THE PARABOLA AS A LOCUS: PAVING THE WAY TO THE COMMON CORE STATE STANDARDS

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Abstract

This article presents teaching investigations conducted in a geometry class for prospective secondary mathematics teachers. The main goal is to derive the equation of the parabola given the focus and the directrix. Additionally, students use GeoGebra to investigate the effect of the focus and the directrix on the graph.

Keywords: *GeoGebra* (4.4.2), Preservice Secondary Teachers, Geometry, Parabola

1 INTRODUCTION

Parabolas are often the first graphs of non-linear functions that students meet in the mathematics curriculum. Initial encounters with the parabola happen in the middle school when students are introduced to the function $f(x) = x^2$. Later, students learn that the graph of the function $g(x) = ax^2 + bx + c$ is a parabola, and are able to determine the effect of the coefficient a on the graph. Rarely, however, students are shown (or told) why the graph of a quadratic equation in x is a parabola. In this paper, we approach parabolas using the geometric locus. We employ dynamic geometry software [3] to illustrate properties of the parabola, illustrate real-life situations, and dynamically manipulate parabolas. Most of the ideas presented here are discussed in a college geometry class for prospective secondary mathematics teachers taught by one of authors. Students regularly use dynamic software for constructions, problem solving and conjecture making. Parabolas and some applications to maximization problems are the last topics covered in the course. Here, we focus on the determination of the equation of the parabola.

1.1 The Parabola as a Locus

There are at least three characterizations of the parabola. The first characterization, as a conic, involves a cone intersected by a plane (Apollonius of Perga, c.262-c.200 BCE), [1]. The second characterization is algebraic in nature. In this case, the parabola is defined by the second-degree polynomial $f(x) = ax^2 + bx + c$. The third characterization is geometric and was suggested by Pappus (ca. 290 – c. 350AD) in terms of a fixed line called the directrix and a fixed point called the focus. In this paper, we discuss Pappus' characterization and establish a connection between the geometric and algebraic representations of the parabola. We introduce the notion of geometric locus or locus before continuing.

The geometric locus (place in Latin) is a set of points whose location satisfies a set of conditions. Ancient Greek mathematicians studied various curves defined by certain conditions, i.e., as a locus, an approach that is still in use. For instance, it is common to define the circle as the locus of all points that are equidistant from a fixed point, called the center of the circle, and the fixed distance the radius of the circle. The locus can be used to characterize other curves. For instance, the perpendicular bisector of a segment $\overline{AB}$ is the locus of all points in a plane that are equidistant to both endpoints A and B of the segment. Pappus, in one of his collection of papers, "first found the focus of the parabola and suggested the use of the directrix," ([4], p. 309). In current language, we define the parabola as the locus of all points equidistant from a fixed line called the directrix and a fixed point called the focus.

1.2 GeoGebra Tools

In the next section, we provide directions to construct the parabola using the directrix and the focus. Since we will be using *GeoGebra* [3] in the discussion, we provide a succinct overview of the software and the location of some commands. We start by showing the *GeoGebra* Window (Figure 1). Notice that the *GeoGebra* window has two parts: the Algebra View (left side) and the Graphics View (right side). The x icon on the upper right corner of each view closes that view.

