BROWNIE POINTS

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Abstract

In this article, we explore an open–ended problem that integrates fair division, geometry, algebra, and its extensions in higher dimensions. GeoGebra[1] is an essential tool used to further investigate this problem through its juxtaposition of algebraic and geometric representations. The reader is invited to try similar activities with his or her own students through the purposeful use of dynamic geometry.

Keywords: geometry, problem solving, connections, modeling

1 INTRODUCTION

Imagine you just baked a fresh batch of brownies. Removing the rectangular pan from the oven, you let it cool overnight before you savor your treat. You awake the next day in anticipation, only to realize that a thief has cut out a rectangular portion from the cooling brownies! Still, you decide that it is only fair to divide the remainder of the baked good into two equal areas, one for you, and one for your best friend. What mathematical questions can one pose based on this situation?

This mouth-watering problem was presented recently at a Math Teacher Circle [1] Training by American Institute of Mathematics (AIM). Naturally, the first question is to determine how to properly subdivide the remaining configuration into two equal areas? However, before this task is tackled, perhaps two more important questions about mathematical modeling are:

1. What do we know/what are we given about the situation?

2. What are the unknowns and what assumptions can be made?

Relative to the first question, we are given that both the overall shape as well as the removed portion are rectangular. There are many unknowns including the dimensions of either rectangle, the orientation of the removed brownie, the shape of cut, and the density of the brownie mix. We need to make assumptions about these unknowns in our model.

To get started, let us assume the easiest scenario: the removed portion shares a vertex with the original brownie, that is, a "corner" of the original rectangle was removed. To get started, let us assume a 3×6 brownie with a 1×2 portion removed, as shown in Figure 1. Areas were measured showing that each subdivided area must measure 8 square units.

[1]Editors' note: The student activities and figures contained within this paper were generated with GeoGebra 4.2.56

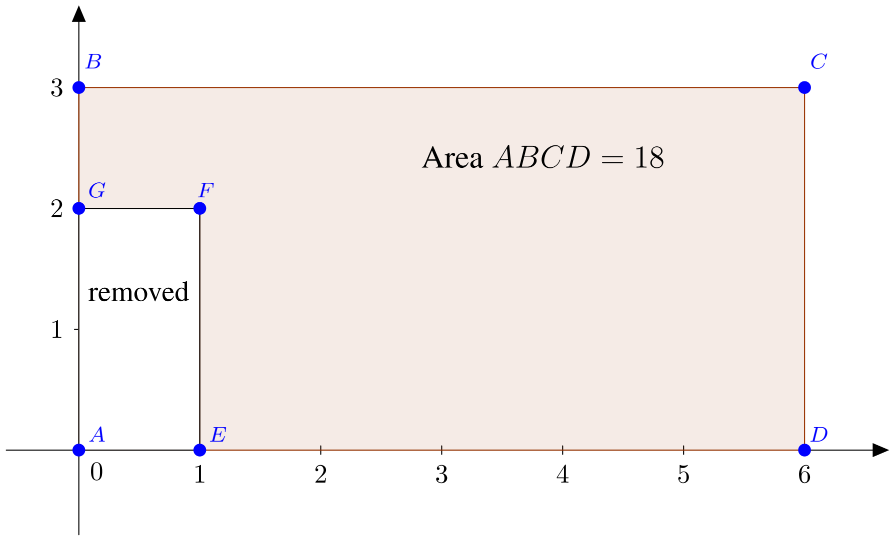

Figure 1. The easiest scenario: removal of a corner

Using *GeoGebra*, we can experimentally find a linear cut by placing two moveable points (using the "Point On" feature) along the polygon perimeter and dragging until the area enclosed measures approximately 8 square units. Such a cut is illustrated in Figure 2.

