GeoGebra in financial education

Roman Hašek and Vladimíra Petrášková

Abstract: In this paper, the authors discuss the use of GeoGebra in the teaching of financial issues in basic and secondary schools and in teacher preparation courses. By means of example, the authors illustrate the use of GeoGebra as a vehicle for explaining present-day financial issues while providing students with skills necessary to make well-informed monetary decisions. The dynamic features of GeoGebra encourage experimentation and deep conceptual understanding of the mathematics underpinning various financial formulas.

Keywords: reflection, primary education, GeoGebra, transformations, dynamic geometry

1. INTRODUCTION

Fostering the development and improvement of financial literacy among students is one of the aims of basic and secondary education in the Czech Republic (OECD, 2005). To be successful in life beyond school, graduates should be familiar with all typical contemporary financial issues; they should have all necessary skills to manage money responsibly and make reliable personal financial decisions. These requirements are properly reflected in basic and secondary school mathematics and civics curriculums (VÚP, 2009a; VÚP, 2009b). For these reasons, financial education is an emphasis within both pre-service and in-service teacher preparation courses in schools of education across the Czech Republic.

The authors of this paper have participated in the production of various teaching materials devoted to issues of financial education (Dvořáková and Smrčka, 2011; Hašek and Petrášková, 2010; Petrášková and Hašek, 2012). In these materials, we've utilized GeoGebra as a means for presenting financial phenomena and for solving numerous applications-oriented problems. For example, in Dvořáková and Smrčka (2011) the graphical tools of GeoGebra were used to illustrate the relation between compound and simple interest, as shown in the next example.

Example 1 (Compound vs. simple interest)**:** Suppose you deposit $1000 into an account that pays a 5 % annual interest rate. Determine the total account balance after five years of saving using both compound and simple interest. Solve the problem for various interest periods per year (e.g., consider the payment of the interest for every half year, every month, week, day, etc.). Describe the effect of the growing number of interest periods per year.

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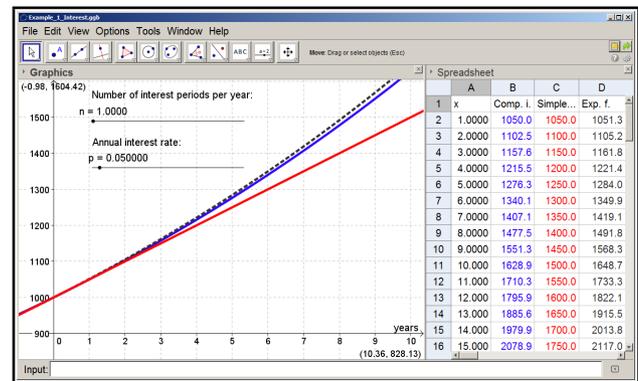
Fig 1: GeoGebra textbook example (Dvořáková and Smrčka, 2011)

As Figure 1 suggests, exploration of the task within a GeoGebra spreadsheet using sliders helps students explore the difference between the simple and compound interest. The dynamic nature of the sketch - in particular, the use of sliders to vary the number of compounding periods - enables students to uncover a number of noteworthy mathematical features related to the task. For example, as the number of interest periods per year increases, the relationship between the account balance and the number of years more closely approximates an exponential function. We explore the use of sliders and methods for linking spreadsheets to graphical elements in GeoGebra sketches in the *Examples* section of this paper. The interested reader is also encouraged to refer to Hašek (2012).

2. GEOGEBRA AS A MEANS OF FINANCIAL EDUCATION

Arguably, the spreadsheet is the single most important type of software for solving financial problems. Spreadsheets provide enormous capacity for conducting quantitative analyses and, unlike traditional statistical programs, they provide users with the ability to change numerical values in cells and instantaneously observe the impact without tedious manual recalculation (Wikipedia, 2013). Spread-

sheets provide users with a variety of tools to solve common financial problems such as creating a budget or comparing advantages and disadvantages of various financial services.

