Study of functions in a GeoGebra environment during "learning week"

Antonio Criscuolo and Adriana Gnudi

Abstract: In this paper, the authors summarize their experience using GeoGebra during an intensive, one-week course (i.e., "learning week"). High school students engaged in collaborative problem-solving activities and explorations involving the limits and derivatives of elementary functions and function graphs. The "hide and show" features of GeoGebra were used to conceal (and reveal) analytic function forms from students. The final assessment and evaluation test suggests that GeoGebra is a useful tool in the study of function and calculus concepts.

Keywords: function, calculus, problem solving

1. CONTEXT AND AIMS OF THE COURSE

During the last few years, the MatNet team at Bergamo (Italy) University has worked with area high schools to offer mathematics instruction for students and professional development for in-service teachers. In June 2012, the team worked with twenty-four high school seniors from E. Majorana Technical School in an intensive "learning week" on campus. In the following paper, we describe MatNet's work with these students. In particular, we describe ways in which the team used GeoGebra to build student understanding of function.

At the end of the previous school year, ten of the 24 learning week participants were identified as having an "educational deficit" in mathematics by their teachers. The remaining 14 students were asked to participate in order to review and remediate mathematical skills. Learning week is focused on the exploration of mathematics content that has been previously studied - but not mastered - by students. Because conventional teaching methods were not entirely successful with students during the school year, the MatNet team typically revisits content using alternative approaches that stress collaborative problem solving and the use of technology to explore important mathematical concepts. The learning week course was overseen by a MatNet professor from the University of Bergamo along with a mathematics teacher from E. Majorana Technical School. The school teacher created and conducted a number of classroom activities; in addition, the teacher helped construct and administer entrance and exit tests during the week.

The primary goal of the learning week was to strengthen students' conceptual understanding of function and its application. To reach these goals, the instructors found it helpful to create activities and explorations that required students to "make sense" of mathematical objects. During the week, "Number sense" is taken to mean the knowledge of integers, decimals, fractions, irrationals, and the ability to operate with them, as well as the estimation skills for an order-of-magnitude, error, significant and decimal figures and the ability to determine a percentage or to make approximate calculations (Sowder, 1992). "Symbol sense," is taken to mean the ability to read, using symbols, the relations between variables, and to choose the best way to display them (Arcavi, 1994). "Graph sense" indicates both the ability to read and interpret graphs and the ability to present data and data equation functions (Robutti, 2003). As students worked to develop their mathematical senses, they worked in collaborative settings that fostered the development of their metacognitive skills, mathematical curiosity, and positive attitudes.

Sense-making skills are particularly limited among pupils with learning difficulties. Shortcomings in these areas are revealed when students grapple with concepts that require a high level of abstraction (such as limits and derivatives in calculus). The multi-representational capabilities of GeoGebra - with algebraic, graphical and spreadsheet views - enable students to explore abstract concepts in multiple ways, stimulating various "mathematical senses" for each topic. Exploring ideas from multiple representations helps students make abstract ideas, such as those related to functions and calculus, more concrete.

For instance, as students explore the concept of limit in GeoGebra, a dynamic notion of the concept (potential infinity) is fostered - one that is more accessible to secondary school students than static representations (actual infinity, epsilon-delta definition) commonly found in school texts. Research (Maschietto, 2006; Paola, 2005) has shown that manipulating graph presentations is useful for learning basic concepts of mathematical analysis - primarily

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GeoGebra constructions relating to this activity and others may be downloaded from GeoGebraTube at: www.geogebratube.org/collection/show/id/1826 because the technology allows the student to interact with concepts on a more concrete level. Software is also useful when studying the concept of derivative: a dynamic approach using GeoGebra allows perceptive-active learning and the construction of mental images and meanings leading to conceptualization (Paola, 2005). Students, particularly those with learning difficulties, fail to achieve this conceptualization with logical, formal approaches and procedures.