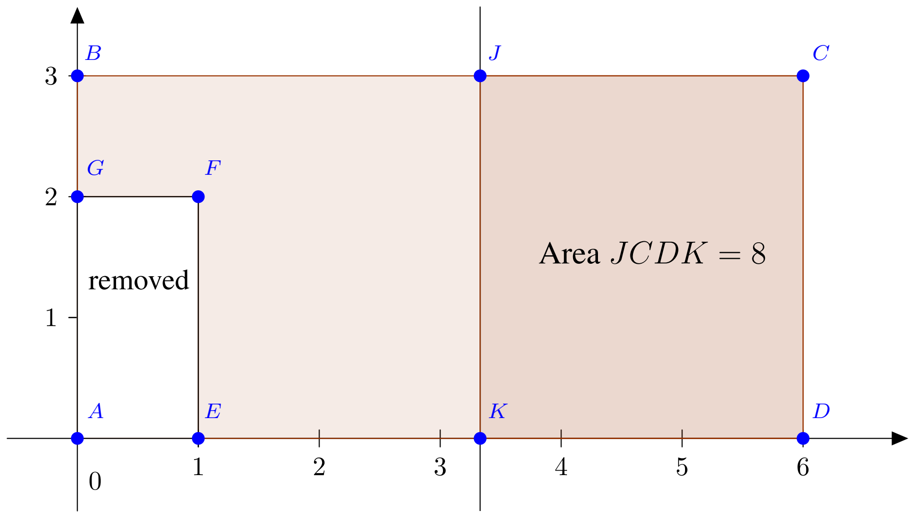

Figure 2. An exploratory cut subdividing the remaining area equally

After dynamic exploration using *GeoGebra*, students could be asked to use what they know about algebra and geometry to determine the exact point on the base $\bar{AD}$ where the perpendicular cut should occur. *GeoGebra* functions as a conceptual tool bridging the concrete to the abstract. Letting b denote the base of the rectangular portion, we require $(b)(3) = 8$, hence $b = \frac{8}{3}$ units and the point of interest, denoted by K in Figure 2, is $(\frac{10}{3}, 0)$.

While the initial problem of determining a fair cut has been solved, what other questions could students investigate? For one, are there any other cuts that would also work? Since points J and K are generally constructed on the perimeter, students can move these points, showcasing the dynamic power of *GeoGebra*. Another such cut is demonstrated in Figure 3. forming a red trapezoidal region.

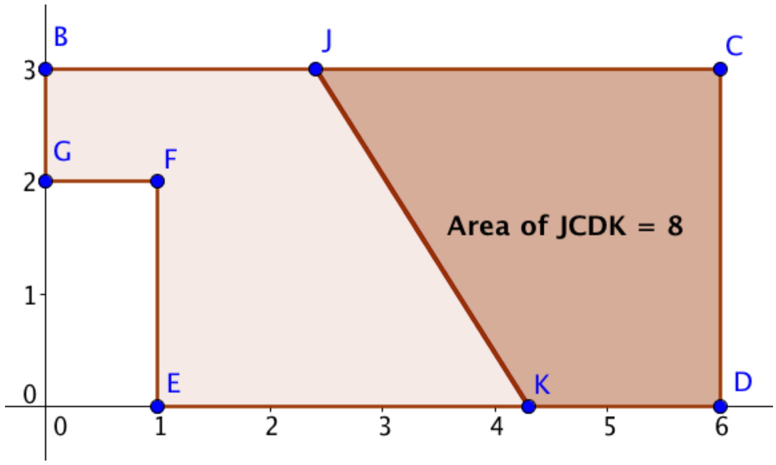

Figure 3. Another possible cut found by dragging points

As before, students could be asked to algebraically verify the location of point K. In Figure 3, let b_1 and b_2 denote the bases of the trapezoid (namely $\bar{JC}$ and $\bar{KD}$ from Figure 3). Any such trapezoid will have a height of 3 units. Using the trapezoid area formula, $\frac{1}{2}(b_1 + b_2)(3) = 8$, we find $b_1 + b_2 = \frac{16}{3}$. Using *GeoGebra*, we can redefine point J using the following formula in the input bar:

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J=(6-(16/3-distance(D,K)),y(C)).
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This formula is valid since it is forcing the sum of the bases to be $16/3$. Using point K as the driver, point J will move and a family of lines will be created as shown in Figure 4. It is especially fun to trace the line formed by J and K as K moves to form a family of cuts.

