A CELESTIAL PLANISPHERE IN GEOGEBRA

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Abstract

This article presents a digital celestial planisphere for the northern hemisphere created with GeoGebra. It describes all the necessary elements of the tool and how it is constructed as a projection of the celestial sphere onto a plane. It also details its development process and presents examples of educational activities that can be implemented in astronomy classrooms and workshops.

Keywords: GeoGebra, Science, Planisphere, Stereographic projection.

1 Introduction

A celestial planisphere is a map that represents the stars of the sky in a circular chart that shows the visible stars for an observer located at a particular latitude. Along with this circular chart, the planisphere incorporates an overlay of the same size and shape that rotates around the center of the planisphere, and that simulates the apparent daily movement of the celestial sphere. This overlay is opaque, except for a window (a transparent hole) that represents the horizon of the observer, and that allows the identification of the visible stars at a particular time of the day.

With this intention, a traditional planisphere graduates an annual calendar on the rim of the star chart and a 24-hour cycle on the rim of the overlay. The visibility window of the overlay is marked with the cardinal points and, depending on the model, also with azimuth marks. Thus, by synchronizing the date of the star chart with the time of the overlay ruler, the planisphere displays the visible stars for that latitude, date, and time.

These types of maps are made as a projection of the celestial sphere onto a plane, with the most used methods being the stereographic projection and the polar azimuthal equidistant projection (do Carmo, 1976). The stereographic projection has interesting mathematical properties, such as angle preservation, which makes it suitable for constructing navigational charts (Instituto Superior de Navegación Darío Fernández, 2024). Additionally, it properly plots the shape of those constellations near the center of the planisphere, the North Star, but there is considerable distortion in the size of those constellations at the edge. The distance between parallel circles is not constant and increases as they move away from the center. The polar azimuthal equidistant projection is an alternative to the stereographic projection that adjusts the distance of the elements from the center of the map so that all the parallels are equidistant from each other, but at the expense of flattening the shape of the constellations farther from the center of the projection.

Using a planisphere is not only valuable for learning the positions of the stars, but also because it enables students to visualize the observer's spatial location on Earth, to relate it to the abstract concept of projecting onto a plane, and to interpret two-dimensional representations as they would appear in three-dimensional space. Today, educational and simulation software help teachers enhance students' learning capabilities. GeoGebra stands out as an excellent dynamic geometry system (GeoGebra; Acikgul and Onuk Keskin, 2025) for implementing these methodologies across STEM disciplines, such as Mathematics (Xing et al., 2024; Romero Albaladejo and García López, 2024), Physics (Solvang and Haglund, 2021), and Astronomy (Langendorf et al., 2022). Within astronomy education specifically, it is a highly valuable resource for exploring scientific topics such as Kepler's laws (Matos et al., 2023), Earth's precession (Zhang et al., 2025), and even the reconstruction of ancient astronomical models (Dhakulkar and Nagarjuna, 2011).

This article presents a digital version of the traditional celestial planisphere, developed in GeoGebra, with the following educational objectives:

- To provide students with a digital replica of paper planispheres commonly used in science education centers and astronomy workshops, but which are not often available in many schools.
- To create a resource that reproduces the classical diagrams typically found in the introductory chapters of astronomy textbooks (Equipo editorial de Susaeta, 2002).
- To bridge the gap between paper-based models and advanced simulation software.

Both teachers and students can use the app to study and manipulate a fully functional planisphere. Moreover, the versatility of GeoGebra enables the incorporation of features that traditional planispheres lack. For example, the visibility window can be adjusted to any latitude in the northern hemisphere. The paper also describes the following features of the digital planisphere:

- Planisphere for an observer in the northern hemisphere.
- Adjustable for latitudes between 1° and 90° north.
- Possibility of using stereographic projection and polar azimuthal equidistant projection.
- Model of a year with 365 days and duration of the sidereal day accordingly.
- Uniform and circular motion of the Sun along the ecliptic.
- Date and time display, avoiding the graduation of the star chart and the overlay.
- Representation of the eighty-eight modern almagest constellations, and the most important elements of a planisphere: parallels, meridians, celestial equator, cardinal directions, azimuth marks, ecliptic, equinoxes, and solstices.

We divide the article into sections, beginning with the description of the projection of the elements of the celestial sphere onto the plane and the necessary mathematics to implement either the stereographic or the equidistant projection. We then explain how to construct and adjust the planisphere in GeoGebra. We devote the last section to an educational application, presenting activities and exercises which help students to learn important astronomy concepts: rising and setting of the Sun, equinoxes and solstices, and the seasonal visibility of the constellations.

2 The planisphere as a projection of the celestial sphere

The celestial sphere (Karttunen et al., 2007; Martín Asín, 1990) is a classic model that arranges the stars of the sky in a fixed-radius sphere with the Earth at its center. The celestial sphere apparently rotates clockwise, opposite to the rotation of the Earth around its axis. The rotation axis of the sphere is the extension of the Earth's rotation axis and passes through two fixed positions in the sky named the celestial north and south poles. The celestial equator is the great circle perpendicular to the axis that divides the celestial sphere into two equal parts. The sphere also incorporates meridians (great circles through the north and south poles) and parallels, which divide the sky and allow using coordinates to locate the stars.

The apparent position of the Sun also appears on the celestial sphere, but, unlike the position of the stars, it is not fixed because it is a consequence of the Earth's motion around the Sun. The curve that the Sun traces on the sphere is called the ecliptic, and it is modeled as a great circle slightly inclined with respect to the celestial equator. On the sphere, we also model the observer‘s horizon as a great circle perpendicular to its position vector.

