

GeoGebra as a tool to explore, conjecture, verify, justify, and prove: The case of a circle

Yip-Cheung Chan

Abstract: GeoGebra is a good platform for experimentation, which supports the development of mathematical concepts and the abilities to explain geometric properties. In this paper, a series of GeoGebra activities which aims at consolidating students' understandings on the concepts of center and radius and developing the concepts of locus and perpendicular bisector through experiencing the process of exploring, conjecturing, verifying, justifying and proving is described.

Keywords: GeoGebra, circle, exploring, conjecturing, verifying, justifying, proving

1. INTRODUCTION

The purpose of this paper is to describe a series of GeoGebra activities that aim to develop students' geometric concepts and experiencing the process of exploring, conjecturing, verifying, justifying, and proving. This series of activities focuses on the following question, which is mentioned in Leung (2003) as a demonstration of the exploration power in dynamic geometry software.

The question: *Let A and B be two fixed points. How many circles can be constructed through A and B ?*

2. TARGET AND OBJECTIVE

The target groups of this series of activities are senior primary and junior secondary, i.e., Grades 5-10. It is assumed that the students have already known the basic concepts of a circle, including center and radius. This series of activities is designed for the students to work with the computer individually, or in pairs, or in small groups of 3-4. The objective of the activities is to consolidate their understandings on the concepts of center and radius, as well as develop new concepts such as locus and perpendicular bisector. Furthermore, it provides an opportunity for experiencing the process of exploring, conjecturing, verifying, justifying, and perhaps also proving.

3. THE ACTIVITIES

3.1. Activity 1

The simplest way to solve the question is to apply the command circle through three points on point A and point B . First select the tool as shown in Figure 1. The teacher

Yip-Cheung Chan is a faculty member in the Department of Curriculum and Instruction at The Chinese University of Hong Kong - Hong Kong, China (e-mail: mathchan@cuhk.edu.hk).

may ask the students to use this command and see what happens.

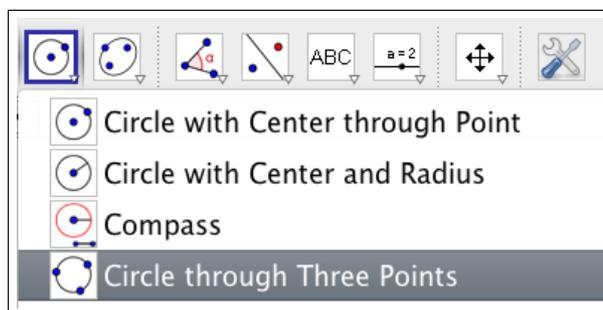


Fig 1: Selecting the **Circle Through 3 Points** Tool

Click on points A and B then move the cursor away from the points without clicking. Although the third point has not been given, a graphic preview of the final object (the circle) can be observed as shown in Figure 2.

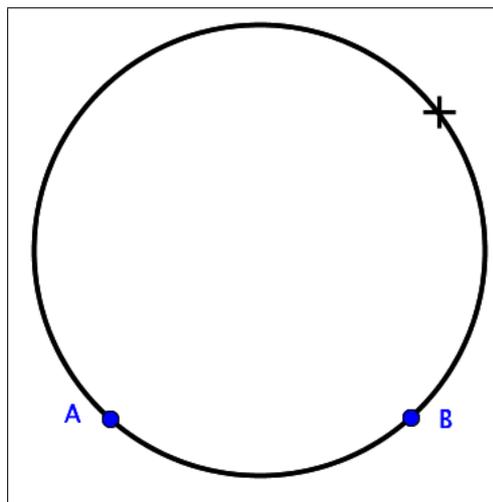


Fig 2: Apply the **Circle Through 3 Points** tool on *two* points A and B

By dragging the (previewed) circle randomly (without click-

ing to select the third point), one can observe that this circle varies. Indeed, the size of the circle looks to assume any values. This observation has already provided an answer to the question, namely there are infinitely many circles passing through A and B . It is natural to ask how these circles are alike or how they are different. In other words, it motivates us to investigate the relationship among these circles.

