THE PARABOLA AS THE LOCUS OF THE PRODUCT OF TWO LINES: BUILDING FUNCTIONS FROM FUNCTIONS

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Abstract

In this article, authors explore quadratic expressions as the product of two linear expressions. In particular, authors use graphing, locus, and Input Box tools within GeoGebra to investigate graphical relationships that exist between pairs of graphed lines and the cooresponding parabola.

Keywords: conic sections, linear factors, quadratic equation

1 INTRODUCTION

Students first learn about factors and prime factorization in elementary grades. In high school, they work with variables and expressions and learn to factor these expressions. In particular, they learn several methods to factor quadratic expressions (i.e., difference of squares, perfect square trinomials, quadratic formula) and write these expressions as the product of linear terms, when possible. However, students rarely investigate the converse of this situation graphically – namely, how the product of two lines (not necessarily different) in x produces a second-degree polynomial in x. As graphing technology becomes more powerful and accessible, opportunities to investigate such relationships are within reach in secondary-level mathematics classrooms.

In this paper, we use the locus in GeoGebra to construct the parabola as the product of two lines and determine the location of the focus and directrix of the parabola. We assume minimal familiarity with basic constructions in GeoGebra and include construction details of lines and point plotting. Other tools like the locus and use of parameters are introduced as needed. We highlight each step in the process with screenshots from the software. Our constructions and investigations easily extrapolate to the product of n linear factors and their relationship to polynomials.

We've explored parabolas as the product of two lines before (Authors, 2002). With older dynamic geometry software applications, locus tools were cumbersome. In our earlier work, we were unable to discuss the locus and the directrix in a simple and friendly manner. Thankfully, the tools are relatively simple to use in GeoGebra. When locus tools are paired with the GeoGebra **Input Box**, students are able to alter various parameters of their locus constructions while exploring how changes in either (or both) lines result in changes in the parabola.

2 The Parabola as the Product of Two Lines

The Fundamental Theorem of Algebra states that any polynomial of degree n with complex coefficients has at least one complex root. When we limit polynomials to real numbers solutions (as most high schoolers and early-college students do) and think of parabolas, we can claim that parabolas either have no real zeros (irreducible) or have two real zeros. In our discussion, we address the case when the quadratic function (a vertical parabola) $f(x) = ax^2 + bx + c$ has two real zeros: $x = r_1$ and $x = r_2$. In this case, we can express the quadratic function $f(x)$ in terms of its zeros, namely $f(x) = ax^2 + bx + c = a(x - r_1)(x - r_2)$. Each of these factors can be determined using the quadratic formula as $x = -b + \frac{\sqrt{b^2-4ac}}{2a}$ and $x = \frac{-b-\sqrt{b^2-4ac}}{2a}$. Hence $f(x) = ax^2 + bx + c = a(x - \frac{-b-\sqrt{b^2-4ac}}{2a})(x - \frac{-b+\sqrt{b^2-4ac}}{2a})$. So, the parabola can be seen as the product of two lines. This motivated our initial discussions (Authors, 2002) and this follow up.

3 Steps to Construct the Parabola Using the Locus

To construct the parabola as the locus of the product of two lines in GeoGebra, first we construct two lines using two slope and two $y-$intercept measures. Rather than using sliders, we construct these values from four points on the $x-$axis. Next, using one point, we produce two points with the same abscissa. After this, we determine the product of the ordinates of these points, constructing a new point, `(abscissa, product of ordinates)`. Finally, we produce the locus of this last point.

Step 1 *Define the slope and y-intercept as parameters.* Construct four points, A, B, C and D on the $x-$axis. In the GeoGebra **Input Bar**, define $m1$, $m2$, $b1$, and $b2$ by typing the following commands: `m1 = x(A)`, `b1 = x(B)`, `m2 = x(C)`, and `b2 = x(D)`. Results are shown in Figure 1.