2. GENERATING A GRAPH OF A RATIONAL FUNCTION

In the following section, we describe an in-class activity that we designed to strengthen student understanding of function. In the activity, students construct a graph of a rational function from its analytical (i.e., symbolic) form. Provided with only an equation, students use an "identikit" - mathematical "clues" gleaned from the analytical form - to construct the graphical form of the function within GeoGebra.

Students work in pairs, applying analytical methods to construct the graph of a function from its analytic form. Using the construction protocol feature built into GeoGebra, students analyze (and interact with) the graph construction in a step-by-step manner. Assisted by university student tutors, the student teams sketch predictions by hand at each stage of the process, comparing their predictions with additional information provided by the construction protocol. A final discussion of the curve sketching process is facilitated by the classroom teacher.

The purpose of the graph construction activity is to foster the development of a general graph sketching procedure. As students construct their graphs, GeoGebra makes computations and representations of data more accessible. Freed from burdensome computational tasks, students are provided with additional time to reflect on their predictions, on the meaning of each step of the process, on the sequence of steps that they invoke, and on their interpretations of the final graphs.

As Figure 1 suggests, students are initially provided with the symbolic definition of function $y = \frac{3-x^2}{x+2}$. The function is defined as an auxiliary object; its graph is not immediately visible.

As the play buttons at the bottom of Figure 1 suggest, the sketch construction has been broken down into 7 discrete steps. Users advance to the next stage of the construction by clicking the "forward" button once. Likewise, the "rewind" button moves one back to previous stages of the construction. We encouraged our students to discuss and sketch by-hand predictions at each of these 7 stages using GeoGebra's built-in Pen tool.

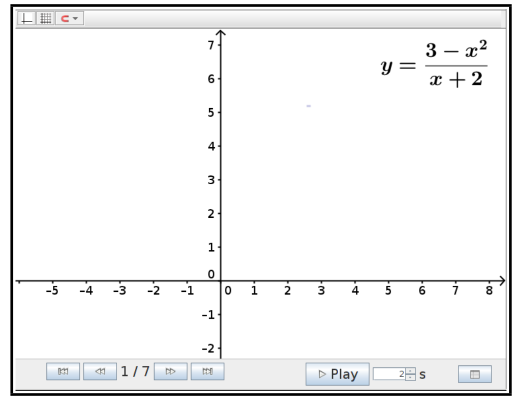

Fig 1: Initial problem

Figure 2 illustrates a student team's first attempt at constructing the graph of function $y = \frac{3-x^2}{x+2}$. The students conjectured that the graph would resemble $y = \frac{-x^2}{x} = -x$, noting that the constant terms in the numerator and denominator will have relatively impact on the shape of the graph for large values of x. One of the students noted that the function yields division by 0 when $x = -2$, so she placed an "open" circle at this location on the graph.

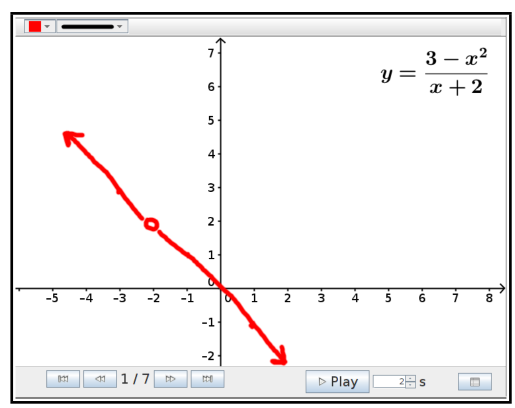

Fig 2: A first student conjecture generated with the GeoGebra Pen tool

After sketching hypothesized graphs and discussing them with university student tutors, students used the "forward" button to advance to the next stage of the construction. As Figure 3 illustrates, roots of the numerator (points N_1 and N_2) and denominator (D_1) and the y-intercept of the function (generated by evaluating the function at $x = 0$) are made visible. At this point, students are asked to discuss and revise their initial predictions.