A planisphere represents the stars as seen from a given observer's latitude; therefore, different latitudes require different planispheres. The lower the observer's latitude, the larger the part of the sky visible throughout the day, and thus the larger the star map on the planisphere. A circle on the sphere limits this visible sky, and its projection defines the boundary of the planisphere that can be constructed for that latitude. Furthermore, the observer's horizon limits the stars visible at any given moment, which are those located on the celestial hemisphere determined by the horizon containing the celestial north pole. As the sphere rotates, the group of visible stars changes, and all together form the visible sky that the planisphere represents.

The stereographic projection for a northern hemisphere planisphere projects from the celestial south pole onto a plane parallel to the equator and tangent at the celestial north pole (figure 1). Each point on the sphere is projected to the intersection of the projection line and the projection plane. The elements of the celestial sphere, stars, constellations, meridians, parallels, ecliptic, and horizon, when projected, form the corresponding elements of the planisphere. The projected horizon depicts a closed curve whose interior represents the visible stars at any one time. This region is known in astronomy as the visibility window of the planisphere. The area outside the visibility window is usually partially opaquely colored to highlight the visible stars at once. Along the horizon line, the cardinal points and azimuth marks are identified, indicating the subtended angle with respect to the north direction.

The polar stereographic projection has the following mathematical properties:

- The meridians are straight lines from the center of the planisphere.

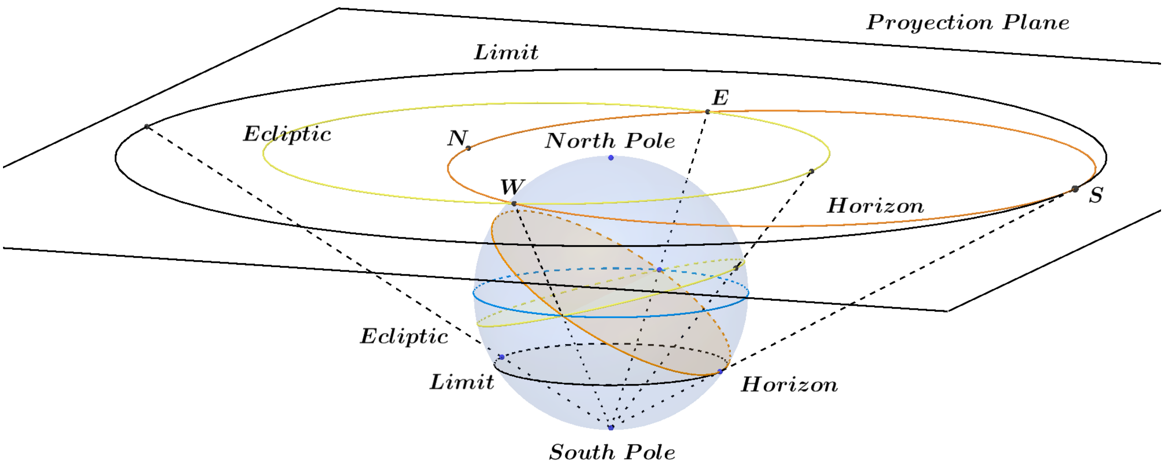


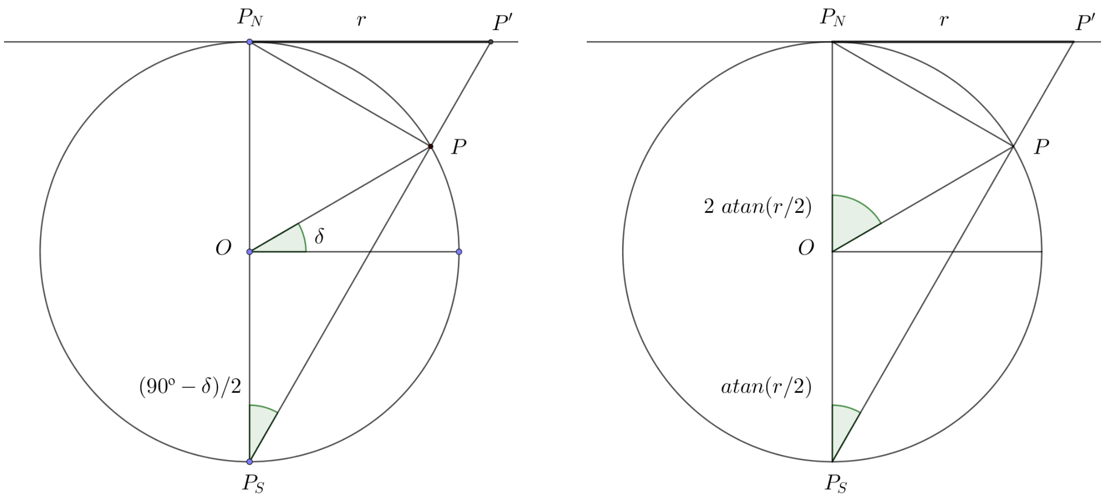
Figure 1. Planisphere as a projection of the celestial sphere onto a plane

- The parallels are concentric circles.
- The projection is conformal, meaning that it preserves angles at which curves meet. This makes it suitable for navigation (Instituto Superior de Navegación Darío Fernández, 2024).
- Every circle on the sphere not passing through the projection pole maps onto a circle on the plane (Gaussianos, 2009).
- Distances increase as we move away from the center of the projection. The closer the object on the sphere is to the south pole, the greater its distance from the center of the map, and the larger it is represented. In particular, the distance between parallels is not constant.