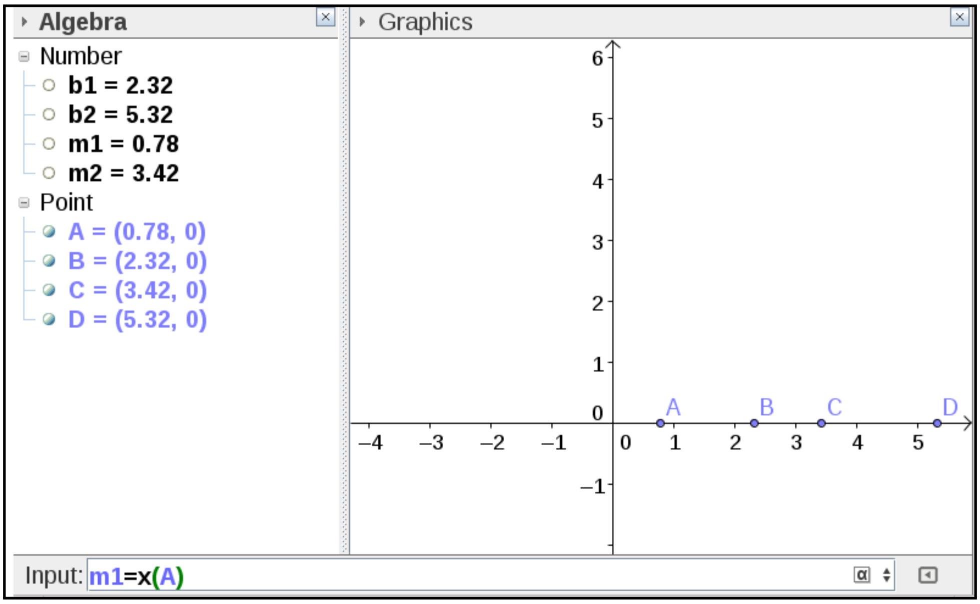

Figure 1. Using the $x-$coordinate of a point to define parameters for slope and $y-$intercept

Step 2 *Construct two Lines, $f(x)$ and $g(x)$, given their slope and y-intercept.* In the **Input Bar** define the function f by typing the following command: `f(x)=m1*x+b1`. Make certain to insert an asterisk (*) between $m1$ and x. Proceed similarly to define g as suggested in Figure 2. Observe how dragging a point (either slope or $y-$intercept) affects the graph of the corresponding line.

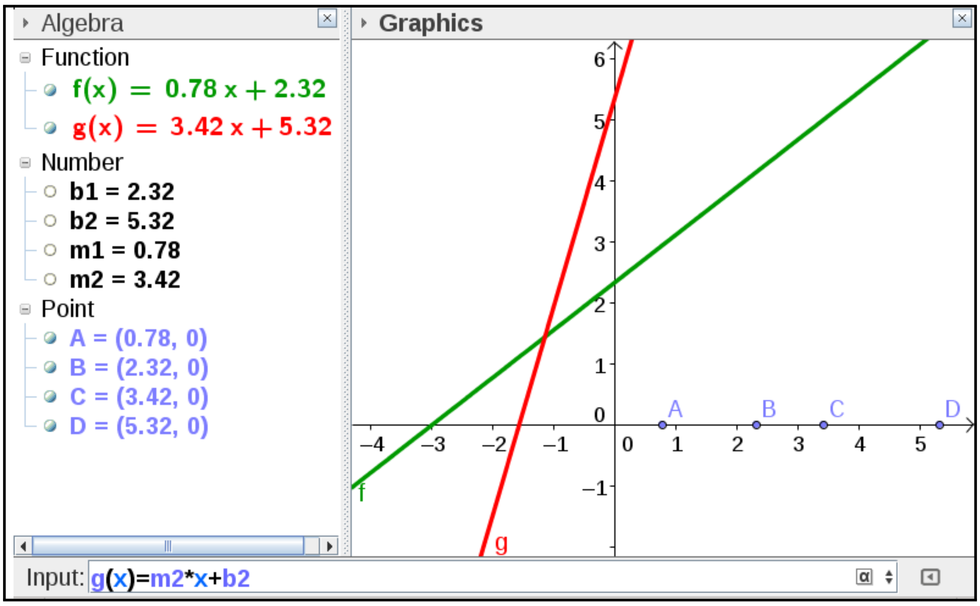

Figure 2. Construction of a Line given the slope and $y-$intercept

Step 3 *Construct point E on one line*. In Figure 3, we construct E on function $g(x)$. In the **Input Bar** define $k = x(E)$ by typing `k=x(E)`. Doing so will allow students to type `k` in subsequent commands. This makes typing less cumbersome while promoting student use and understanding of variables.