Through conversation, students recognize that roots of the numerator are x-intercepts of the function. They revise their sketch to reflect this. Moreover, the position of N_1 on the x-axis leads students to rethink the shape of the sketch on the interval $(-2, -1)$. Students explore various shapes that can pass through points N_1, N_2 and *yint* without crossing the vertical asymptote at $x = -2$. As shown

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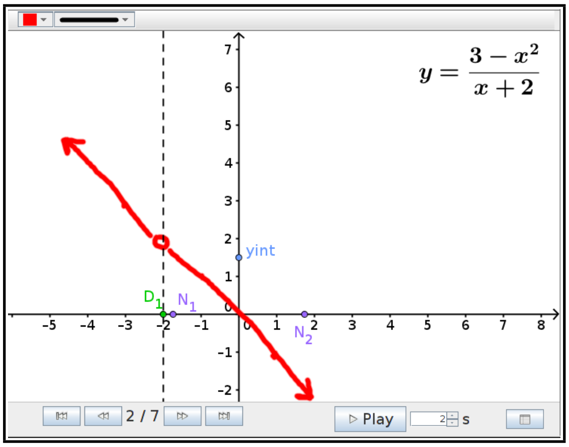


Fig 3: Student conjecture with intercept and roots of numerator and denominator visible

in Figure 4, students ultimately construct a "u-shape" on the interval $(-2, \infty)$. The portion of the graph on the interval $(-\infty, -2)$ is constructed to match the left-most portion of the "u-shape" and the original sketch as closely as possible.

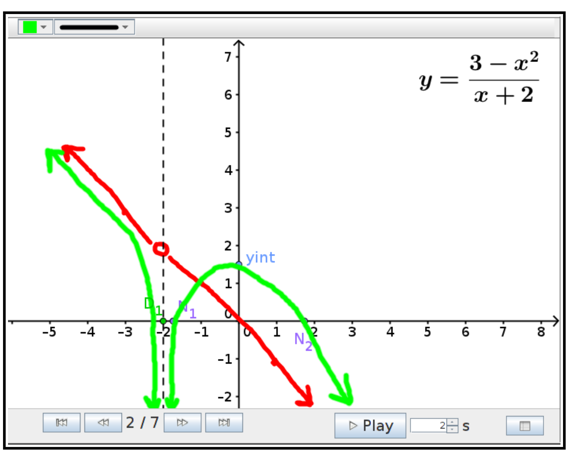


Fig 4: Revised conjecture based on intercept and root information

After students are comfortable with their revised conjectures, the university student tutor advances to the next step of the construction. In this step, we reveal the graph of $sgn\left(\frac{3-x^2}{x+2}\right)$ (depicted as a violet curve in Figure 5). This graph encourages students to consider the domain, zeros, and signs of the graph more carefully. Wherever the graph of $sgn\left(\frac{3-x^2}{x+2}\right)$ is negative, the sketch of the function $y = \frac{3-x^2}{x+2}$ should fall below the x-axis. Likewise, when the *sgn* graph is non-negative, the function sketch should fall on or above the x-axis.

After some discussion, students noted that the portion of their sketch from $(-2, \infty)$ was consistent with sign of the *sgn* function; however, the portion from $(-\infty, -2)$ was not. Although the *sgn* function was positive on this interval, their hypothesized graph (in green) dipped below the x-axis. Figure 6 illustrates revisions based on this information (shown in brown). Note that students are mindful of their original linear sketch as they make revisions.