The former feature makes planispheres based on stereographic projections little useful. Most objects are concentrated near the center of the map, while others appear quite dispersed. The polar azimuthal equidistant projection overcomes this difficulty by preserving the distance between parallels. This projection, traditionally used in planispheres, generates maps of reasonable size for practical use. However, the projection's conformity is lost, and the shape of the constellations is flattened when compared to the stereographic projection. The digital planisphere presented in this article offers both types of projections, allowing users to compare these features as we show in the activities section.

3 Mathematics of the stereographic projection

We model the celestial sphere (López García and Cerisola, 2022) as a sphere of radius 1 centered at $O = (0, 0, 0)$, $S \equiv x^2 + y^2 + z^2 = 1$. We consider the projection plane $\pi \equiv z = 1$ and the celestial south pole $P_S = (0, 0, -1)$ as the origin of the projection. Using rectangular coordinates in $\mathbb{R}^3$ and in $\mathbb{R}^2$, the stereographic projection maps each point (x, y, z) on the sphere to the point on the projection plane given by $\left(\dfrac{2x}{1+z}, \dfrac{2y}{1+z}\right)$.



(a) Radius calculus given δ **(b)** Angle calculus given r

Figure 2. 2-dimensional stereographic projection

In our model of the sphere, the west point is located at $(1, 0, 0)$ and the east point at $(-1, 0, 0)$. The previous expression projects the west point onto the positive side of the horizontal axis and the east point onto the negative side. Since this orientation does not correspond to that of a conventional planisphere, we apply a symmetry with respect to the vertical axis, obtaining the expression that we use in the construction of a planisphere based on stereographic projection.

$$\begin{array}{rcl} T : S\backslash P_S \subset \mathbb{R}^3 & \to & \mathbb{R}^2 \\ (x, y, z) & \to & \left(-\dfrac{2x}{1+z}, \dfrac{2y}{1+z}\right) \end{array} \tag{1}$$

The stereographic projection can also be described using spherical (also known as equatorial) coordinates on the celestial sphere -*right ascension* α and *declination* δ- and polar coordinates -the radius r of the projected point and the angle θ with respect to the OX axis- on the projection plane. In this formulation, the projection assigns the right ascension to the angular coordinate $\theta = \alpha$, and the required symmetry modifies this value to $\theta = 180 - \alpha$. The radius of the projected point is given by $r = 2\tan((90 - \delta)/2)$. This expression is derived from the geometric construction in figure 2a, which models the stereographic projection in two dimensions, noticing that the inscribed angle subtends half the arc of the central angle $90 - \delta$. Thus, the expression of the stereographic projection using spherical coordinates in $\mathbb{R}^3$ and polar coordinates in $\mathbb{R}^2$ is

$$\begin{array}{rcl} T : [0, 360] \times (-90, 90] \subset \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (\alpha, \delta) & \to & \left(2\tan\left(\dfrac{90-\delta}{2}\right), 180 - \alpha\right) \end{array} \tag{2}$$

The stereographic projection has the disadvantage of inducing some distortion in objects further from the center of the map. The polar azimuthal equidistant projection overcomes this difficulty, scaling the distance of each projected element from the center, so that the celestial parallels remain equidistant. More precisely, the radius of every projected point is adjusted to be the arc length of the meridian from the point to the north pole of the sphere. For a given point on the plane with polar coordinates $P' = (r, \theta)$, image of a point in the sphere with coordinates (α, δ), the scaling modifies the value of the radius to the length of the arc of meridian, which is $2 \arctan(r/2)$ radians (figure 2b).

The relations between rectangular coordinates (x, y) and polar coordinates (r, θ) given by $r = \sqrt{x^2 + y^2}$ and $\theta = arctan(y/x)$, lead to the expressions in $\mathbb{R}^2$ that transform stereographically projected points into equidistant projected points. In rectangular coordinates, we have

$$\begin{aligned} T : \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ (x, y) &\rightarrow \frac{2}{\sqrt{x^2+y^2}} \arctan\left(\frac{\sqrt{x^2+y^2}}{2}\right)(x, y) \end{aligned} \tag{3}$$

and in polar coordinates, we have

$$\begin{aligned} T : [0, \infty) \times [0, 360] \subset \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ (r, \theta) &\rightarrow \left(2 \arctan\left(\frac{r}{2}\right), \theta\right) \end{aligned} \tag{4}$$

The equidistant projection results as the composition of the stereographic projection in (1) or (2) and the scaling functions described in (3) and (4). We combine (1) and (3) to project those elements onto the sphere modeled with rectangular coordinates. In contrast, we combine (2) and (4) to project those elements modeled in spherical coordinates, such as the stars. This composition admits the following closed-form expression (5), noticing that the central angle in figure 2b, given by $2 \arctan(r/2)$ corresponds to $\pi\left(\frac{1}{2} - \frac{\delta}{180}\right)$, for a known value of declination δ.

$$\begin{aligned} T : [0, 2\pi] \times [-\pi/2, \pi/2] \subset \mathbb{R}^2 &\rightarrow \mathbb{R}^2 \\ (\alpha, \delta) &\rightarrow \left(\pi\left(\frac{1}{2} - \frac{\delta}{180}\right), 180 - \alpha\right) \end{aligned} \tag{5}$$

4 Construction of the planisphere in GeoGebra

The construction of the planisphere organizes the commands into three pieces of JavaScript code and GeoGebra functions. The first piece of code creates the celestial chart and the overlay, using both projections described in the previous section. The code accesses the star data catalog stored in the GeoGebra spreadsheet, similar to the Celestial Sphere application (López García and Cerisola, 2022). The second piece of code defines the date and time of the observer by adjusting the motion of the Sun to that of the celestial sphere, for a simplified 365-day school model. The third piece of code generates a grid of buttons that provides functionality to the application and facilitates its use by teachers and students.