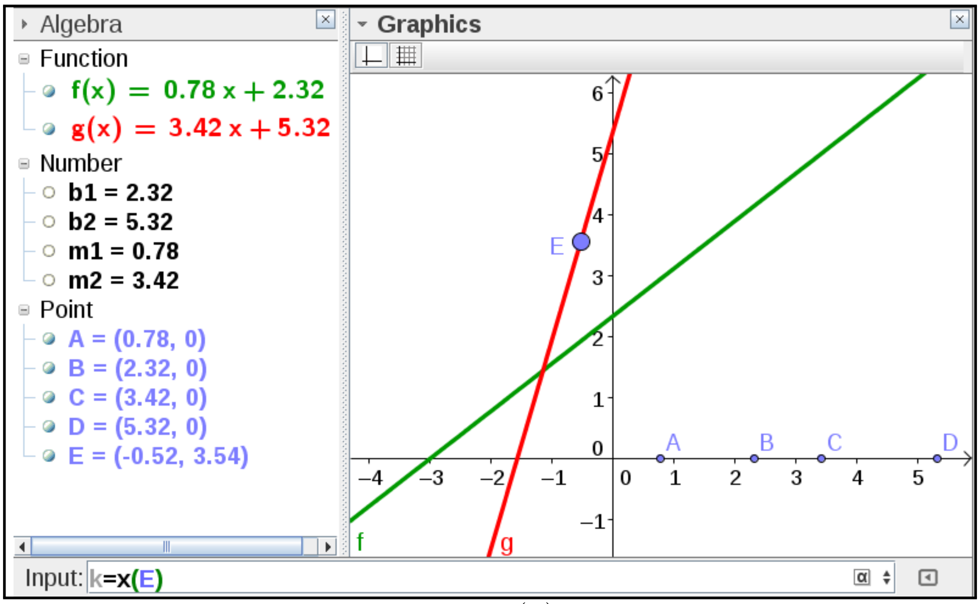

Figure 3. Point E on line $g(x)$ with definition of k

Step 4 *Construct a point P.* Define P by typing the following command in the **Input Bar**: `P=(k, f(k)*g(k))` as shown in Figure 4.

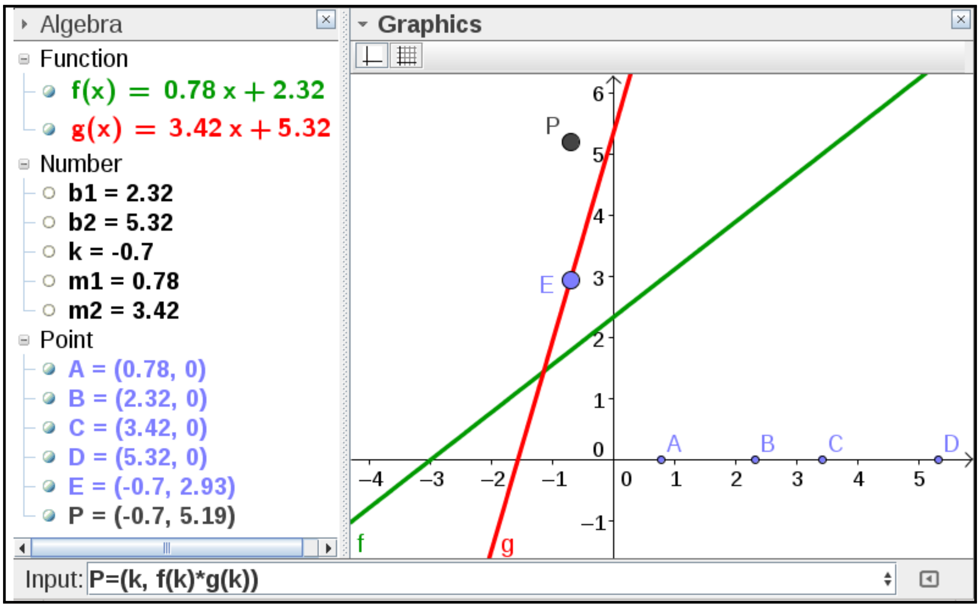

Figure 4. Construction of P

Step 5 *Construct the locus of point P as E moves along the line.* This step uses the **Locus** Tool. In the **Input Bar** type the following command: `Locus[P,E]` to produce the parabola as a locus. If P does not show on the screen (because the product of the ordinates falls out of the window), drag one of the lines to make it visible. As Figure 5 illustrates, we colored the locus purple and changed its style to a dotted line to make it more visible for students.