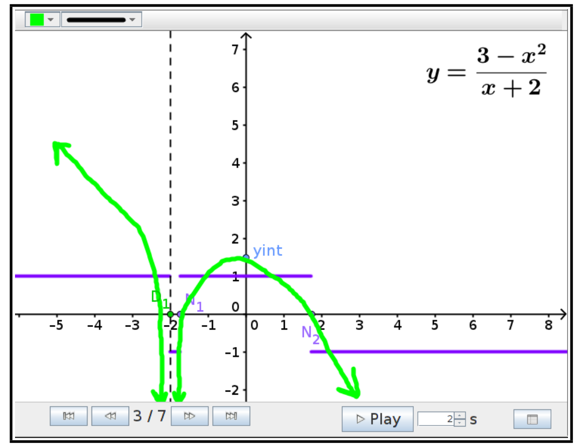


Fig 5: The next step of the construction reveals sign information

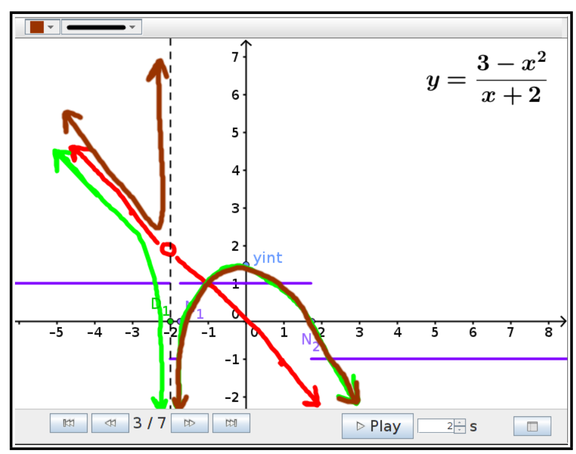


Fig 6: Revised conjecture based on sign information

In the next step of the construction, the oblique asymptote of the graph is revealed. Typically, construction of the oblique asymptote of a rationale function requires students to perform polynomial long division. With GeoGebra, such computations are relegated to the computer. This affords students with more time to consider the *process* of constructing an accurate function graph. As Figure 7 suggests, we hid the students' first two sketches by clicking on the hide button adjacent to each "stroke" object within `Algebra` view.

The students noted that the oblique asymptote was parallel (though not incident with) their original function sketch (the line $y = x$ with a removable discontinuity at $x = -2$). Furthermore, they noted that although the portion of their sketch from $(-2, \infty)$ fell within the restrictions of the vertical and oblique asymptotes, the portion from $(-\infty,-2)$ did not. As suggested in Figure 7, students shifted the mismatched portion of their sketch upward vertically (shown in pink) to conform to the newly revealed constraints.

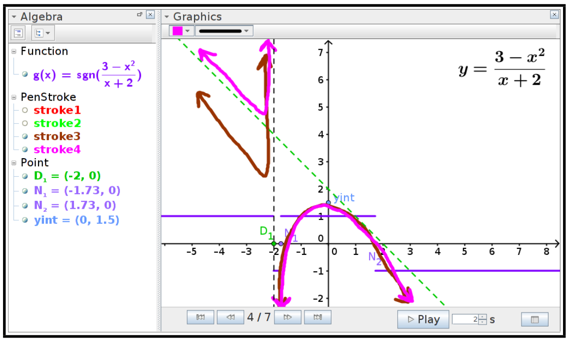


Fig 7: Revised conjecture based on oblique asymptote information

Next, students are shown the derivative of the original function, which they note is also a rational function. The roots of the numerator of the derivative are plotted as M_1 and M_2. These are locations at which the derivative is zero. A graph of the *sgn* function of the derivative is also revealed. This is shown in Figure 8.