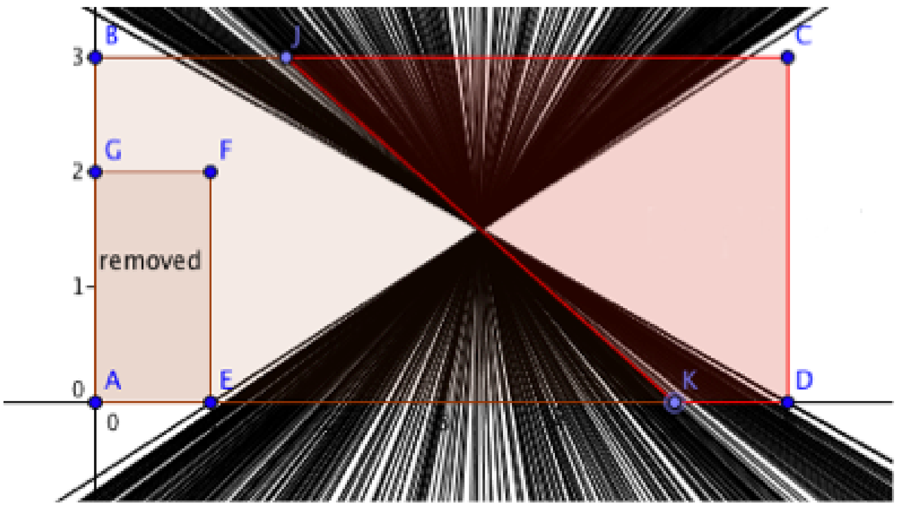

Figure 4. A locus of cuts forming trapezoidal regions

Using a more algebraic approach, denote the coordinate $K = (a, 0)$ so that $KD = b_1 = 6 - a$ yielding $b_2 = \frac{16}{3} - (6 - a) = a - \frac{2}{3}$. Hence, the point J defining the other base of the trapezoid can be identified by the ordered pair $J(6 - (a - \frac{2}{3}), 3)$ or $(\frac{20}{3} - a, 3)$ when simplified. The line defining the cut passes through $J(\frac{20}{3} - a, 3)$ and $K(a, 0)$ and has slope $\frac{3-0}{\frac{20}{3}-2a} = \frac{9}{20-6a}$. The equation of this line is $y = \frac{9(x-a)}{20-6a}$ where a denotes the $x-$coordinate of K.

If point K is free to move, then $y = \frac{9(x-a_1)}{20-6a_1}$ and $y = \frac{9(x-a_2)}{20-6a_2}$ could both be viable lines where $a_1 \neq a_2$. These two lines intersect when $\frac{9(x-a_1)}{20-6a_1} = \frac{9(x-a_2)}{20-6a_2}$. With some algebra, we find $x = \frac{20}{6} = \frac{10}{3}$, precisely the value of a that makes $\frac{9}{20-6a}$ undefined! Hence, the point $(\frac{10}{3}, \frac{3}{2})$ is the center of rotation for all lines that form a trapezoid like that in Figure 4. Let's call this important point a **brownie point**, namely, an intersection point of different possible cuts. Does a given pan of rectangular brownies with a rectangle removed have other brownie points?

There is nothing stopping students from applying the same type of analysis using the *vertical sides* intersected by a cut to form a trapezoidal region. Upon similar analysis, we can find other brownie points based on the sides intersected by our cut, three of which are shown in Figure 5. How many total brownie points can a figure have?