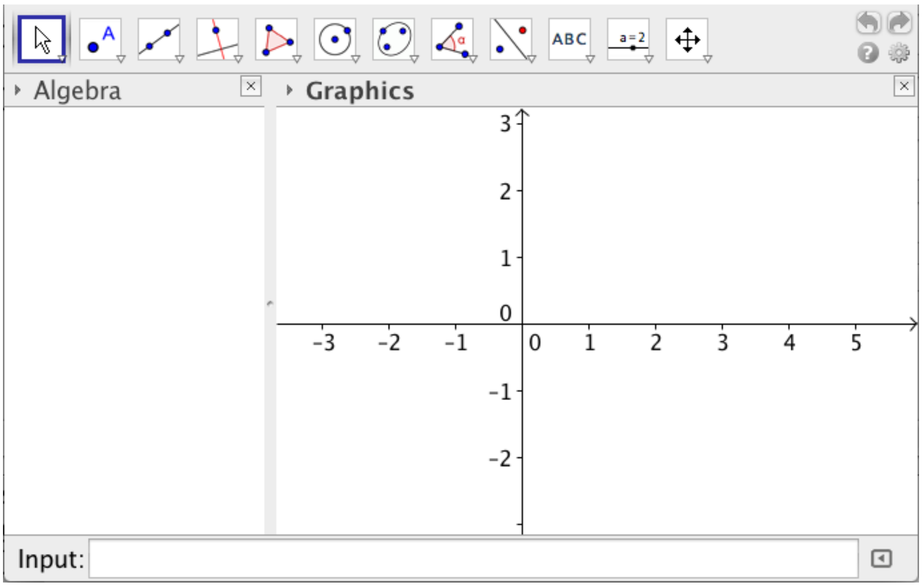

Figure 1. The *GeoGebra* window consists of the Algebra View and the Graphics View

Tools are located horizontally above the Algebra and Graphics View. When the cursor is placed on a tool, *GeoGebra* [3] displays the name of the tool (in bold) and directions to use it (below the name). Figure 2 shows the *GeoGebra* [3] tools.

We will be using the Locus tool, located under the Perpendicular Line tool, in this paper. The triangle pointing down (here and in other tools) indicates that there are more tools in this category. To access those other tools, one simply left clicks the original tool and moves the pointer (see Figure 3). We construct the parabola in the next section. One can notice and appreciate how few software tools are needed to construct it.

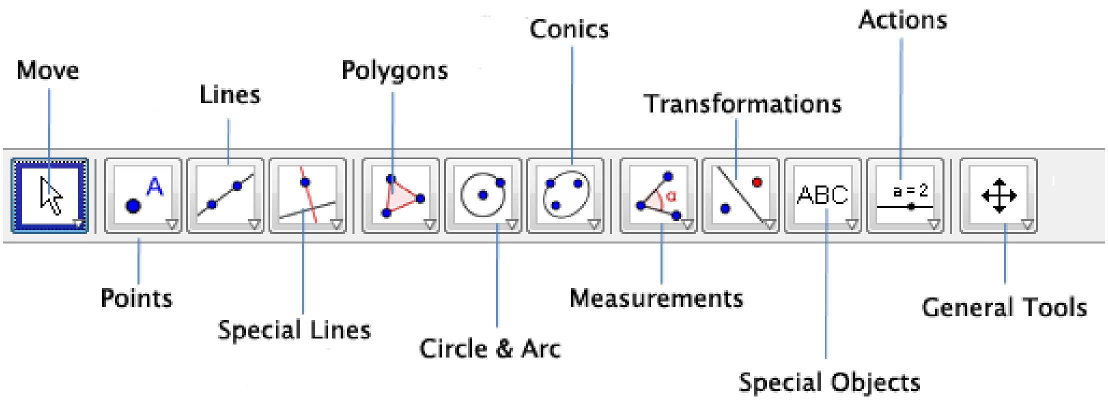

Figure 2. *GeoGebra* Tools

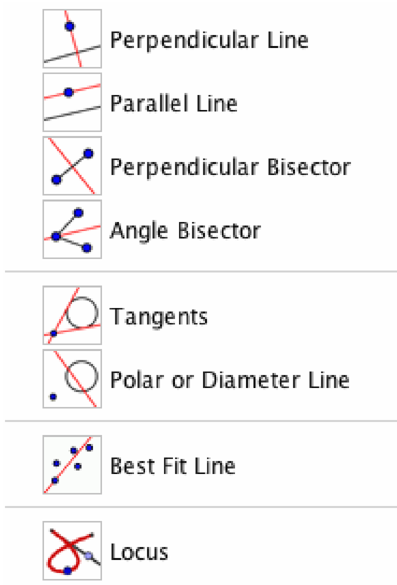

Figure 3. The Locus tool is under the Perpendicular Line Tool

1.2.1 Steps to Construct a Parabola

Remember that the parabola is the locus of all points that are equidistant from a fixed line (directrix) and a fixed point (focus). Start by closing the Algebra View (click on the x icon) and hiding the Axes in the Geometric View. To hide the axes, right click on the screen and click on the Axes command.