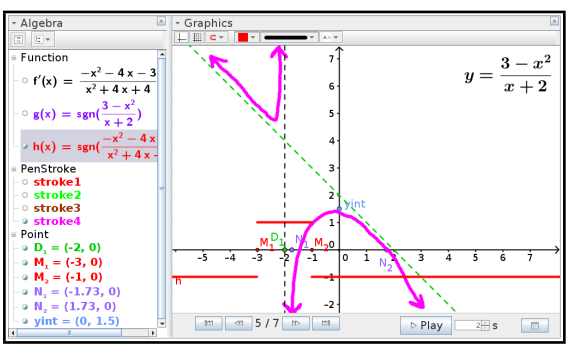


Fig 8: The next step of the construction reveals information about the derivative of the function

The university student tutors encourage students to interpret the location of M_1 and M_2, specifically asking how their location will impact the shape of the graph of $y = \frac{3-x^2}{x+2}$. The *sgn* graph of the derivative (labeled as $h(x)$ in Figure 8) helps students answer such questions. For instance, the sign of the derivative to the left of M_1 is negative; to the right, positive. This suggests that the function has a relative minimum (i.e., a "turning point") at M_1. A similar observation suggests a relative maximum at M_2. Students use this information to revise their sketch yet again, as shown in Figure 9.

In the 6th step of the construction, critical points of the function are shown; and in the 7th (and final) step, the graph of $y = \frac{3-x^2}{x+2}$ is ultimately revealed. These last two steps are suggested in Figure 10. Note how closely the

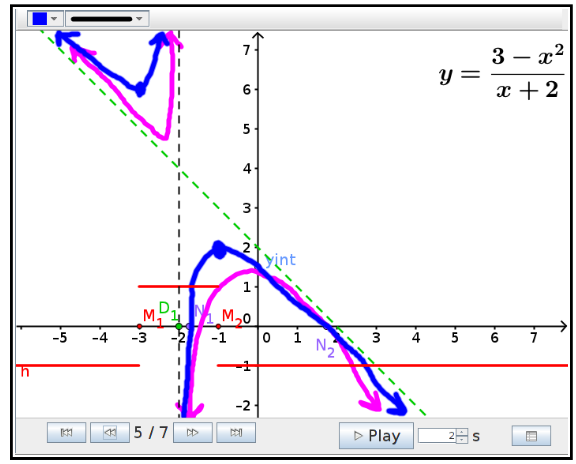


Fig 9: Revised sketch based on information about the derivative of the function

graph of the function matches the students' final conjecture.

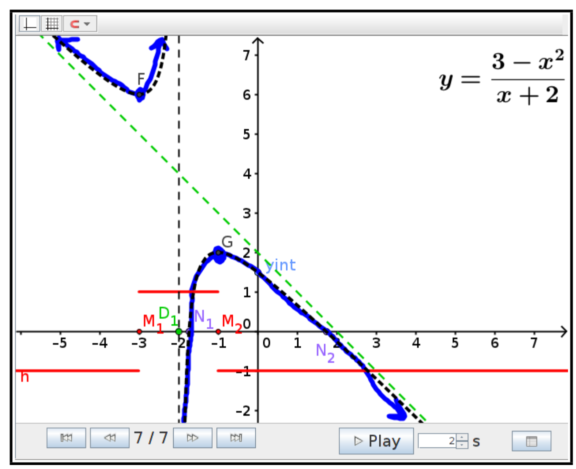


Fig 10: Graph of function compared with students' final sketch

After the final graph is revealed, students discuss the phases of constructing an accurate graph in a whole group setting facilitated by the classroom teacher. In particular, the following steps are stressed.

1. Determine the roots of the rational function. Roots of the denominator indicate locations of discontinuity; roots of the numerator indicate locations where the graph may cross the x-axis.

2. Determine the sign of the function on the intervals created by the roots of the rational function. The graph of the function must fall below the x-axis on intervals where the sign is negative. Likewise, the graph must fall on or above the x-axis where the sign is non-negative.

3. Determine the oblique asymptote of the rational func-

tion (if one exists). This asymptote, together with the function roots and y-intercept, will dictate the general shape of the graph.

4. Determine the derivative of the function. The roots of the numerator of the derivative will help determine where the function "turns" (i.e., where slope of the graph changes sign). The roots indicate where relative extrema and inflection points of the graph occur.

5. Determine y values of the function at the roots of the numerator of the derivative. The function will pass through these points.