In the construction of the star map, we take into account that the visibility of many elements of the planisphere depends on the observer's latitude. Therefore, we define the slider *lat* in the range $[1°, 90°]$ and formulate the mathematical formulas as functions of this variable. This is the case for the boundary of the planisphere, the horizon, the cardinal points, and the azimuth marks on the overlay rim. Besides, some elements of the planisphere are modeled in polar coordinates while others are in rectangular coordinates. We use polar coordinates for elements whose description on the celestial sphere is straightforward in spherical coordinates. For instance, the boundary of the planisphere is a circle with center $O = (0, 0)$ and radius $r = 2\tan(90° - \text{lat}/2)$, which corresponds to the declination of the cardinal south point $\delta = \text{lat} - 90°$. Similarly, the stars, parallels, celestial equator, and meridians are modeled in polar coordinates. We limit the meridians to represent declinations between $-80°$ and $80°$, noticing that the declination point $-90°$ cannot be represented in stereographic projection. In addition, the celestial points that indicate the change of seasons are also described in polar form: the vernal equinox $(\alpha, \delta) = (0°, 0°)$; the summer solstice $(\alpha, \delta) = (90°, \epsilon°)$; the autumnal equinox $(\alpha, \delta) = (180°, 0°)$; and the winter solstice $(\alpha, \delta) = (270°, -\epsilon)$. The table 1 summarizes the used mathematical expressions, and Listings 1 gives part of the code that creates these objects.

In contrast, we model the ecliptic, the horizon, and the cardinal points with rectangular coordinates. We create the ecliptic on the celestial sphere as an ϵ-rotation of the celestial equator $Equator = (\cos(\theta), \sin(\theta), 0)$ around the OX axis (6). The projection of this circle gives the ecliptic of the planisphere, which is a circle when using the stereographic projection (7) but not after applying the equidistant scaling.

$$Ecliptic_{Sphere} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\epsilon) & -\sin(\epsilon) \\ 0 & \sin(\epsilon) & \cos(\epsilon) \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\theta) \\ \cos(\epsilon)\sin(\theta) \\ \sin(\epsilon)\sin(\theta) \end{pmatrix} \tag{6}$$

$$Ecliptic_{Planisphere} = \left(\frac{-2\cos(\theta)}{1 + \sin(\epsilon)\sin(\theta)}, \frac{2\cos(\epsilon)\sin(\theta)}{1 + \sin(\epsilon)\sin(\theta)} \right), \quad 0 \leq \theta \leq 360° \tag{7}$$

Similarly, we model the horizon on the celestial sphere as a $(90° - lat°)$-rotation of the celestial equator around the OX axis (8). Under the stereographic projection, it turns into a circle (9), while the equidistant projection transforms the horizon into a closed curve whose shape varies depending on the latitude. At high latitudes, the horizon resembles a circle; at mid-latitudes, it resembles an ellipse; and at low latitudes, it resembles a semicircle that spans approximately half of the celestial map.

$$Horizon_{Sphere} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin(lat) & -\cos(lat) \\ 0 & \cos(lat) & \sin(lat) \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{pmatrix} = \begin{pmatrix} \cos(\theta) \\ \sin(lat)\sin(\theta) \\ \cos(lat)\sin(\theta) \end{pmatrix} \tag{8}$$

$$Horizon_{Planisphere} = \left(\frac{-2\cos(\theta)}{1 + \cos(lat)\sin(\theta)}, \frac{2\sin(lat)\sin(\theta)}{1 + \cos(lat)\sin(\theta)} \right), \quad 0 \leq \theta \leq 360° \tag{9}$$

Object	Stereographic projection	Equidistant projection
Star	$\left(2\tan\left(\frac{90°-\delta}{2}\right); 180° - \alpha\right)$	$\left(\pi\left(\frac{1}{2} - \frac{\delta}{180}\right); 180° - \alpha\right)$
Limit	$circle(O, 2\tan\left(\frac{180°-lat}{2}\right))$	$circle(O, \pi(1 - \frac{lat}{180}))$
Parallel_k	$circle(O, \frac{2\cos(k)}{1+\sin(k)})$	$circle(O, \pi(\frac{1}{2} - \frac{k}{180}))$
Equator	$circle(O, 2)$	$circle(O, \frac{\pi}{2})$
Meridian_k	$segment[(2\tan(5°); k), (2\tan(85°); k)]$	$segment[(\frac{5\pi}{90}; k), (\frac{85\pi}{90}; k)]$
Equi_{Spring}	$(2; 180°)$	$(\frac{\pi}{2}; 180°)$
Sols_{Summer}	$\left(2\tan\left(\frac{90°-\epsilon}{2}\right); 90°\right)$	$\left(\pi\left(\frac{1}{2} - \frac{\epsilon}{180}\right); 90°\right)$
Equi_{Fall}	$(2; 0°)$	$(\frac{\pi}{2}; 0°)$
Sols_{Winter}	$\left(2\tan\left(\frac{90°+\epsilon}{2}\right); 270°\right)$	$\left(\pi\left(\frac{1}{2} + \frac{\epsilon}{180}\right); 270°\right)$