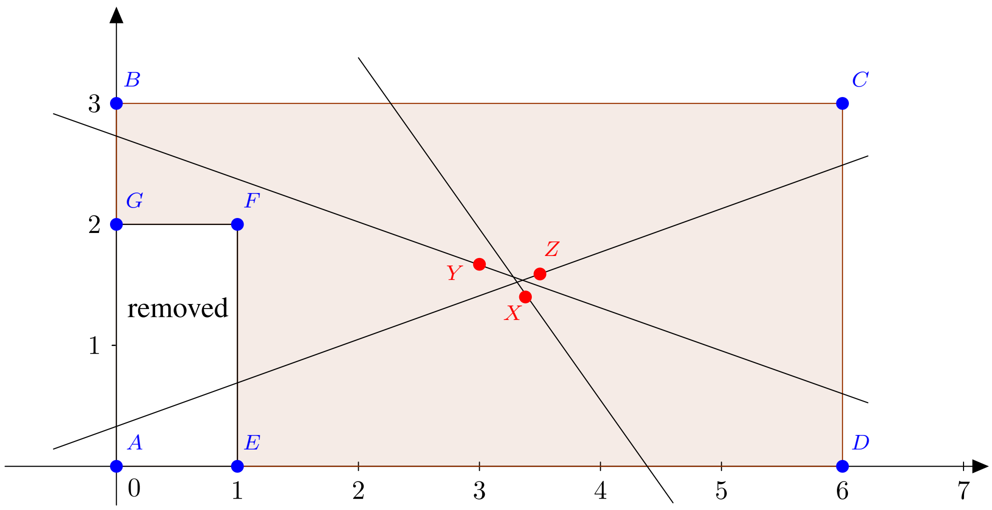

Figure 5. Three distinct brownie points for a figure based on the orientation of the cut

While we have discovered a lot of information about this problem, especially thanks to *GeoGebra*, there are still many stones left unturned. For example, how does the problem change as the removed rectangular portion changes orientation? *GeoGebra* makes it easy to construct a general rectangle. Figure 6 shows a general rectangle $ABDC$ constructed in the interior of the larger rectangle. Points A, B were placed at random in the interior and perpendicular lines were used to construct $ABDC$. Students could be challenged to investigate other ways to construct this general rectangle so that its properties as maintained as one of its vertices is dragged.

With this general rectangle constructed, students could conduct a similar investigation as before by finding general cuts based on the area to be divided and the sides intersected by the cut. The situation quickly becomes complicated when a cut enters into the "negative space" left by the portion removed, such as the cut shown in Figure 7 forming a concave heptagonal region. Students could conduct an exploratory investigation using the polygon, area, and intersect tools, or more advanced students could make this investigation more algebraic.

So far we have moved from a specific rectangle removed to a general one, using *GeoGebra* to find different cuts that will solve the puzzle. However, there is a very important property of rectangles