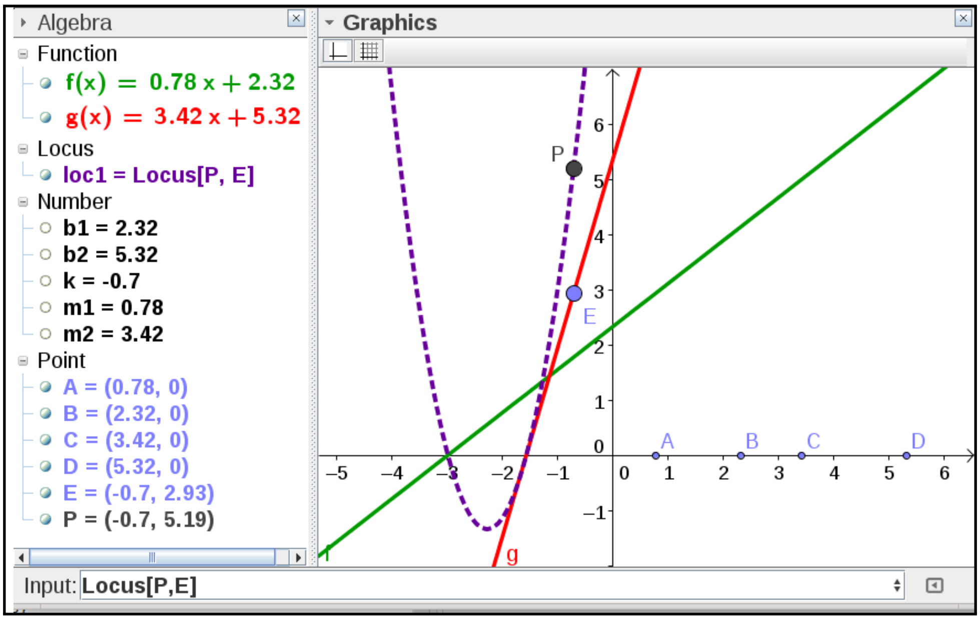

Figure 5. The locus of H when E moves along the line is a parabola

In summary, below are five steps we use to construct the parabola as the locus of the product of two lines.

- Construct points on x-axis and use their abscissa as parameters for slope and y-intercept (Figure 1).
- Construct two lines, $f(x)$ and $g(x)$, given their slope and y-intercept (Figure 2).
- Construct a point E on one of the lines (Figure 3). Define k as the abscissa of E (Figure 3).
- Construct point P such that $P = (k, f(x) * g(x))$. In other words, the abscissa of P is the same as the abscissa of point E, and its ordinate is the product $f(k) * g(k)$ (Figure 4).
- Construct the locus of H when E moves along the line (Figure 5).

Invite your students to construct the sketch, then change the slopes and $y-$intercepts (by dragging points A, B, C and D). Encourage them to observe the changes in the parabola as points are dragged. We've found the following questions helpful for students (Authors, 2002).

- What is the relationship between the zeros of the parabola and the zeros of the lines?
- Where is the vertex of the parabola with respect to the zeros of the parabola?
- What is the relationship between the $y-$intercepts of the lines with the $y-$intercept of the parabola?
- What happens to the parabola when one, or both of the lines are constant?
- What happens if the lines are perpendicular or parallel? What happens to the parabola when one, or both, line(s) are vertical?

In the next section, we determine the location of the focus and the directrix. Our 2002 paper did not include their location.