Table 1. Expressions for stereographic and equidistant projection

```
function CreatePlanisphere(){
   ...
if(stereographic){
a.evalCommand("Equator = Circle((0,0),2)");
a.evalCommand(\"Limit = Circle((0,0),2*cos(lat-90)/(1+sin(lat-90)))\");
a.evalCommand("Parallels =
  Sequence[Circle((0,0),2*cos(k)/(1+sin(k))),k,-80,80,10]");
a.evalCommand("Meridians = Rotate(
  Sequence[Segment((2*tan(45-0.5*80);k),(2*tan(45+0.5*80);k)),k,0,345,15],r)");
a.evalCommand("SSum = Rotate((2*tan(45-0.5*eps); 90),r");
a.evalCommand("SWin = Rotate((2*tan(45+0.5*eps); 270),r)");
a.evalCommand("EFall = Rotate((2;0),r)");
a.evalCommand("ESpr = Rotate((2;180),r)");
...
}else{
a.evalCommand("Equator = Circle((0,0),0.5*pi)");
a.evalCommand("Limit = Circle((0,0), pi*(1-lat/180))");
a.evalCommand("Parallels =
  Sequence[Circle((0,0),pi*(0.5-k/180)),k,-80,80,10]");
a.evalCommand("Meridians = Rotate(
  Sequence[Segment((pi*(0.5-80/180);k),(pi*(0.5+80/180);k)), k,0,345,15],r)");
a.evalCommand("SSum = Rotate((pi*(0.5 - eps/180); 90),r)");
a.evalCommand("SWin = Rotate((pi*(0.5 + eps/180); 270),r)");
a.evalCommand("EFall = Rotate((pi*(0.5); 0),r)");
a.evalCommand("ESpr = Rotate((pi*(0.5); 180),r)");
...
}
...
}
```

Listing 1. Planisphere construction function

5 Time modeling and planisphere tuning in GeoGebra

We use a slider t for modeling time and define some objects of the planisphere as functions of this variable. We consider a 365-day year, with $t \in [0, 365]$, and assume a circular and uniform motion of the Sun. Therefore, the Sun traces the ecliptic at that time. This assumption implies that the Sun covers 1/365 of the ecliptic every day, which corresponds to an angular range of $\theta = 360 \cdot t/365$ in t days. Our parametrization of the ecliptic (6) maps the value $\theta = 0$ to the point $(1, 0, 0)$, which our modeling assigns to the vernal equinox. Thus, to adjust the position of the Sun (so that the beginning of the year does not correspond with spring), we consider the function

$$y = 91.25t + phase_{Sun} \tag{10}$$

and take real dates of the spring equinox, summer solstice, autumn equinox, and winter solstice (y_0, y_1, y_2, y_3) from (U.S. Naval Observatory, 2024). We solve the least squares problem that fits the function (10) to the data points $(0, y_0)$, $(1, y_1)$,$(2, y_2)$, and $(3, y_3)$. The solution of this problem is $phase_{Sun} = -80.34$, and let us model the position of the Sun as

$$Sun = E\left(\frac{360}{365}\,t + phase_{Sun}\right) \tag{11}$$

where E is the curve that describes the ecliptic (7). Note that $phase_{Sun}$ indicates the number of days from the beginning of the year until spring.

Date	**U.S. Naval Observatory Data**	**Planisphere aproximation**
Spring equinox	20 March 9:01	21 March 8:16
Summer solstice	21 June 2:42	21 June 14:16
Autumn equinox	22 September 18:19	20 September 20:16
Winter solstice	21 December 15:02	21 December 2:16

Table 2. Approximations of equinoxes and solstices in the Planisphere

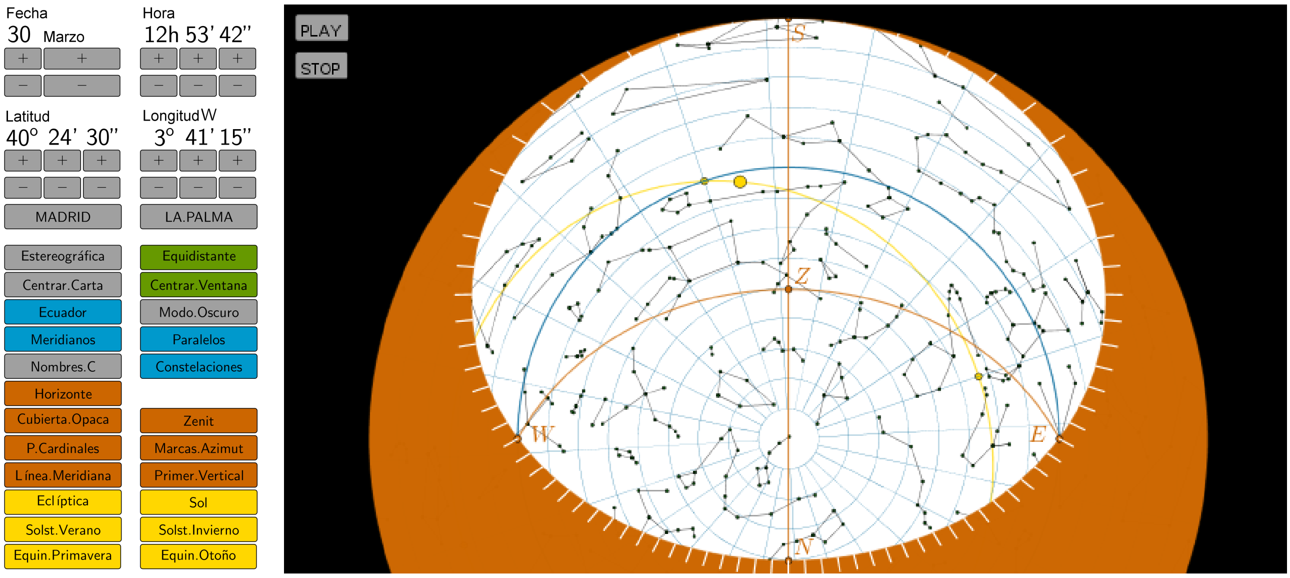
Additionally, the rotation motion of Earth induces an apparent motion of the celestial sphere that our app models by applying the command *rotate* to each element related to the sky (stars, constellations, meridians, ecliptic, and Sun). We do not apply this rotation to the elements related to the Earth (horizon line, cardinal points, meridian line, and first vertical). Considering that a model of 365 solar days implies 366 sidereal days (Martín Asín, 1990), in a solar day, the Earth rotates 366/365 turns against the background of the stars. We model this motion in reverse sense for the objects of the sky and obtain as a function of t the parameter r that animates the rotation of the sky $r = \frac{366}{365} \cdot t \cdot 360$. Besides synchronizing this rotation parameter with the time slider t, we tune its value so that the position of the Sun is correct along the day. We get this by shifting the rotation