In addition to the utility of spreadsheets in business contexts, the software plays an important role in school mathematics classrooms. As students change values within cells of the spreadsheet, they gain a better understanding of variables, parameters, and functions. The role of spreadsheets as a medium for experimentation in school classrooms is illustrated by the following example.

Example 2 (Installment plan)**:** Create your own installment plan for a \$20,000 loan with a monthly interest rate of 2.22 % and installments of \$1,600. Determine total amount of the loan debt assuming no additional administration fees.

A solution to the above example using a Microsoft *Excel* spreadsheet is shown in Figure 2.

	A	B	C	D	E	F
1		**Loan - Installment plan**				
2						
3	Interest rate:		0,0222			
4	Installment:		1600,00	Total:	23688,00	
5						
6	**Period**	**Installment**	**Interest**	**Amortization**	**Balance**	
7	0				20000,00	
8	1	1600	444,00	1156,00	18844,00	
9	2	1600	418,34	1181,66	17662,34	
10	3	1600	392,10	1207,90	16454,44	
11	4	1600	365,29	1234,71	15219,73	
12	5	1600	337,88	1262,12	13957,61	
13	6	1600	309,86	1290,14	12667,47	
14	7	1600	281,22	1318,78	11348,68	
15	8	1600	251,94	1348,06	10000,62	
16	9	1600	222,01	1377,99	8622,64	
17	10	1600	191,42	1408,58	7214,06	
18	11	1600	160,15	1439,85	5774,21	
19	12	1600	128,19	1471,81	4302,40	
20	13	1600	95,51	1504,49	2797,91	
21	14	1600	62,11	1537,89	1260,03	
22	15	1288,0004	27,97	1260,03	0,00	
23						
24						

Fig 2: Installment plan in Microsoft *Excel*

The Czech national curriculum mandates that students at all levels use spreadsheets to solve practical problems. A number of mathematics teachers and students have opted to use the GeoGebra spreadsheet because of its strong multiple-representational capabilities and immediate graphical representation of content. Unlike *Excel* and *Calc*, the GeoGebra spreadsheet provides users with the ability to control cells with sliders and includes rich computer algebra (CAS) abilities. Moreover, GeoGebra includes simple export features and compatibility with Microsoft *Excel*. Thus, GeoGebra provides mathematics teachers and students with a robust alternative to *Excel* and *Calc* for teaching and learning. Although we do not see GeoGebra as a replacement for *Excel* and *Calc*, the software's unique features provide a worthwhile complement for these well-established business applications.

In the following section, we illustrate the usefulness of GeoGebra in the exploration of various financial mathematics applications. The examples we provide have been used successfully with classroom teachers and middle grades and lower-level secondary students.

3. EXAMPLES

3.1. Doubling an investment

Example 3: A clay tablet from Mesopotamia, dated to about 1700 B.C. and now in the Louvre, poses the following problem: "How long will it take for a sum of money to double if invested at a 20% interest rate compounded annually?" (Maor, 1998). Solve the given problem.

This seemingly simple problem can play different educational roles depending on students' familiarity with interest rates and their capacity to consider problems abstractly using algebraic methods. We have assigned the problem in both basic and secondary schools with great success. Below, we discuss some classroom observations.

Pupils of the seventh grade of basic school (13–14 years old) mostly solved the example as a simple interest problem. Typical solutions were as follows: "If you add 20 % of the initial investment five times you have 200 %, therefore the doubling of an investment will take five years." In this situation, the differing interpretations of interest (namely, simple versus compounded interest) provide us with a useful springboard for whole-class conversations leading to a comparison of linear and exponential growth. Since a significant number of pupils were also confused by the absence of an initial deposit amount, we begin by addressing this within GeoGebra.