In summary, note that during the aforementioned learning activity, students used GeoGebra's construction protocol and drawing features rather than the software's direct commands to construct their own graph in a step-by-step manner. Underneath the hood, our sketch made use of the following GeoGebra features: (1) the `Root` and `Intersection` commands; (2) the function `Sgn` (to determine the domain, to find the zeros and to study the signs); (3) the commands `Limit` and `Derivative`; and (4) the instrument **Pen**.

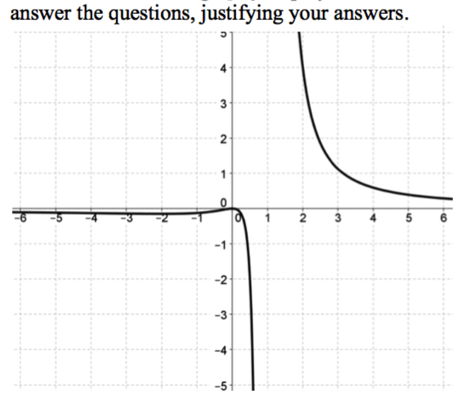
In the activity, the function graph was completely hidden from the student. Students were only provided with the function to be studied in its analytical form, defined as a hidden auxiliary object. Given the lack of general information derived from this (type, properties, possible symmetries) students have to rely solely on the correct application of the procedure and especially on the interpretation and synthesis of the information from the analysis. Using GeoGebra in this way, the study of a function is simplified in some ways, since by-hand computation is not required. At the same time, the students' work is more stimulating and demanding conceptually. Students must work to interpret information provided by GeoGebra and make use of "identikit" clues that ultimately lead to the construction of an accurate graph. In our opinion, these aspects are the most significant and conceptually important elements of the study of functions.

3. EXERCISES FOR THE ENTRANCE AND EXIT TEST

Following our hands-on explorations of function graphs, students carried out group activities aimed at consolidating their "graph sense" and "symbol sense" relating to the concept of function and how to display it.

In Figure 11, we provide an example of an exercise that we used with our students to encourage careful analysis of a function graph. The exercise was developed specifically to revisit and build upon previous analyses that students carried out in small groups with their university student tutor. Working individually to complete the task, students used GeoGebra to check their answers.

Consider the function graph f displayed below and answer the questions, justifying your answers.



a. What is the best approximate value of $f(2)$?

b. Find, if it exists, an x which makes $f(x)$ undefined.

c. Are there values for x for which the function has the same value?

d. In the interval displayed, how many solutions are there for the equation $f(x) = 2$? How many solutions are there for the equation $f(x) = -0.01$?

e. Using the graph, approximately identify:
   1. The intervals where the function is positive;
   2. The intervals where the function is increases or decreases monotonically;
   3. The potential maximum and minimum points

f. The graph displays a fractional-rational function. Which of the following could be its equation:

   1. $y=\dfrac{(x-1)^2}{x^3}$ 2. $y=\dfrac{x^2}{(x-1)^3}$

   3. $y=\dfrac{x}{(x-1)^2}$ 4. $y=\dfrac{x^2}{(x+1)^3}$

g. Determine precisely, if they exist, the maximum and minimum points of the function.

h. Determine, if they exist, the asymptotes of the function.

Fig 11: Exercise on the concepts of function and graph trends.

At the start and the end of learning week, our students completed similar pre- and post-experience questionnaires covering the basics of functions and Calculus. The 15 multiple-choice test items contained within each test were designed / selected by the classroom teacher according to academic standards set by E. Majorana Technical School.

As a group, student participants answered 40% of the entrance test questions correctly and 84% in the exit test questions correctly. We consider the increase a positive result, although we recognize that in the entry test students were less familiar with the material than they were at the conclusion of the week. Nevertheless, the test results suggest to us that our use of GeoGebra led to significant learning gains among a group of students who had struggled with the content during the regular school year.

