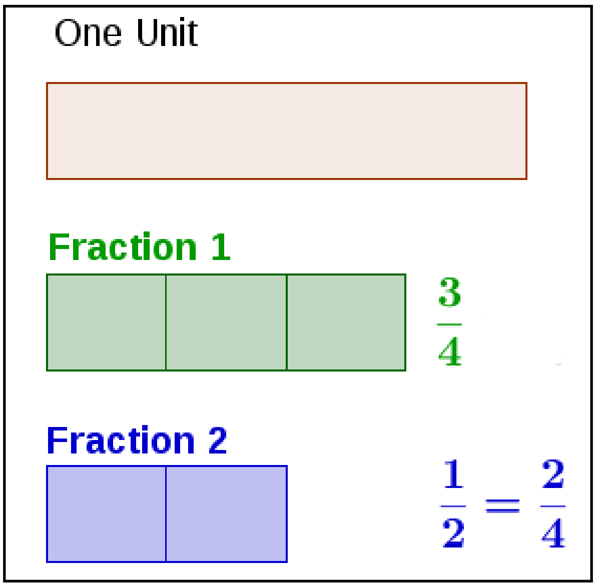

Figure 29. Fraction bars showing a common denominator for $\frac{3}{4}$ and $\frac{1}{2}$

5 MULTIPLICATION

A common fraction bar model for multiplication starts with a unit rectangular bar with an area of 1 square unit and constructs a rectangle with dimensions corresponding to the fractions. The subdivided rectangle can be used to visualize the product. For example, consider the following problem, which is modeled in Figure 30. On Saturday, Ralph ate $\frac{2}{5}$ of a candy bar. On Sunday, he ate $\frac{1}{3}$ of what was left. What fraction of the original candy bar did Ralph eat on Sunday?
In Figure 30, the unit is divided into thirds and fifths with the overlap of $\frac{2}{5}$ and $\frac{1}{3}$ shaded. We can see that the number of shaded boxes is the product of the numerators, and the total unit has been divided into 15 pieces, which is the product of the denominators. So Ralph ate $\frac{2}{15}$ of the whole candy bar on Sunday.

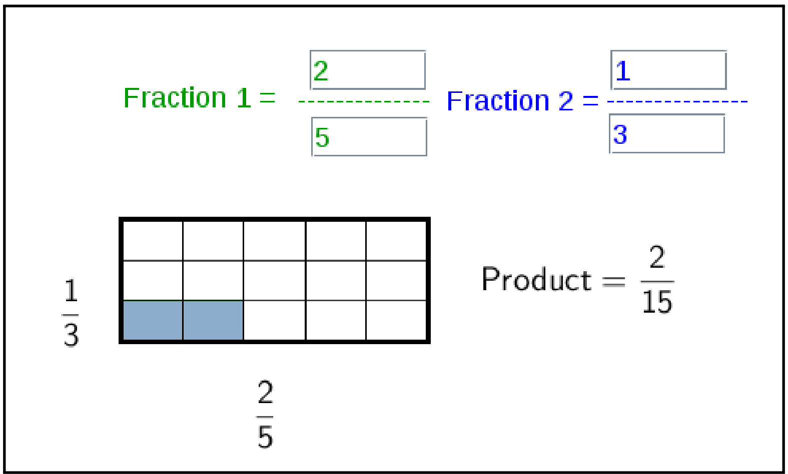

Figure 30. A multiplication model for $\frac{1}{3}$ x $\frac{2}{5}$

Figure 31 shows that improper fractions can be used as well. This figure could help a user figure out how much of a whole candy bar Ralph ate if he started with $\frac{4}{3}$ candy bars and ate $\frac{2}{5}$ of that amount.