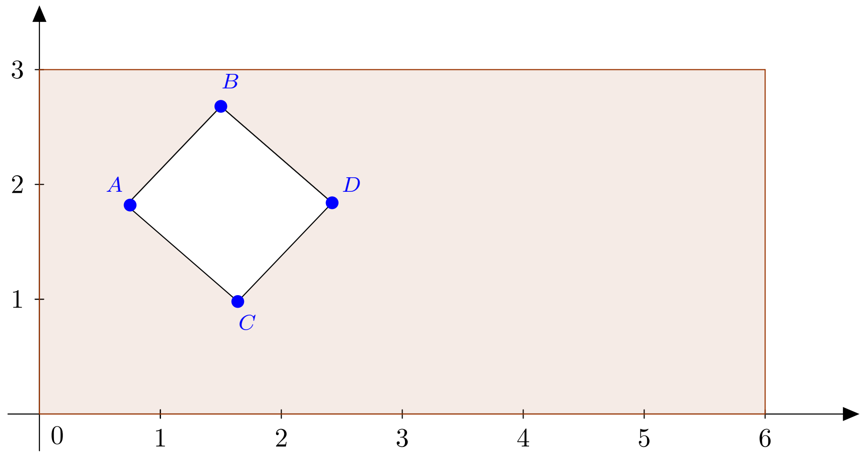

Figure 6. A general rectangular portion constructed and removed

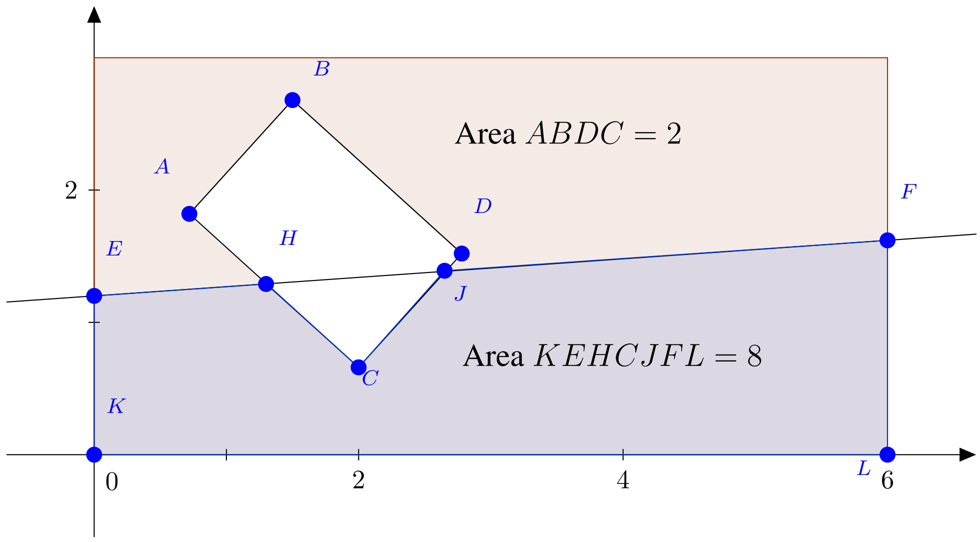

Figure 7. A cut forming a heptagonal region

that we have yet to employ. Using *GeoGebra*, students can informally verify that *any* line through the center of the rectangle divides the rectangle into two regions of equal area. Students could first investigate how to construct the center of a rectangle, such as using midpoints of sides or intersections of diagonals. Now form any cut through the center and construct a region bounded by the cut. The area of this region should be half the rectangle, as shown in Figure 8. Further, as the cut is moved, the area of the resulting region should not change.

With this property in hand, we find yet another solution to the original puzzle. Construct a line connecting the center of the original rectangle and the center of the removed rectangle. This, too, will be a possible cut to solve the original puzzle, as demonstrated in Figure 9. Students can use *GeoGebra* to informally verify this or provide a more rigorous geometric or algebraic argument. GeoGebra can be used to investigate other shapes that have a similar property to rectangles. Let us call this the *Center Property*: There exists a point such that *any* line through that point divides the remaining areas equally. For example, a circle naturally has this center property with the special point being the center. Similarly, a regular hexagon is a compelling choice as shown in Figure 10.