$$r = \frac{366}{365} \cdot t \cdot 360° + phase_{Rotation} \tag{12}$$

and adjusting the previous function so that on some dates $t_i : i : 1, \ldots, n$ the angle formed by the Sun with the meridian line coincides with the expected azimuth values. These expected azimuth values $r_i : i : 1, \ldots, n$ are taken from Stellarium (Stellarium contributors, 2025) and we fit the function (12) to the point cloud $(t_i, r_i) : i : 1, \ldots, n$ by means of solving a least squares problem. The solution obtained is $phase_{Rotation} = 11.06°$, which is the value we use in the modeling. With this modeling, a maximum error of 6 minutes is made in the sunrise and sunset.

6 Description of the application and didactic activities

The digital celestial Planisphere allows students to identify the stars and constellations for a given time of the day and for different observer latitudes. It incorporates buttons for animating, showing, and hiding the different elements of the planisphere. The appearance of the application is shown in Figure 3.



(a) Buttons **(b)** Star chart

Figure 3. Screenshot of the application Celestial Planisphere

The buttons on the panel execute the following actions:

- Date and time: selects the date and time of the observation.
- Latitude and longitude: positions the observer.
- Stereographic vs equidistant: selects the projection.
- Center chart: centers the image to visualize the entire planisphere.
- Center window: centers the image to visualize the entire window.
- Blue buttons: show or hide the elements of the sky: stars, constellations, meridians, and parallels.
- Orange buttons: show and hide the elements of the horizon: cardinal points, meridian line, first vertical, and azimuth marks.
- Yellow buttons: show and hide the elements related to the Sun: ecliptic, equinoxes, and solstices.
- Play and stop buttons: start and stop the simulation of the daily motion of the sky.