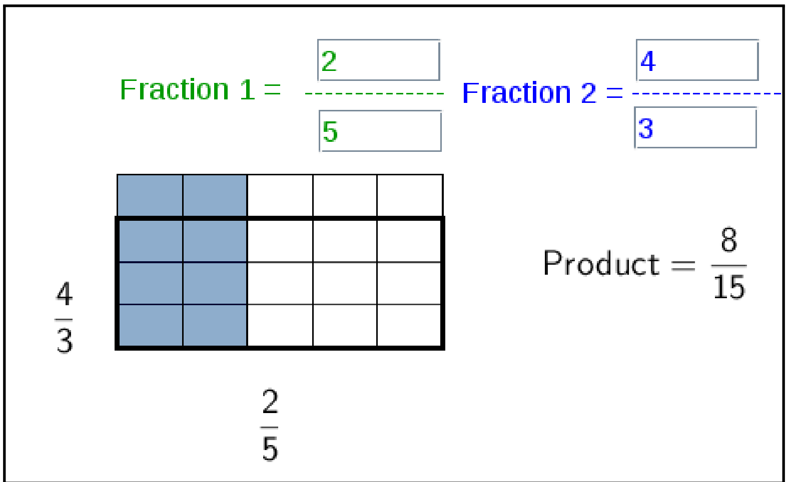

Figure 31. A multiplication model for $\frac{4}{3}$ x $\frac{2}{5}$

Step-by-step instructions for constructing a multiplication sketch in GeoGebra can be found in Appendix B. Because the construction uses techniques similar to those discussed in previous sketches, these construction steps are provided without comments or screenshots. A completed version of the sketch can be found at GeoGebra Tube `http://www.geogebratube.org/student/m140805`.

6 CONCLUSION

In order to help teachers and students better understand fractions and their operations, mathematics educators have recommended the use of fraction bars to provide visualizations. With the use of GeoGebra, we can create dynamic constructions in which users can quickly and easily create images of fraction bars. In this article, the author has presented examples and instructions for making such demonstrations in GeoGebra for showing equivalent fractions and the four basic operations. In addition to the explicit use of GeoGebra to explore fractions, the author hopes these constructions will help others to see the powerful potential that GeoGebra has for building virtual manipulatives.

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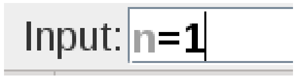
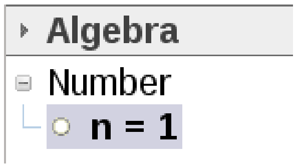
APPENDIX A: SOME GEOGEBRA INPUT TOOLS

6.1 Input Boxes

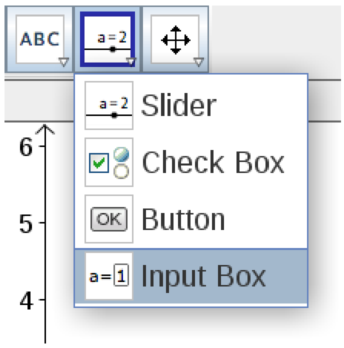
Input boxes provide users with a quick way to change the values of variables. Before creating an input box, one must first define a variable. Suppose that we want to have a variable, n, controlled by an input box. First use the command $n = 1$ (or any other initial value) in the GeoGebra **Input Bar**.



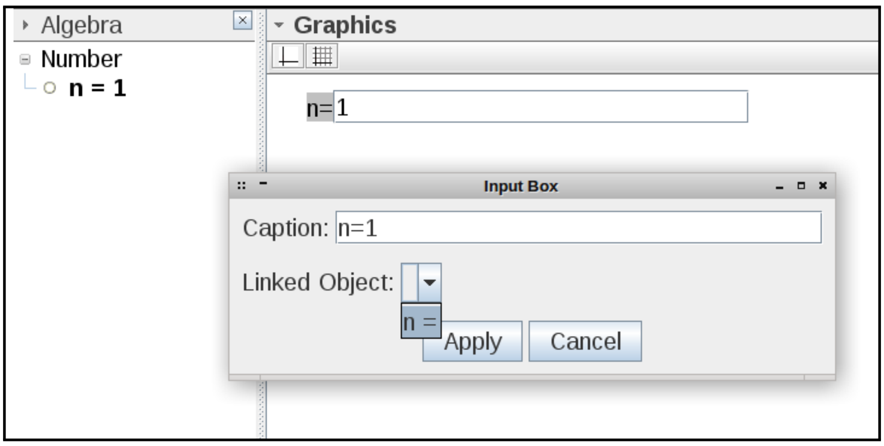
This will initialize the variable and add it to the algebra view.



Next select the **Input Box** tool from the drop down menu and left click in the graphics view near the desired location of the input box.



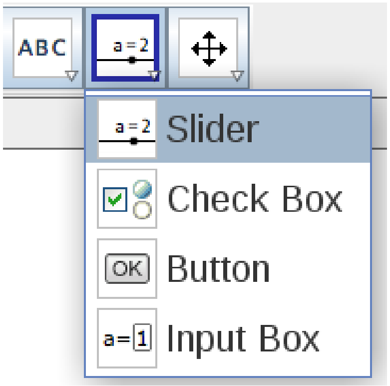
A dialog box will appear for completing the input box. Enter an appropriate caption such as "n=?" and select the variable from the drop down menu to link it to the input box. Then click "apply" to complete the input box.