First, we ask students to create a spreadsheet in GeoGebra with two columns, namely `Year` and `Savings`. The `Year` column contains the number of years that have elapsed since the initial investment. As Figure 3 suggests, students construct this column by typing the initial years, 0, into cell $A2$ then defining cell $A3$ as $A2 + 1$.

	A	B
1	**Year**	**Savings**
2	0	
3	= A2 +1	
4		
5		
6		

Fig 3: Defining `Year` column

By dragging the definition of cell $A3$ down to subsequent cells in column A, students define the contents of column A recursively. A similar approach is used to define the sav-

ings column, B. Students type an arbitrary initial investment into cell $B2$, then define cell $B3$ as $B2 + 0.20 * B2$ (i.e., the amount from the previous year plus 20% interest). This is step shown in Figure 4.

Fig 4: Defining `Savings` column

We purposely choose to define $B3$ in a manner that generates compound interest since doing so encourages student discussion and debate. Dragging down the definition of cell $B3$, students quickly discover that their initial solution conflicts with the one provided by GeoGebra. According to the spreadsheet, the investment doubles in slightly less than 4 years regardless of the initial amount of the investment. This is illustrated in Figure 5.

Fig 5: Completed spreadsheet

To help students visualize the growth of the investment, we instruct them to make the GeoGebra `Graphics View` visible. Within the spreadsheet, students highlight the `Year` and `Savings` columns, then select the `Create List of Points` menu option as shown in Figure 6.
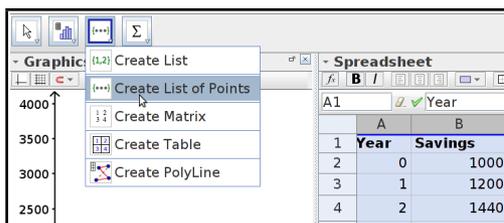

Fig 6: Generating a plot from spreadsheet columns

This generates the plot of `Savings` with respect to `Year` shown in Figure 7.

We ask our students to consider why their initial approach and ours yield different solutions. To help students understand the distinction between their approach (simple interest) and our approach (compound interest), we have them create a second list of data using their initial method. We begin by relabeling the `Savings` column in our spreadsheet as `Compound` and adding a new column, `Simple`.
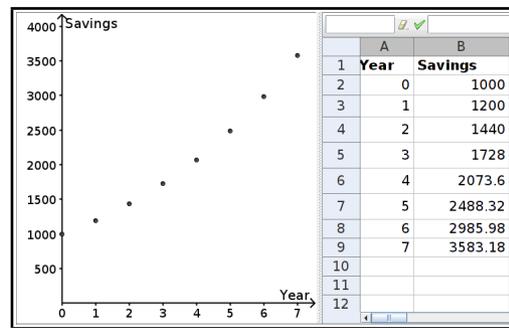

Fig 7: Simultaneous view of investment data as spreadsheet and graph

To generate values within the `Simple` column, we define $B3$ as $B2 + 0.20 * 1000$ and fill down as suggested in illustrated in Figure 8.

Fig 8: Revised spreadsheet with simple and compound columns

A simultaneous plot of the two investment scenarios, such as that shown in Figure 9, helps students recognize that the value of the investment grows at a constant rate with simple interest. On the other hand, the value added to the `Compound` column increases each month since compound interest earns *interest on interest*.
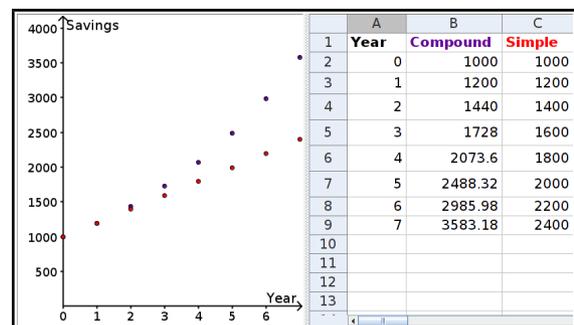

Fig 9: Simultaneous plot of compound and simple interest scenarios

The example in Figure 9 illustrates that an initial investment of $1000 doubles in slightly less than 4 years, whereas simple compounding doubles the principal in precisely 5 years. What if the principal were a value other than $1000? How would doubling time be impacted? A slider is a valuable tool for exploring such questions within GeoGebra.