1. Construct a point, A. This is the focus.
2. Construct line $\overleftrightarrow{BC}$ (not through A). This is the directrix.
3. Construct point D (different from B and C) on the directrix.
4. Construct the perpendicular line to the directrix through D.
5. Construct segment $\overline{AD}$.
6. Construct the midpoint, E, of segment $\overline{AD}$.
7. Construct the perpendicular bisector of segment $\overline{AD}$.
8. Construct the point of intersection, F, of this perpendicular bisector with the perpendicular to the directrix.
9. Construct the locus of F when D moves along the directrix.

Figure 4 shows the first five steps to construct the parabola as a locus. In this figure, A is the focus of the parabola, and line $\overleftrightarrow{BC}$ is the directrix.

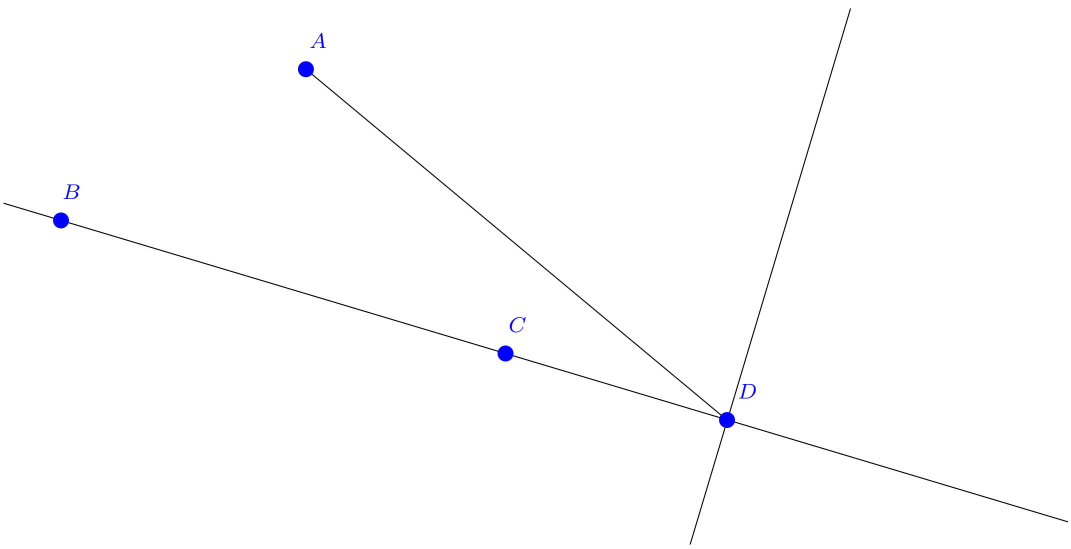

Figure 4. First five steps to construct the parabola as a locus

Our next plan of action is to construct the perpendicular bisector to segment $\overline{AD}$. First, we construct the midpoint of segment $\overline{AD}$. To do so click on the New Point Tool (second from left), and scroll down to the Midpoint or Center tool. Click on it to select it and then click on segment $\overline{AD}$. This produces the midpoint of segment $\overline{AD}$, point E. Next, construct the perpendicular to $\overline{AD}$ through E.

To do so, click on the Perpendicular Line tool (fourth from left), then on point E and next on segment $\overline{AD}$. Using the New Point tool, construct the intersection, point F, of the perpendicular bisector of segment $\overline{AD}$ with the perpendicular to the directrix through D. Finally, to produce the locus, select first the Locus tool (under Perpendicular Line), and then click on F and then on D in that order. Figure 5 shows these remaining steps.