3.2. Activity 2

By reflecting on what is observed in Activity 1, we note that it is not easy to tell the values of the radii of the circles passing through the two points. As center and radius are basic concepts of a circle, it is natural to consider the positions of the centers instead of the values of the radii. Looking through the **Circle** menu, one will likely uncover the **Circle With Center Through Point** tool illustrated in Figure 3. This command first requires the selection of a

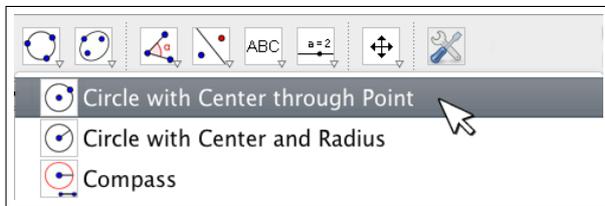


Fig 3: The **Circle With Center Through Point** tool

center point followed by the selection of a point on circle. As we do not have any idea on the position of the center point of our circle, we choose to implement a trial-and-error approach, selecting an arbitrary point as the center point C and the given point A as a point on circle. It is most likely that the circle produced does not pass through another given point B . However, we can drag point C so that point B appears to lie on the circle. This strategy of dragging point C is called “drag-to-fit” (Lopez-Real and Leung, 2006). (The teacher may like to teach the students this dragging technique first.) We may also try another “drag-to-fit” strategy. Let us produce one more circle by applying the **Circle With Center Through Point** command again with the same point C as center point and then point B as a point on circle. This approach is suggested in Figure 4.

Since we want to find a circle passing through both point A and point B , we drag point C so that the two circles overlap. While dragging point C , both circles will move accordingly. We may try to investigate the invariants under varying the position of C . What happens to the figure if the two circles continuously overlap while point C is dragged? By contrasting the configurations when the two circles coincide against non-coincide, it is not difficult to realize that point C should be located somewhere in the midway between A and B in order to keep the two circles

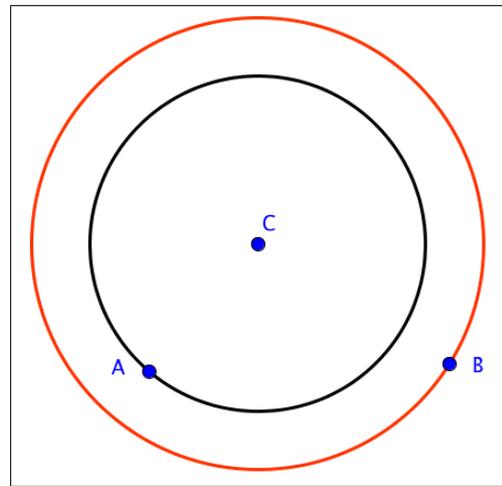


Fig 4: Two circles with center C that pass through A and B , respectively

coincide. We may try to associate this observation with our prior mathematical knowledge and may realize that \overline{CA} and \overline{CB} are actually the radii of the two circles.

3.3. Activity 3

Activity 2 motivates us to ask several follow-up questions. For instance, what happens if C is dragged midway between A and B (i.e., along the perpendicular bisector of \overline{AB})? What happens if C is dragged so that the two circles continuously overlap? In this case, will the path of C trace along the perpendicular bisector of \overline{AB} ? To answer these questions, we invoke “dummy-locus dragging” (Arzarello, Olivero, Paola and Robutti, 2002). First, we activate the trace of center point C by right-clicking on the point (**control** and clicking on a Mac) then choosing the **Trace On** option from the pop-up contextual menu as illustrated in Figure 5.