Figure 12 displays several multiple-answer test questions asked at the end of learning week.

1 Out of the following functions, find the one that has an increasing trend over its entire existence:

A $y = -x^2 + 3$ C $y = \log_{\frac{1}{2}} x$

B $y = \dfrac{3}{x}$ D $y = \dfrac{1}{e^{-2x}}$

2 The function with a positive first derivative in $x_0 = 1$ is:

A $y = \dfrac{3}{x^2}$ C $y = x^3 - 3x$

B $y = e^{-x}$ D $y = \sqrt{x}$

3 The equation of the tangent line at the curve $y = 2x^2$ at point P(1,2) is:

A $y - 2 = 4 \cdot (x - 1)$ C $y - 2 = 1 \cdot (x - 1)$

B $y - 2 = 2 \cdot (x - 1)$ D $y - 1 = 4 \cdot (x - 2)$

4 Find the **incorrect** statement about the function: $y = \dfrac{x^3}{x^2 - 4}$

A it has vertical asymptotes

B the function has a horizontal asymptote *y=0*

C it is an odd function

D $y > 0$ if $x > 2$

Fig 12: Multiple-answer test questions asked at the end of the learning week.

4. SATISFACTION QUESTIONNAIRE

The satisfaction questionnaire, filled in at the end of the course, showed that 96% of students found the experience to be "interesting" or"very interesting," 88% found it "engaging" or "very engaging." With regard to using GeoGebra, 22 out of 24 students "agreed" or "strongly agreed" with the statement "I saw the usefulness of the software in studying mathematics."

Out of the various activities that were carried out, the highest level of satisfaction was for the computer guided exercises with GeoGebra, where 42% of students answered with the maximum level of satisfaction and for the problem-solving activities (3.3 average on a scale of 1-4). This feedback suggests that the activities involved the project, such as the problem-solving activities and the use of GeoGebra, were among the most appreciated by the students.

Equally interesting were the suggestions provided by several students in the "personal opinion" section of the questionnaire. In particular, multiple students noted that their interest in mathematics had increased as a result of the learning week. Others mentioned the effectiveness of the experience. One student made the following comment: "The learning week was interesting, above all it made me experience a new way of learning maths." Another noted that "This increased my mental strength, reassuring me and allowing me to improve in reasoning and completing exercises."

5. THE BENEFITS OF GEOGEBRA

GeoGebra was used by our students in an essentially intuitive and independent manner. Because the classroom explorations we created made use of pre-constructed files and sketches, our students used GeoGebra immediately with little - if any - difficulty. In no instance were they required to build their own GeoGebra sketches from "scratch." Therefore, it was not necessary to carry out a preliminary GeoGebra training with our students. This made it possible to focus on classroom activities rather than on mathematical procedures and the use of electronic tools and button pushing.

Student conceptual understanding of potentially troublesome mathematics content - such as limits and derivatives - was strengthened by GeoGebra's dynamic display. In particular, the multiple-representational capabilities of the software truly engaged our students - generating much discussion among student groups and university student tutors. The software-based investigations lightened up the study of functions, making Calculus less tedious while enriching the study of mathematics content through visualization.

6. CONCLUSIONS

Our experience suggests to us that the difficulties that our students encounter in learning basic analytical concepts can be positively addressed through laboratory activities based on cooperative learning and supported by the use of GeoGebra.

The software provides students and teachers with the opportunity to use and coordinate different methods of representation (text, numeric, tabular, graphical, algebraic and symbolic), to propose problems and conceptual issues related to learning the concepts of limit and derivative, in a way that is stimulating for students and effective in terms of learning. Moreover, our experience suggests that the use of GeoGebra in the classroom does not require any prior training for students in their final year of secondary school. Students can learn how to use the main features directly during problem-solving activities and exercises. This helps to keep the focus on the activity and not on the tool, which can be used autonomously by students also to support group activities once the roles and responsibilities within the group have been defined.

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