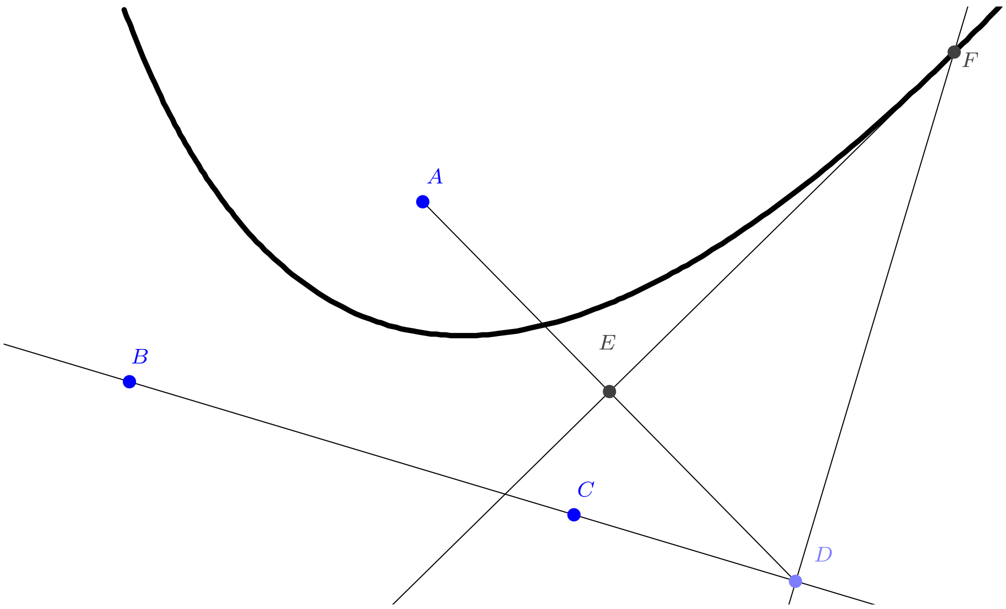

Figure 5. The parabola is the locus of all points that are equidistant to both the directrix and the focus

The alert reader will have noticed that we have not proved yet that F is equidistant to the focus and the directrix. To do so, we invoke a property of the perpendicular bisector, namely, that any point on the perpendicular bisector of segment $\overline{AD}$ is equidistant to both endpoints, A and D. Since F is on the perpendicular bisector of $\overline{AD}$, we conclude $\overline{AF} \cong \overline{FD}$. But AF is the distance of point F to the focus, and FD is the (perpendicular) distance of point F to the directrix (D is on the directrix and on the perpendicular to line $\overleftrightarrow{BC}$). We can conclude now that F satisfies the parabola definition, and is a point on the parabola having focus A and directrix $\overleftrightarrow{BC}$.

Now, that we have constructed the parabola we invite the reader to carry out some investigations. For instance, drag the focus and observe what happens to the parabola when the focus moves closer to or farther from the directrix. Similarly, drag the directrix and observe what happens when the directrix is horizontal, vertical or slant. Also, drag point D on the directrix and observe what happens to point F.

1.2.2 The Equation of the Parabola Function

The locus definition of the parabola can be used now to determine the equation of the parabola. We restrict ourselves to the vertical parabola that opens up since graphically this represents a function. However, the reasoning is the same to determine the equation of a horizontal parabola; the equation of a slant parabola is out of the scope of this paper.

Suppose that the vertex of the parabola is at the point $V(h,k)$. The vertex is the midpoint of the perpendicular segment from the focus to the directrix. Let's assume that the distance from the vertex to the focus and to the directrix is $p > 0$. Then the focus has coordinates $(h, k+p)$ and the equation of the directrix is $y = k - p$. Suppose now that $F(x,y)$ is a point on the parabola. Refer to Figure 6 to visualize these assumptions.