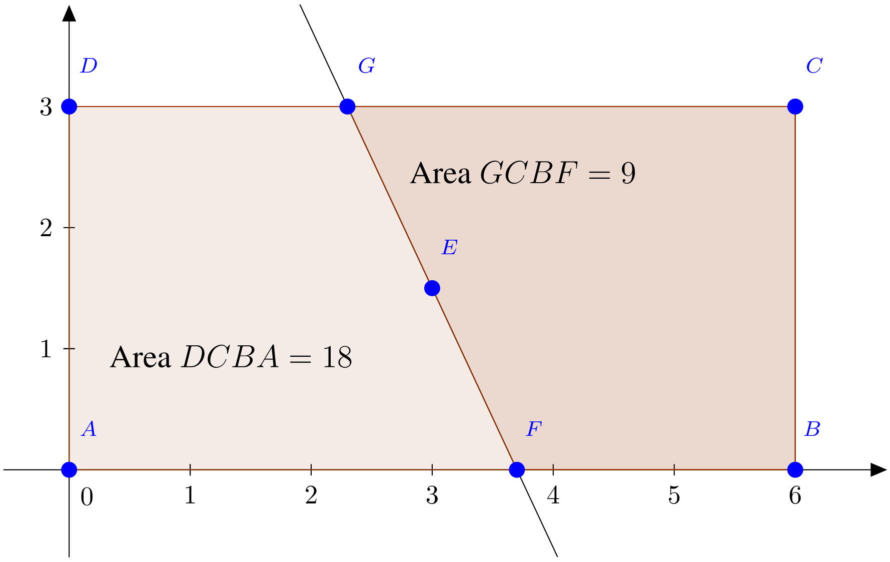

Figure 8. Any line through a rectangle's center divides the rectangle into two equal areas

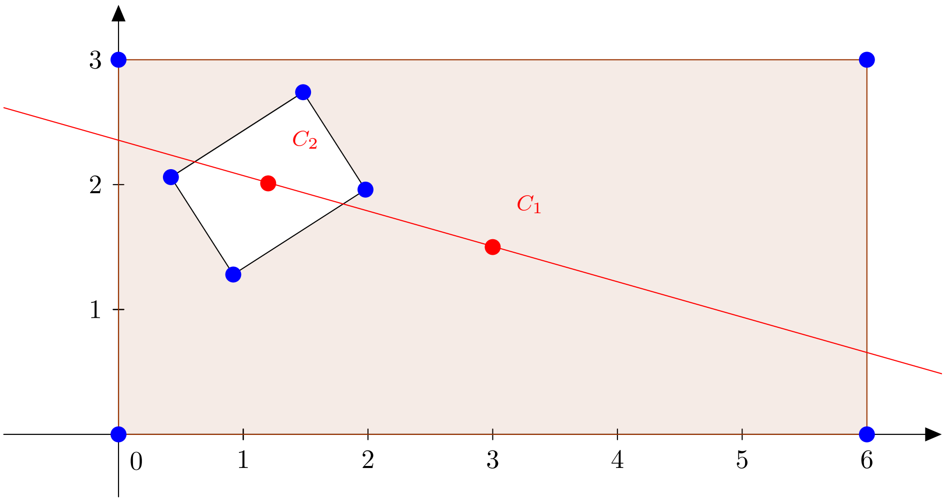

Figure 9. Another cut using the center property of rectangles

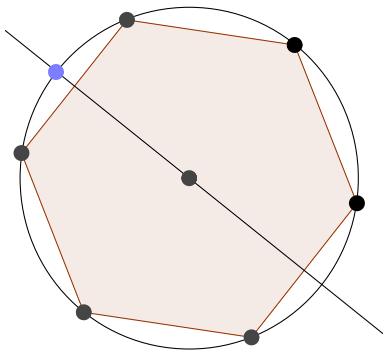

Figure 10. A circle and regular hexagon also have the center property

James Tanton [2] provides a proof that all polygons with the center property are exactly those that have opposite sides parallel and of equal length. Thus, shapes such as parallelograms or regular $n-$gons with an even number of sides too have this property. The benefit of the center property is that similar puzzles such as those in Figure 11 can be solved at least one way by connecting centers.