4 THE FOCUS AND DIRECTRIX OF THE PARABOLA

In a recent paper (Authors, accepted for publication), we used GeoGebra to construct and investigate properties of the parabola as the locus of all points equidistant to a fixed point (i.e., the focus) and a fixed line (i.e, the directrix). We derived the equation of the vertical parabola (see CCSSM, HSG-GPE.A.2) and obtained $y - k = \frac{1}{4p}(x - h)^2$, where (h, k) are the coordinates of the vertex, and $|p| = d(\text{Vertex,focus}) = d(\text{Vertex,directrix})$. In other words, p is the distance between the vertex and the focus, and the distance from the vertex to the directrix.

Given the graph of the parabola, we now determine the vertex knowing that it is midway the two zeros of the parabola. The zeros of the parabola are the $x-$intercepts of lines $f(x)$ and $g(x)$. First, construct those intersections, G and H, using the GeoGebra **Intersect** command. Type the following commands in the **Input Bar**: `G=Intersect[f,0]` and `H=Intersect[g,0]`. Next, determine the midpoint of those intersections, I. Type the command `I=Midpoint[G,H]`. These are shown as large red points in Figure 6.

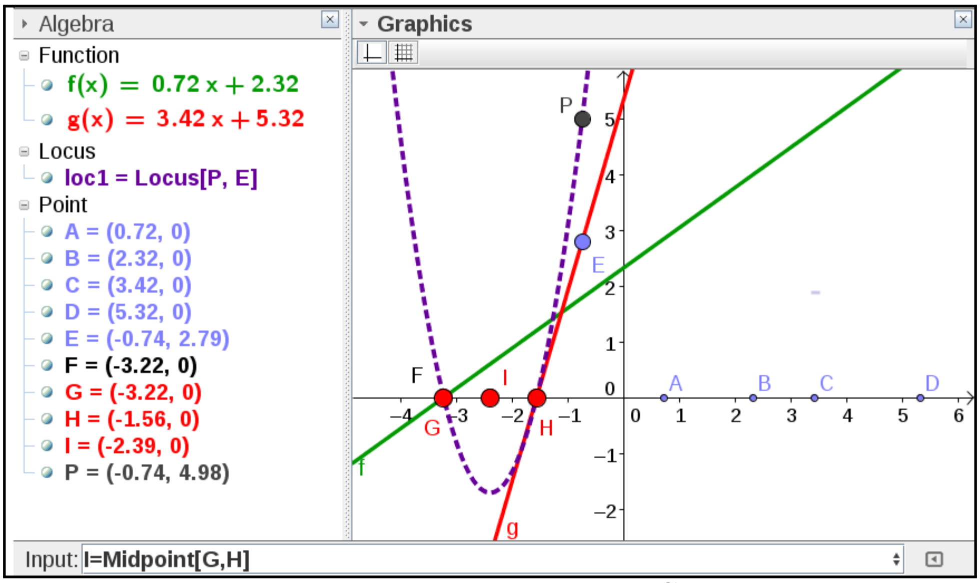

Figure 6. I is the midpoint between the two zeros, G and H, of the parabola.

We now determine the vertex. Using a similar technique to those discussed previously, define j and V by typing the following command in the GeoGebra **Input Bar**: `j=x(I)` and `V=(j,f(j)*g(j))`. Results of these commands are shown in Figure 7.