Using the `Slider` tool, we create slider D with values

ranging from 0 to 5000 and increment 10 (refer to Figure 10). D will control the initial amount of money invested (i.e., the principal).
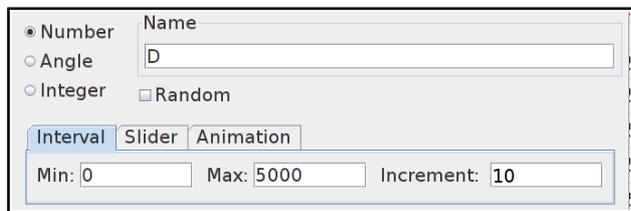

Fig 10: Creating a slider for the principal, *D*

We redefine the cells *B*2 and *C*2 of our spreadsheet (i.e., those containing initial investment information) to equal the value of slider *D*. As students move the slider, the values in `Simple` and `Compound` columns are automatically updated to reflect the new initial value.

Fig 11: Linking slider *D* to the spreadsheet

As students drag on the slider, they recognize that the value of the initial investment is independent of doubling time. For *any* value of *D*, compounding doubles the principal in slightly less than 4 years whereas simple interest doubles the principal in precisely 5 years.

Are *interest rate* and doubling time independent? As Figure 12 illustrates, second slider, *r*, can be constructed to explore this question.
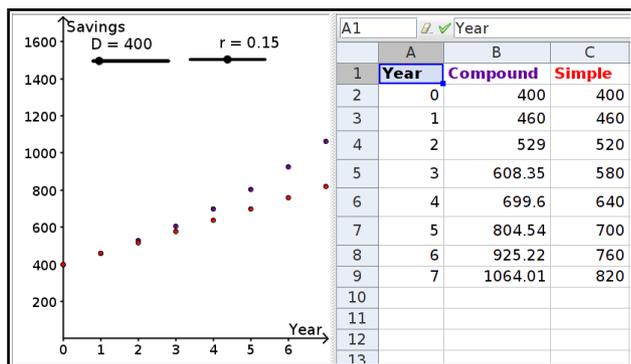

Fig 12: Linking slider *D* to the spreadsheet

Clearly, Figure 12 indicates that interest rate and doubling time are *not* independent. With sliders, it is apparent that doubling time increases as interest rates fall.

As the previous examples illustrate, sliders are useful for exploring relationships among variables. Although slid-

ers are essentially variables, they are far less abstract than conventional variables for students. Sliders provide students with a means to generate and test conjectures in a concrete, hands-on manner.

Once students understand that doubling the principal does not take exactly four years, they begin to ask: "How could we calculate the doubling time more precisely?" In secondary school, such questions motivate an introduction to exponential functions, exponential equations, and logarithms. Through an analysis of recursive formulas generated by compounding, teachers help students understand that an investment that grows by 20% (i.e., by a factor of 1.2) every year results in total growth of $s(x) = 1.2^x$ after x years. To calculate the exact period of doubling, we solve the equation