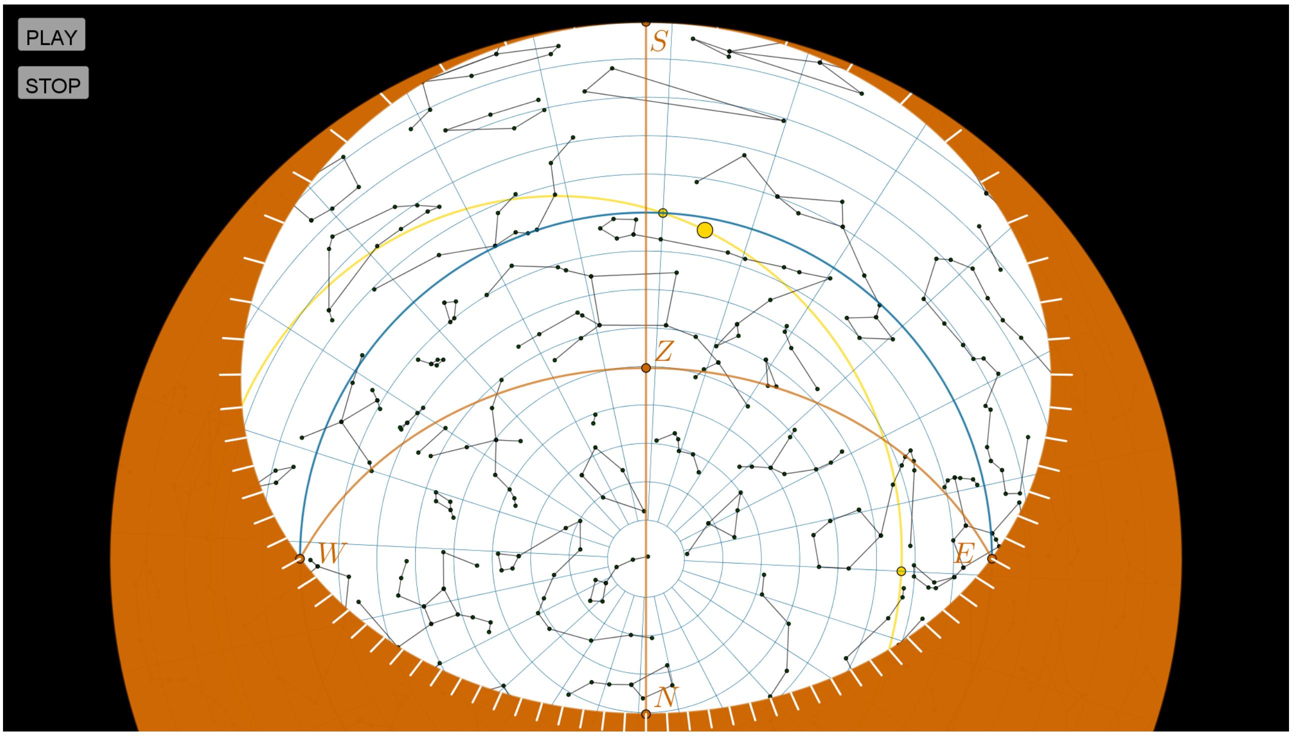
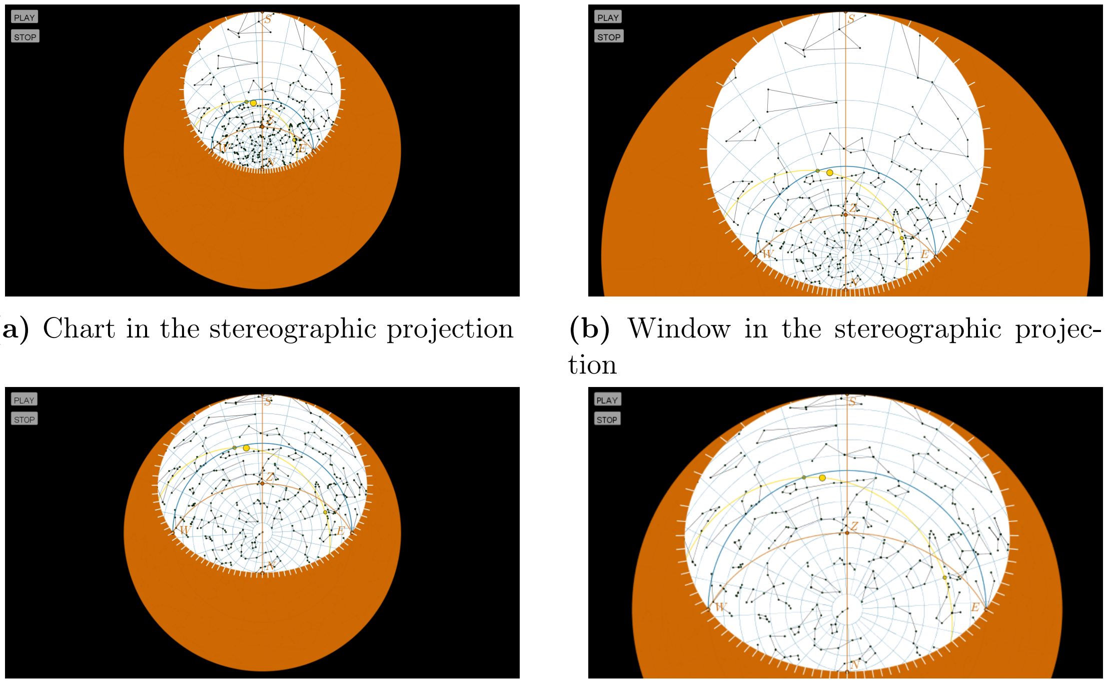


Figure 4. Star chart with Play and Stop buttons

As the reader can test on our group's webpage (López García and Cerisola, 2025), it is a valuable educational resource that fosters the development of spatial reasoning and an understanding of the concept of coordinates. For older students, it reinforces content related to mathematics and technical drawing, while for younger students, it complements the basic learning of some concepts in the early years of the primary science curriculum. As an example of its didactic use, we present the following educational activities that we propose in our workshops and solve using the digital planisphere.

- **Activity 1**. Comparison of the stereographic and equidistant projections.

  Select a latitude (Madrid, for example) and the stereographic projection method. Switch between the options of centering the planisphere and the window. Check in Figure 5a that the visibility window is very small when compared to the star chart. Note also that the distance between parallels grows as we move farther from the center of the map. Modify the configuration and select an equidistant projection. Check in Figure 5c that the visibility window fills more area of the figure and that the parallels keep the distance to their adjacent ones. On the other hand, note that the constellation of Orion (Figure 5d) appears in the equidistant projection flatter than in the stereographic projection, which resembles the direct visual appearance of this group of stars. In general, and more pronounced at low latitudes, the stereographic projection concentrates the celestial objects around the north point, making it difficult to identify stars and constellations. Thus, the equidistant projection may be a good option for analyzing the sky for observers at low latitudes.



(a) Chart in the stereographic projection **(b)** Window in the stereographic projection **(c)** Chart in the equidistant projection **(d)** Window in the equidistant projection