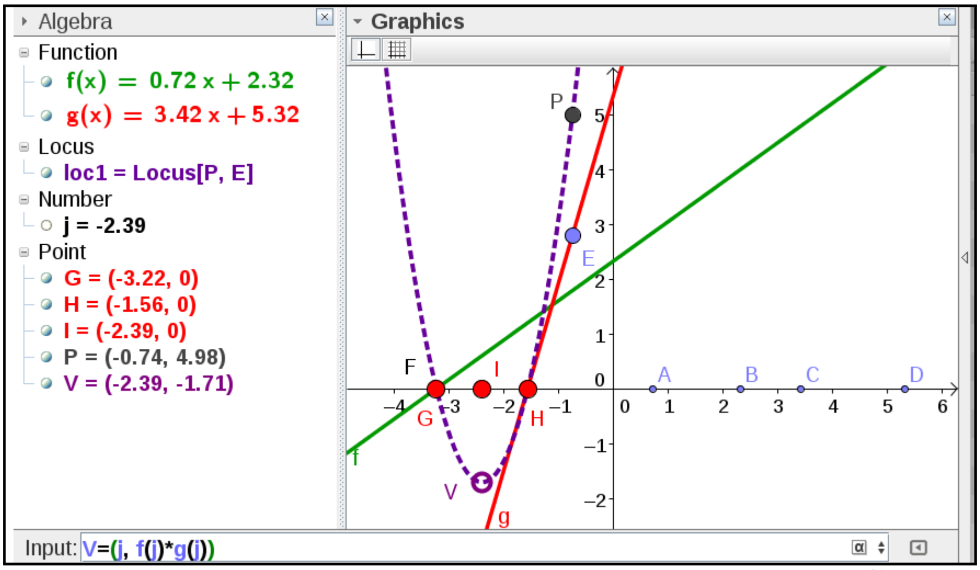

Figure 7. The vertex of the parabola, V, is midway the two zeros, G and H

Next, we determine the directrix and focus of the parabola. Earlier we noted that $f(x) = ax^2 + bx + c =$ (equation of Line1)*(Equation of Line2) $= (m_1 \cdot x + b_1)(m_2 \cdot x + b_2) = \frac{1}{4p}(x-h)^2 + k$.

Expanding the last two expressions and setting the coefficients of x^2 equal, we conclude $m_1 \cdot m_2 = \frac{1}{4p}$. From this, we conclude that $p = 1/(4m_1 \cdot m_2)$. Again, this is the distance from the vertex to the focus, and the distance from the vertex to the directrix. Therefore, we must take the absolute value of this quantity. In the **Input Bar**, type `p=abs(1/(m1*m2))`. See Figure 8. In this expression, we assume that both slopes are non-zero. If at least one of the slopes were zero, the product of the two lines would be another line.

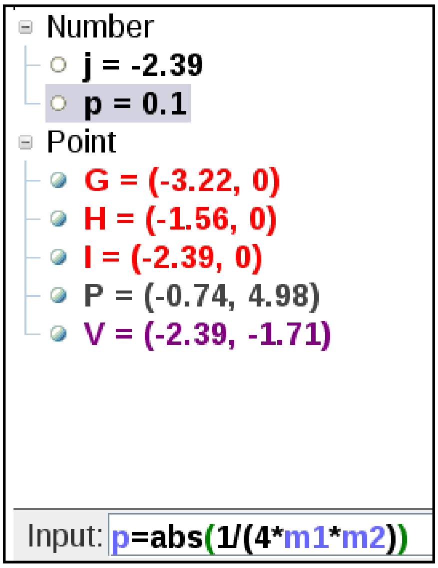

Figure 8. The distance p is determined as `p=abs(1/(4*m1*m2))`

We now graph both the directrix and the focus. To take into account the sign of the product $m1 * m2$, we define a new variable $p1$. Type the following into the GeoGebra **Input Bar**: `p1=sign(1/(4*m1*m2))*p`. Then, the directrix and focus are defined by entering the following commands: `d(x)=x(V)-p1` and `Focus=(x(V),y(V)+p1)`. The results of these steps are shown in Figure 9.