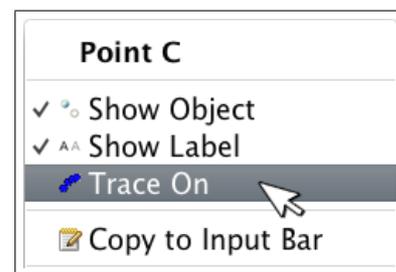


Fig 5: Activating tracing for point C with the **Trace On** contextual menu option

Then, depending on which question we ask, we drag the point by keeping C equidistant from A and B or keeping the two circles coincident. The trace command provides a powerful method to record the path of C and is useful for us in generalizing the geometric ideas. A possible locus

of center C , which we refer to as the “locus of validity” (Leung and Lopez-Real, 2002), is observed directly from the screen. This is shown in Figure 6.

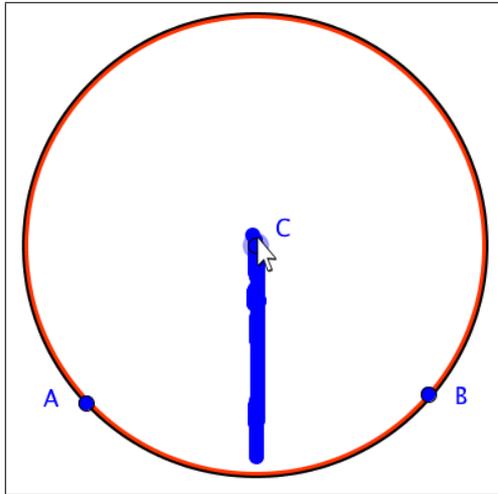


Fig 6: Trace of point C generated by attempting to keep two circles coincident while dragging C

The trace motivates a discussion of mathematical concepts such as locus, midpoint, perpendicular and perpendicular bisector.

3.4. Activity 4

In Activity 3, we generated a trace of possible locations of C when the two circles were held coincident. Afterwards, it is natural to ask students about the nature of the trace. “What is so special to the trace of point C ?” To answer such questions we begin by constructing a line to fit the trace. Students look for geometric relationships between points A and B and the trace of point C . The first observation that many students make is that C appears equidistant from A and B . To explore this conjecture, we construct \overline{AB} . As Figure 7 suggests, the segment appears to be perpendicular to the line.

At this point it is natural for the teacher to introduce the terminologies “midpoint” and “perpendicular.” The teacher may also use measurement commands to make the meanings of these concepts explicit (as suggested in Figure 7). If the students are mature enough, the teacher can also introduce the concept of perpendicular bisector and make the conjecture with the students, such as the locus of C is the perpendicular bisector of \overline{AB} .

3.5. Activity 5

In Activity 4, we conjectured that the locus of C was the perpendicular bisector of \overline{AB} . How can we verify this conjecture? Assuming the concepts of midpoint and perpen-

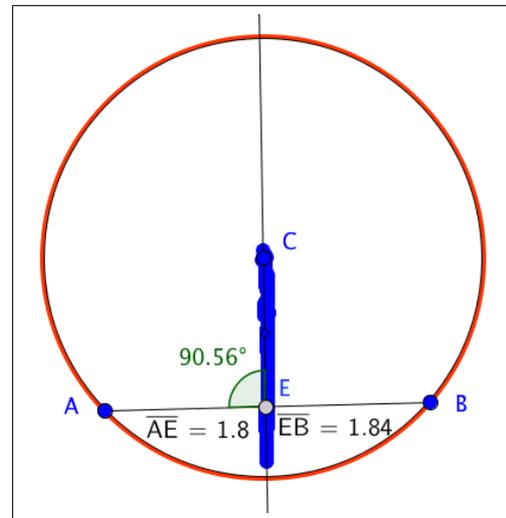


Fig 7: A line to fit the trace of C with accompanying measurements to encourage and support student conjectures

dicular line have been introduced, these geometric objects can be constructed in GeoGebra. If the perpendicular bisector is familiar to students, it can also be constructed directly by the GeoGebra **Perpendicular Bisector** menu option. If our conjecture is correct, this perpendicular bisector should be the locus of the center, C , of the desired circle.