$$2 = 1.2^x. \tag{1}$$

To solve this equation, students can use logarithms or, alterntatively, take advantage of the tools of GeoGebra. Using the `Intersect Two Objects` tool, students approximate the solution using graphical methods. As Figure 13 illustrates, the doubling takes approximately $x = 3.8$ years.
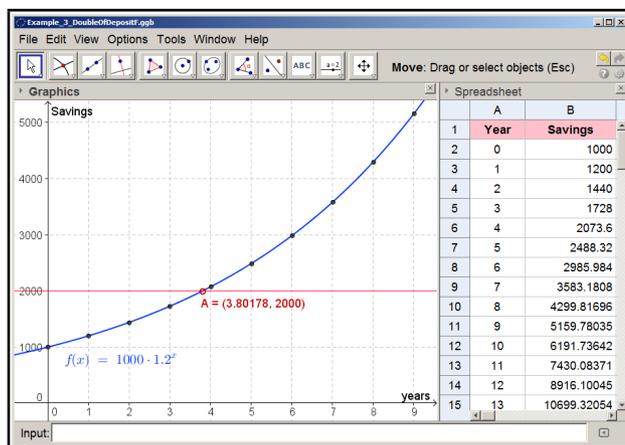

Fig 13: Finding the doubling time graphically

3.2. Installment plan

Now, let us further investigate the problem posed in example 2. As we've discussed previously, experimentation with input values can help students more fully grasp the mathematics underlying common financial calculations.

For example, each installment of a loan is usually split into two different parts, interest and amortization (refer to Figures 2 and 14). The portion of the interest (shown in pink) is greatest at the beginning of a loan repayment then gradually decreases to the benefit of the amortization of the debt. The rate of this change of the installment proportions is given by the input parameters of a loan and is crucial for its repayment. In Figure 14, the
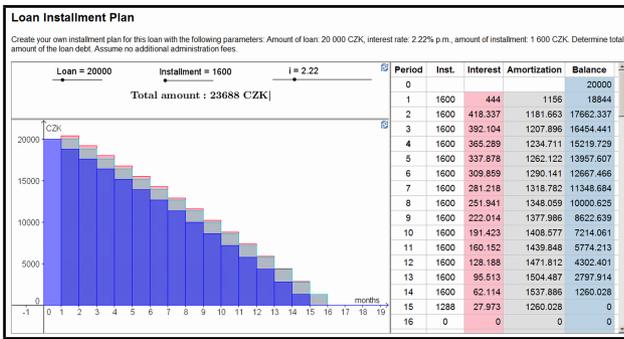
Fig 14: Dynamic worksheet to inquire into the functioning of installments and interest

cells of the respective spreadsheet are filled in as follows: C3: E2 i/100, D3: B3-C3, E3: E2(1+i/100)-B3, etc. The charts were made by the application of the function `Histogram` to the data taken from the spreadsheet. Sliders, which control the values of the loan, installment, and interest, allow a user to change the parameters of the task dynamically and to immediately follow the consequences of these changes.

Once the worksheet in Figure 14 has been constructed by teacher or students, it can be used for other tasks. For example, the gradual change of proportions between the interest and amortization gives us the answer to the question: "Why do mortgage providers require the fixing of the interest rate for several years at the beginning of the mortgage contract?" Another problem, which is developed from the Example 2, is given below.

Example 4: Find the monthly interest rate of a $20,000 loan with 24 monthly installments of $1,600.

The simplest way to the approximate solution to the example is to use the dynamic worksheet from Figure 14.
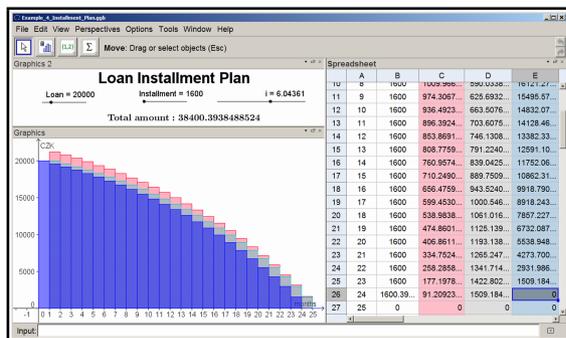

Fig 15: Graphical solution to Example 4

Refer to Figure 15. According to the assignment we first use the appropriate sliders to set the values of the loan and the installment amount. Then, manipulating the interest

rate's slider we find the approximate value $i = 6.04\%$ corresponding to the given number of 24 payments.