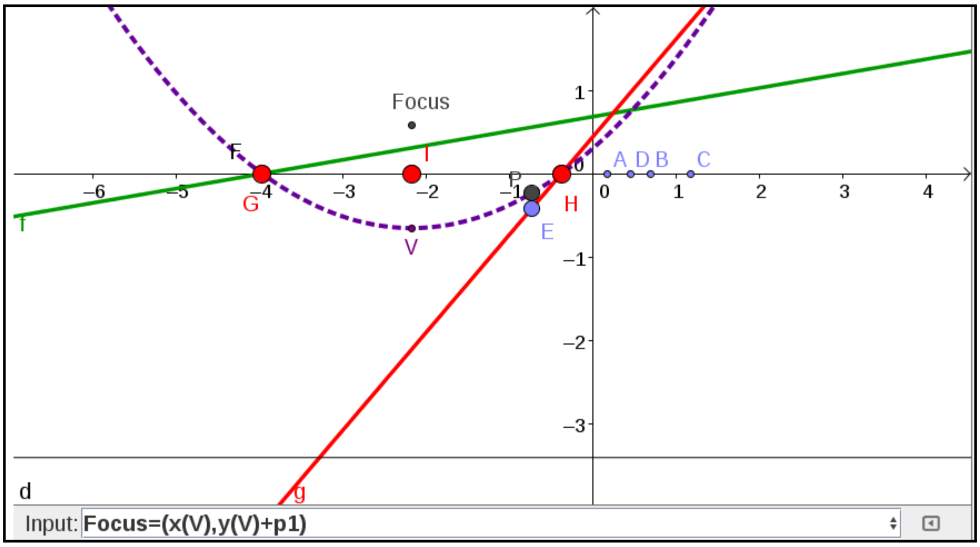

Figure 9. Construction of parabola directrix and focus

5 Concluding Remarks

Functions are a topic in analysis, yet they also appear in the middle grades in high school. Students learn to operate on functions and produce new ones using function operations such as addition, subtraction, multiplication, division, and composition. The importance of this mathematical activity is highlighted in CCSSM in the Functions domain (Building Functions). In this paper, we have addressed how the product of two lines leads to a vertical parabola – a quadratic function. Unfortunately, students rarely study parabolas as the product of two linear terms, despite that fact that such study is accessible from a graphical perspective. With dynamic software such as Geogebra, students can relate properties of the lines with properties of parabolas. The approach we've discussed in this paper can be generalized for more than two factors. We invite readers to investigate the product of several linear factors and the production of horizontal parabolas.

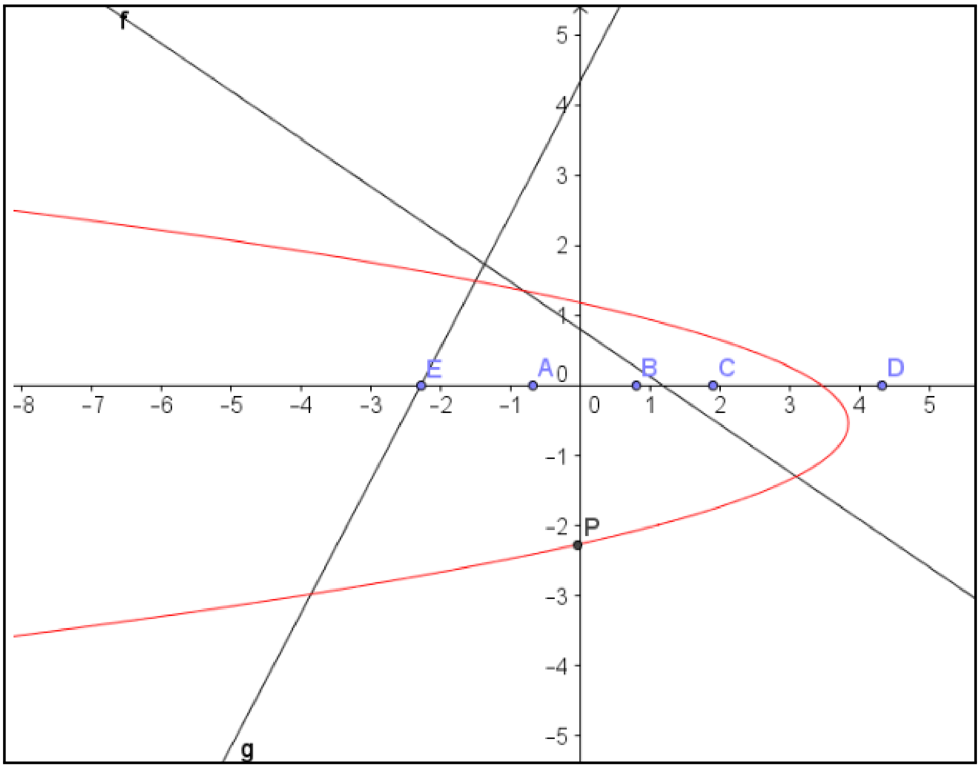

Figure 10. A horizontal parabola as the product of two linear factors

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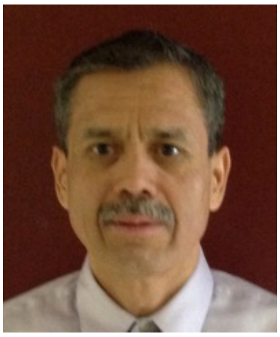

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