To verify the conjecture, we create an arbitrary point C on the perpendicular bisector of \overline{AB} . Then, we construct a circle with center C and point A on the circle using the **Circle With Center Through Point** tool. As Figure 8 illustrates, point B lies on this circle.

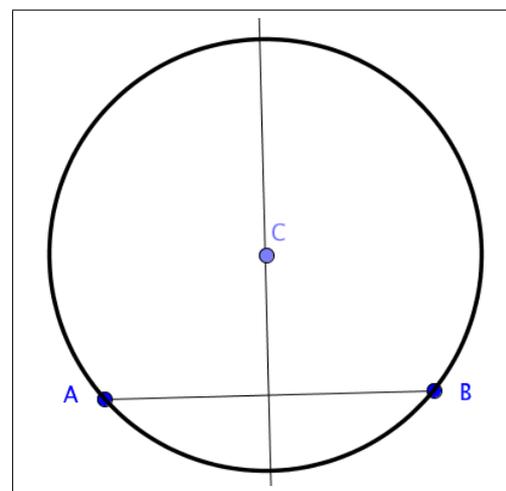


Fig 8: Informal verification the conjecture through the construction of a “robust” sketch.

We call the sketch depicted in Figure 8 “robust” because its essential properties hold regardless of the location of

points A , B , or C . As we drag point C , we see that points A and B remain on the circle. This process, which Arzarello, Olivero, Paola and Robutti (2002) refer to as “drag testing,” students may informally confirm or refute conjectures that they pose. In this case, “drag testing” provides strong evidence that a circle passing through points A and B can be constructed with a point on the perpendicular bisector of \overline{AB} as its center. The “drag test” approach is informal in the sense that it is based on numerous examples. “Drag testing” is an inductive approach. As such, there is always the possibility that a counterexample exists that we haven’t uncovered as we manipulate points in our sketch. Our next step is to prove the conjecture more rigorously, using a *deductive* argument.

3.6. Activity 6

In this activity, the students find relevant geometric relationships to begin to prove the relationships they observed in earlier lessons. Our aim here is to illustrate how technology may be used to *motivate* more formal proof rather than to provide an exhaustive proof.

The teacher may ask the students questions such as, “*What can we say if C is the center and points A and B lie on the circle?*” Many students note that the length of \overline{CA} equals the length of \overline{CB} because these two line segments are actually the radii of the same circle (just as all the spokes in a wheel have the same length). In other words, $\triangle ACB$ is isosceles. This relationship can be visualized if we construct \overline{CA} and \overline{CB} on the figure and then drag C along the perpendicular bisector as illustrated in Figure 9.

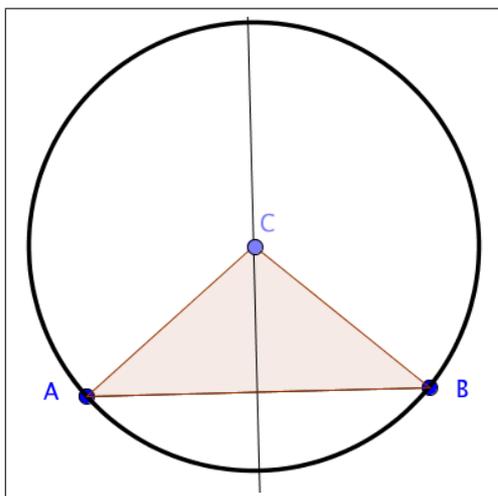


Fig 9: First note that $\triangle ACB$ is isosceles

To complete the proof, it remains to prove that if C is the center of an arbitrary circle passing through points A and B , then C must be lying on the perpendicular bisector of \overline{AB} . Let us construct a circle using the **Circle Through**

Three Points tool with A , B , and one more point, say D . Since D is a free object, this circle is an arbitrary circle passing through A and B . Let us construct the perpendicular bisector of \overline{AB} and the perpendicular bisector of \overline{AD} as shown in Figure 7.