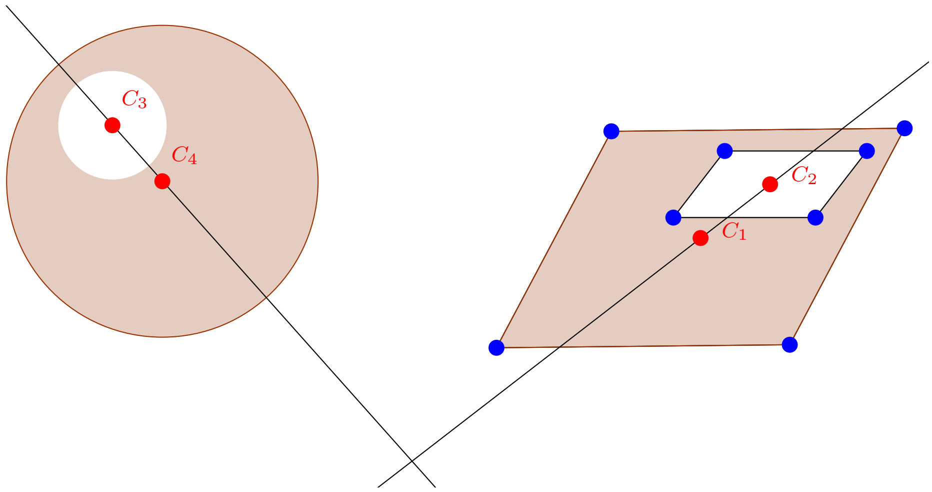

Figure 11. Solving circle and parallelogram puzzles using the center property

A last suggested extension is to analyze the same problem, but in three dimensions. While this sounds much too complicated to analyze, *GeoGebra 3D* makes it realizable. Figure 12 shows a rectangular prism with another rectangular prism removed from its corner. What planar cuts can subdivide the solid into two solids of equal volume?

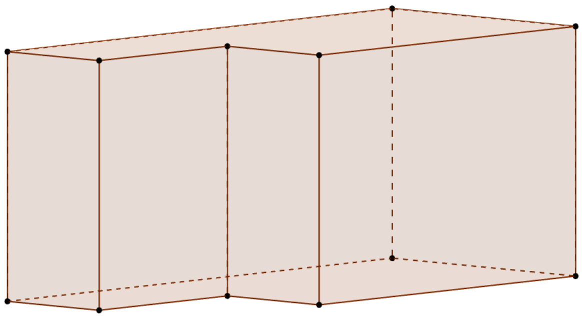

Figure 12. The brownie problem visualized using *GeoGebra 3D*

One obvious cut is to construct a plane parallel to the top and bottom faces through the midpoint of a lateral edge as shown in Figure 13. Similar to the problem in 2–D, do there exist "brownie points," namely those points that generate an infinite number of cuts by rotation? How would this problem differ if other solids were used, such as cylinders, spheres, or cones? It will be left to the reader to experiment with *GeoGebra 3D* to investigate such rich questions.

2 Conclusions

In closing, we have explored a very open-ended task, moving from specific cases to general contexts, from 2–D to 3–D, all with the help of *GeoGebra*. Dynamic geometry software grants us not only

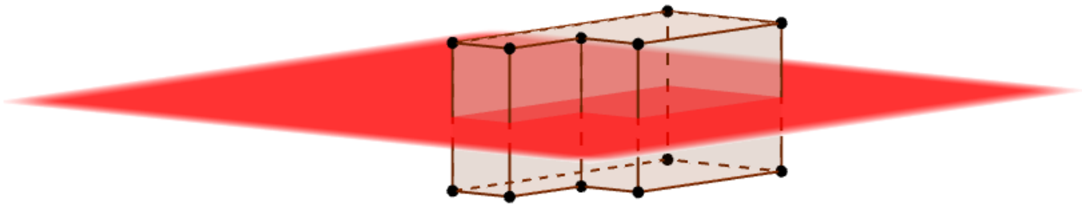

Figure 13. A planar cut bisecting an edge drawn parallel to the base

the power of visualization, but also the ability to move from the concrete to the abstract. Consider investigating any part of this brownie task with your students or create other questions centered around the same context.

References

[1] MTC Session Materials (2013). In *Math Teachers' Circle Network*. Retrieved July 23, 2013 from http://www.mathematicscircle.org/resources/sessionmaterials.html.

[2] Tanton, J. (2012). A Brownie Puzzle. In *Tanton Mathematics*. Retrieved July 23, 2013 from http://www.youtube.com/watch?v=gDO4PJsYRcg.

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