From the distance formula, the distance between the focus and the point F is given as

$$\text{distance(Focus}, F) = \sqrt{(x-h)^2 + (y-(k+p))^2}.$$

On the other hand, the distance of point $F(x,y)$ to the directrix is given as $y-(k-p)$. Since the point $F(x,y)$ is on the parabola, it is equidistant from both the focus and the directrix. Therefore these distances are equal:

$$\sqrt{(x-h)^2 + (y-(k+p))^2} = y - (k-p)$$

Squaring both sides and expanding we get

$$(x-h)^2 + y^2 - 2y(k+p) + (k+p)^2 = y^2 - 2y(k-p) + (k-p)^2$$

Simplify this equation before expanding again. Another simplification produces

$$(x-h)^2 - 2yp + 2kp = 2yp - 2kp,$$

$$(x-h)^2 = 4yp - 4kp = 4p(y-k).$$

Finally, this equation can be written as $y - k = \frac{1}{4p}(x-h)^2$.

This is the equation of a vertical parabola that opens up. The vertex is located at $V(h,k)$, the equation of the directrix is $y = k - p$, and the focus has coordinates $(h, k+p)$. This, on one hand, tells us that the equation of a vertical parabola is a second-degree polynomial, and on the other hand, establishes the connection between the algebraic characterization of the parabola and the directrix-focus characterization of the parabola.

We now discuss the meaning of the term $4p$ in this equation. In Figure 6, construct the segment parallel to the directrix through the focus. This segment is called the latus rectum and intersects the parabola twice. To determine these points of intersection, we substitute $y = k + p$ in the equation $(x-h)^2 = 4p(y-k)$. Substitution leads to the solutions $x = h + 2p$ and $x = h - 2p$. In other words, the latus rectum intersects the parabola at the points $(h-2p, k+p)$ and $(h+2p, k+p)$. The length of the latus rectum is given by the distance between these points. Since the ordinates are the same, we compute the distance between the abscisas. This distance is $4p$, the length of the latus rectum, and it gives the focal width of the parabola. Notice that if we expand the equation $y - k = \frac{1}{4p}(x-h)^2$, the coefficient of x^2 is $\frac{1}{4p}$. Hence, the coefficient of x^2, namely a, in $f(x) = ax^2 + bx + c$ determines the focal width of the parabola.

1.3 A Common Occurrence of a Quadratic Function in the Physical World

We now illustrate a quadratic behavior in real life. "A common occurrence of a quadratic equation function in the physical world is the description of the location, as a function of time, of a body whose

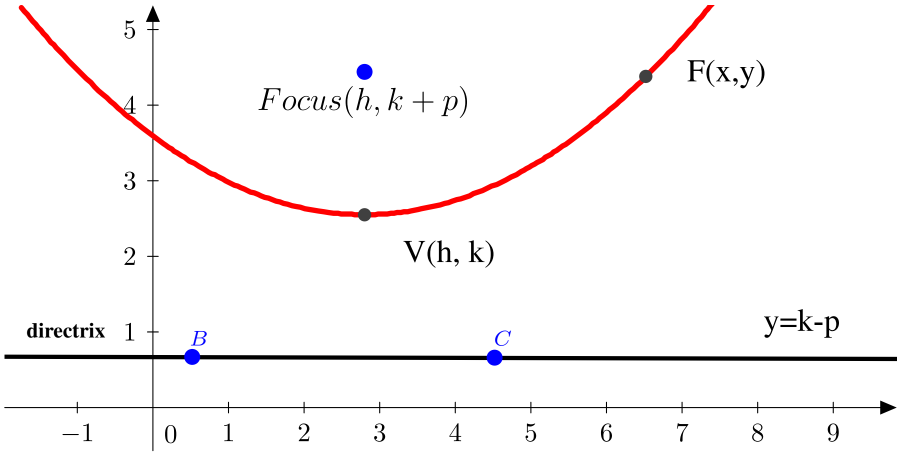

Figure 6. Assumptions to deduce the equation of the vertical parabola that opens up

motion is subject only to the force of gravity," ([5], p. 292). In this case, we took a picture of a stream of water in a water fountain in one of our halls. We inserted the picture into a *GeoGebra* [3]sketch and manipulated the focus and directrix of the parabola in the construction above. See Figure 7. Our students responded enthusiastically to this quadratic behavior. We bet your students will too. See also [5] for an alternative approach to the parabola using a digital camera.