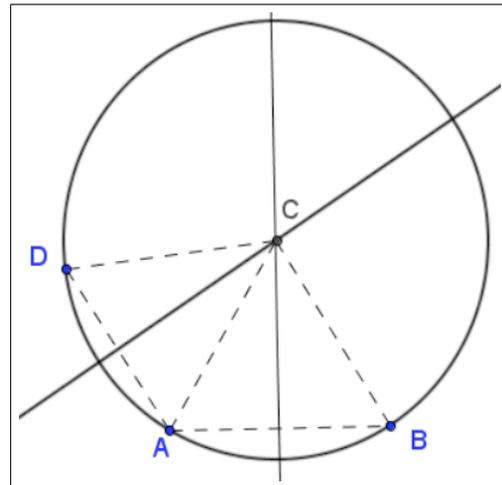


Fig 10: Complete the proof by considering an arbitrary circle passing through A and B

Figure 10 suggests that the intersection point C of these two perpendicular bisectors is the center of the circle. This is true because $\triangle DCA$ and $\triangle ACB$ are isosceles triangles. Hence, the center C of the circle lies on the perpendicular bisector of \overline{AB} . If the students are at higher level - say junior secondary - the teacher can guide the students to write a more rigorous, formal proof.

4. CONCLUSION

In this paper, we described a series of activities by using different tools available in GeoGebra. In Activity 1, the graphic preview of the circle motivates students to look for a *relationship*, namely how are the circles in the sketch alike or different? Through experimentation and *reflection* in Activity 1, we discover alternative explorations for investigation in Activity 2. Here we investigate the center of the circle instead. By means of drag-to-fit strategy, invariants are investigated. We ask questions such as “what happens if the two circles keep coincide while point C is dragged?” Using tracing to record the path of the circle center in Activity 3 provides a visual experience for generalizing the geometric ideas created in Activity 2. The geometric idea is deepened by introducing mathematical concepts such as locus, midpoint, perpendicular and perpendicular bisector. In Activity 4, drawing and measurement tools are used to motivate the formulation of student conjectures. In Activity 5, the conjecture is verified. The geometric situation is constructed using perpendicu-

lar bisector tools, and the property is verified using a “drag test.” In Activity 6, the conjecture is justified and informally proved through careful examination of relevant geometric relationships. Suitable prompting questions and auxiliary lines by construction (drawing) tool are useful for discovering the geometric relationship related to the proof.

The above discussion reveals that the tools available in GeoGebra can be used to actualize the reasoning with relationships, generalizing geometric ideas, and investigating invariants which are important in the process of exploring, conjecturing, verifying, justifying and proving (Driscoll, 2007). Interested readers are encouraged to try out these activities in their classrooms and evaluate the effectiveness and possible difficulties in implementation.

REFERENCES

- Arzarello, F., Olivero, F., Paola, D. & Robutti, O. (2002) A cognitive analysis of dragging practices in Cabri environments. *Zentralblatt für Didaktik der Mathematik*, 34(3), 66-72.
- Driscoll, M. (2007). *Fostering Geometric Thinking Ð A Guide for Teachers, Grades 5-10*. Portsmouth: Heinemann.
- Leung, A. (2003). Dynamic geometry and the theory of variation. In N.A. Pateman, B.J. Dougherty & J. Zilliox (Eds.), *Proceedings of PME 27: Psychology of Mathematics Education 27th International Conference* (Vol. 3, pp. 195-202). Hawaii, USA.
- Leung, A. & Lopez-Real, F. (2002). Theorem justification and acquisition in dynamic geometry: a case of proof by contradiction. *International Journal of Computers for Mathematical Learning*, 7, 145-165.
- Lopez-Real, F. & Leung, A. (2006). Dragging as a conceptual tool in dynamic geometry environments. *International Journal Mathematics Education in Science and Technology*, 37(6), 665-679.