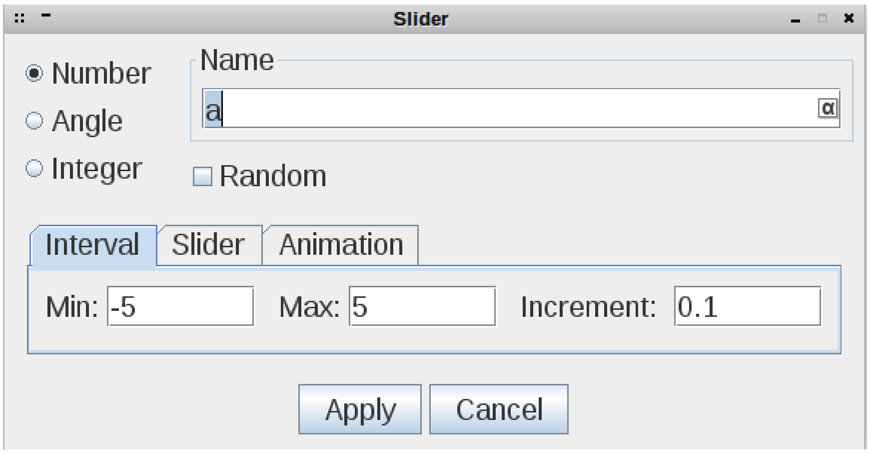
To reposition an input box, use the move tool and click on the caption and drag the input box. To change attributes such as the length of the box, right click on the caption and select **Object Properties**.

6.2 Sliders

Similar to input boxes, sliders allow GeoGebra users to quickly change the value of a variable. Instead of allowing the user to input any value, a slider allows the user to select from a predetermined set of incremental values. For example, suppose that we want a slider that takes on whole number values from 1 to 10. Start by selecting the slider tool and left clicking in the graphics view near the desired location.



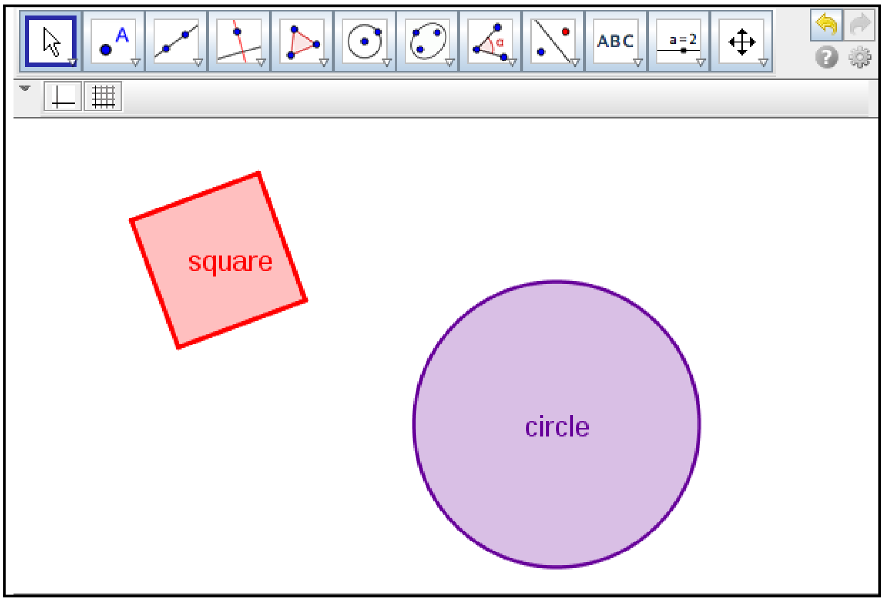
A dialog box will appear for the slider construction.



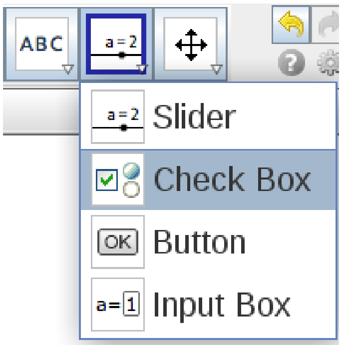
Enter the desired name for the slider; set the minimum, maximum, and increment values for the slider; and click "apply". The minimum and maximum will be the smallest and largest value of the slider, respectively. The increment is the step size for the slider. So this slider will go from -5 to 5 counting by 0.1 each step.