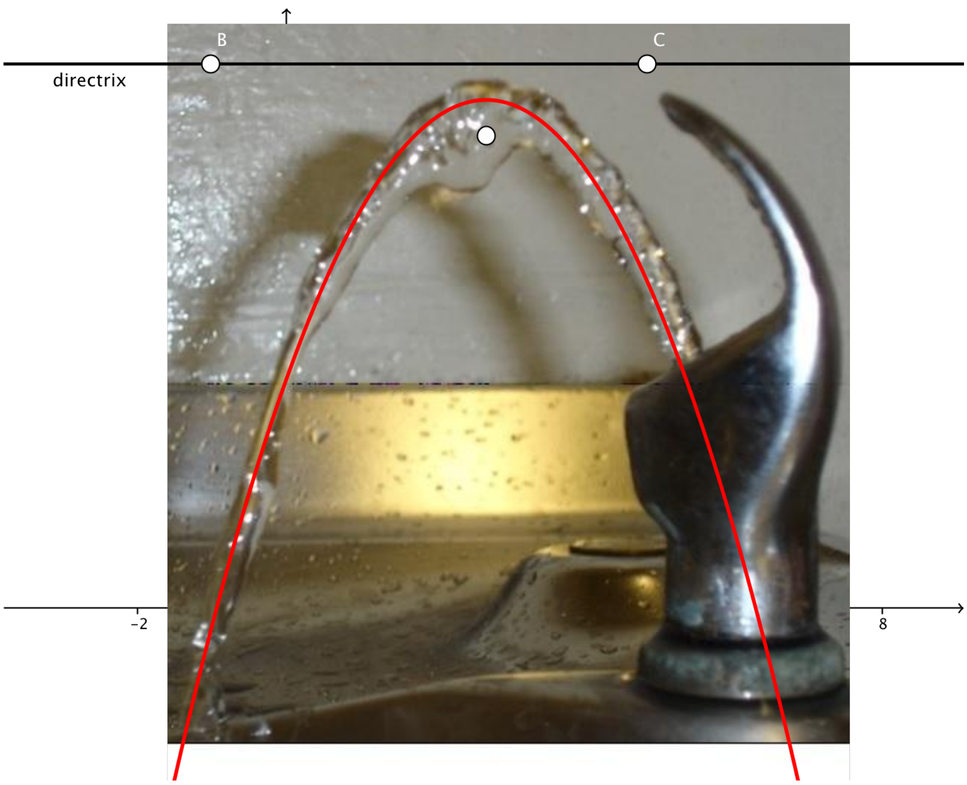

Figure 7. Parabolas in real life

2 Conclusion

The activity described above occurs in a geometry course for prospective secondary mathematics teachers. It is a joint effort of the authors to provide teachers with standards for mathematical practices (model with mathematics, use appropriate tools strategically, and use definitions in constructing arguments) aligned with the Common Core State Standards (CCSS) Initiative [2]. In particular, the focus of the activity is the *CCSS Math Content HSG-GPE.A.2 Derive the equation of a parabola given the focus and directrix*. Students gain intensive experience with dynamic geometry software and locus prior to the discussion of the parabola. *GeoGebra* [3] is particularly appropriate to work with the locus, since the parabola graph emerges complete (in other software the graph appears truncated sometimes). We have heard numerous times how our students appreciate the connections, the content and the preparation for the CCSS.

References

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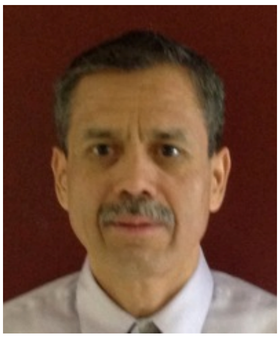

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