Another way to solve the problem is to find the respective mathematical formula that would relate the given parameters' values to the unknown interest rate i. This approach is illustrated in Figure 16.
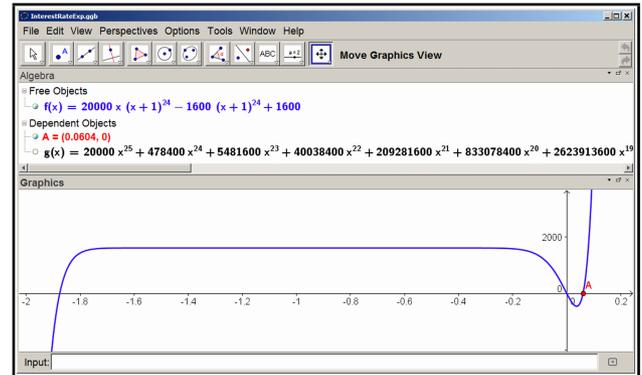

Fig 16: Graphical solution to the polynomial equation (5)

Let D mean the total amount of the debt, a is the installment amount, n is the number of installments and i is the monthly interest rate in its decimal form. Then if we compute the interest from the credit balance $n-$times and pay the installment we zero the debt:

$$(...((D\cdot(1+i)-a)\cdot(1+i)-a)\cdot(1+i)...)\cdot(1+i)-a=0. \tag{2}$$

After expanding the left side of (2) and rearranging the resulting terms we get

$$D\cdot(1+i)^n = a\cdot(1+i)^{n-1}+a\cdot(1+i)^{n-2}+...+a. \tag{3}$$

The terms on the right side of (3) form the geometric progression. Therefore it can be simplified to the form

$$D\cdot(1+i)^n = a\frac{(1+i)^n-1}{i}, \tag{4}$$

or, after removal of fractions

$$D\cdot(1+i)^n i - a(1+i)^n + a = 0. \tag{5}$$

Since i is unknown the latter equation leads to the polynomial of degree $n+1$. Therefore, the problem stated in Example 4 is described by the polynomial equation of degree 25. Its approximate solution, the $x-$coordinate of the zero point $A = (0.0604,0)$, is presented in Figure 16. In contrast to other spreadsheets, GeoGebra also provides a user with CAS tools for solving problems using symbolic algebra. The solution to the equation in GeoGebra CAS View is illustrated in Figure 17.

4. CONCLUSION

In this paper, we've illustrated ways in which GeoGebra can help to explain the underlying mathematics of typical
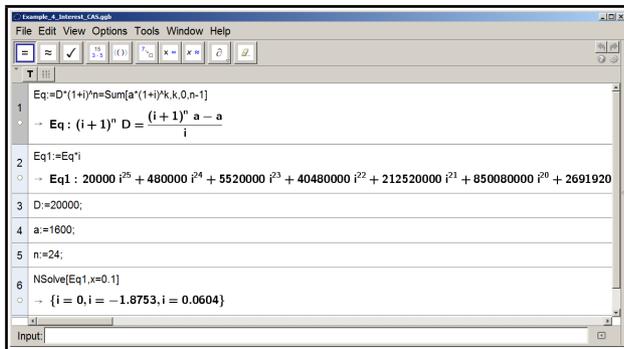
Fig 17: Solving the polynomial equation (5) with Geo-Gebra CAS

present-day financial concepts in a way that is accessible and engaging for school students. In our opinion, Geo-Gebra is a worthwhile tool in the study of financial math in the middle and secondary levels. GeoGebra includes a number of features - most notably, sliders and CAS capabilities - that aren't available in typical spreadsheet programs such as *Excel* or *Calc*. These features provide students with significant tools for experimentation, transforming the traditional classroom into one which encourages active discussion, conjecture, and hypothesis testing. The tools are also useful for teachers in the preparation of the efficient educational materials.

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