6.3 Check Boxes

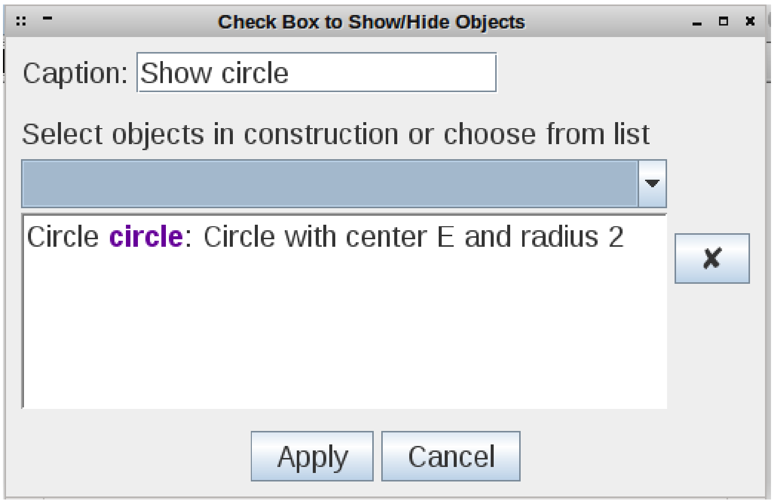
A check box gives a Geogebra user a simple way to hide and unhide objects. Any number of objects can be linked to a single check box. Suppose for example, that we've constructed a sketch containing a square and circle. We'd like to create a check box for the circle to hide and show the object.



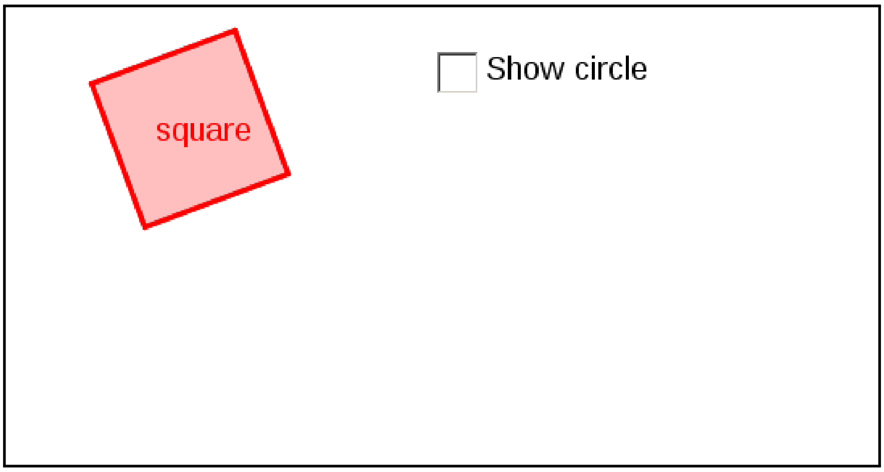
Select the **Check Box to Show/ Hide Objects** tool from the drop down menu and click near the desired location in the graphics view.



A dialog box will open. Enter an appropriate caption such as **Show Circle**, and click the objects you want to show / hide when the checkbox is clicked (in our case, the circle). Note that objects to link can be selected from the drop down menu in the dialog box or by clicking on an object in the graphics or algebra view.



Click "apply" to complete the construction. Note that when the checkbox is "unchecked," the circle disappears.



APPENDIX B: INSTRUCTIONS TO MAKE A MULTIPLICATION MODEL

1. Build the unit fraction model.
   (a) Using the command: `Polygon[(0,-3),(5,-3),(5,-1),(0,-1)]`
   (b) Set the color to black, the opacity to 0, and the line thickness to 6
2. Initialize the fractions.
   (a) Use the commands: `n1 = 1`, `n2 =2`, `d1 = 3`, and `d2 = 5`
3. Use text and input boxes to allow users to easily change the four values in step 2A.
4. Compute the products numerator and denominator.
   (a) Use the command: `n3 = n1*n2`
   (b) Use the command: `d3 = d1*d2`
5. Use text strings to display the product. Those familiar with LaTex can use that to display fractions.
6. Set up the step sizes for the products visualization.
   (a) Use the commands: `step1 = 5/d1` and `step2 = 2/d2`
7. Create the visualization of the product.
   (a) Use the following commands:
   - `L1=Sequence[Polygon[(0,i),(5n1/d1,i), (5n1/d1,i+step2) ,(0,i+step2)],i,-3,-3+step2(n2-1),step2]`
   - `L2=Sequence[Polygon[(0,i),(Max[5n1/d1,5],i),(Max[5n1/d1, 5],i+step2),(0,i+step2)],i, -3,Max[-3+step2(n2-1),-1-step 2],step2]`
   - `L3=Sequence[Polygon[(i,-3),(i+step1,-3),(i+step1,Max [-3+step2*n2,-1]),(i,Max[-3+step2*n2,-1])],i,0,Max[5n1/d1 -step1,5-step1],step1]`
8. For $L1$ set the color to blue, opacity to 25, and line thickness to 2. For $L2$ and $L3$ set the color to black, opacity to 0, and line thickness to 1.