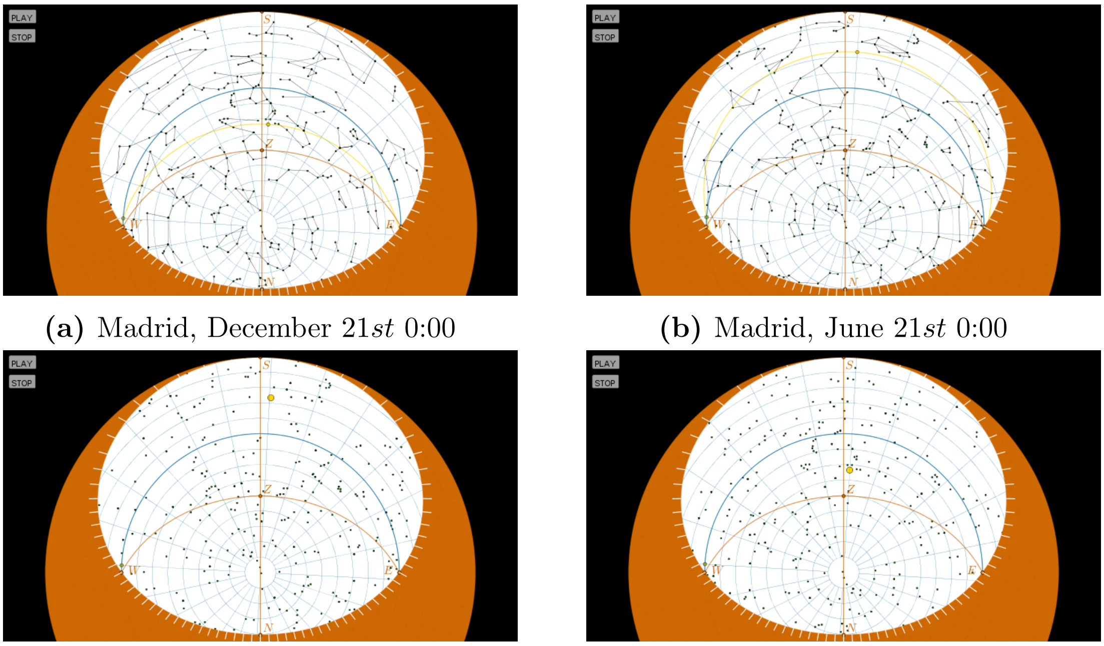
Figure 5. Planispheres in stereographic and equidistant projections

- **Activity 2**. Analysis of the different skies in winter and summer.

  Select a latitude (Madrid, for example) and configure the planisphere for December, the 21st at 00:00 hours, and after that for June the 21st at 00:00 hours. We can observe not only that the visible constellations are not the same, but that those visible constellations appear in different positions relative to Polaris, the North Star. For example, the Big Dipper is located to the right of the Polaris star in winter (Figure 6a), while it is to the left of Polaris in summer (Figure 6b).

  Another exercise consists of analyzing the height of the Sun in winter and summer. The height of the Sun, at a northern hemisphere latitude, measures the angle of elevation from the cardinal point south. To compute the height of the Sun in winter, we configure the planisphere for December the 21*st* and keep visible the objects: Sun, meridian line, cardinal points, and parallels. Animate the application (play) and stop it when the Sun coincides with the meridian line. Count the number of parallels (Figure 6c) from the cardinal point south to the Sun. The parallels in our construction are modeled every 15 degrees, so quick multiplication gives the maximum height of the Sun in winter. Repeat the process for June the 21*st* and notice that the Sun is closer to the zenith point (Figure 6d), which marks the point above the observer's vertical.

  While performing this exercise, we can calculate the number of daylight hours in winter and summer by recording the sunrise and sunset times. With that purpose, select a winter date and keep the overlay of the planisphere opaque, animate the application, and record when the Sun appears in the visibility window and when it disappears. Repeat the experiment for a chosen date in summer.



(a) Madrid, December 21*st* 0:00 **(b)** Madrid, June 21*st* 0:00

(c) Madrid, December 21*st* 12:00 **(d)** Madrid, June 21*st* 12:00

Figure 6. Different skies in winter and summer

- **Activity 3.** Analysis of the skies in different latitudes.

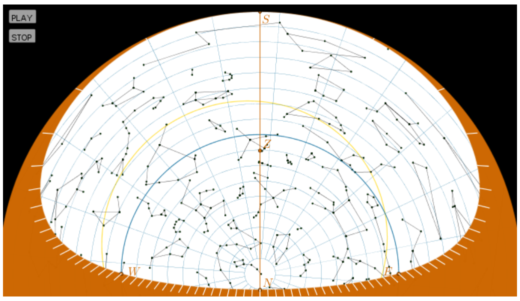
  Set the planisphere for different latitudes. We propose a latitude close to the equator (10°), the latitude of Isla de La Palma (≈ 28°), the latitude of Madrid (≈ 40°), and a latitude of the Arctic Circle (70°).

  The first thing to notice is that the planisphere covers more sky at low latitudes than at high latitudes. Pay attention to the visible parallels in Figure 7. The star chart incorporates more parallels at low latitudes (Figure 7a) than at high latitudes (Figure 7d). The app bolds the celestial equator, which allows the identification of those parallels of negative declinations.

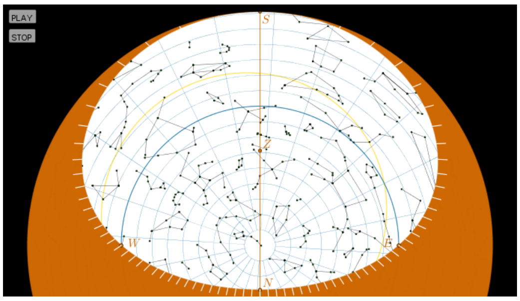
  Closely related to the issue is the position of Polaris, the North Star. It is always located at the center of the star chart, but not at the center of the visibility window. At low latitudes, it is close to the northern cardinal point, which implies that this star appears very low in the observer's vision field (Figure 7a). The North Star height increases as we move to higher latitudes, reaching the center of the visibility window of the planisphere we could build for an observer located at the Earth's North Pole (Figure 7d).

  A similar phenomenon happens to the constellations around the North Star. At high latitudes, all of them are visible, while at low latitudes, some are partially visible in winter and others in summer. For example, for the situation given in Figure 7, the constellation of Scorpio is totally visible from Isla de la Palma (Figure 7b) but only partially visible from Madrid (Figure 7c).

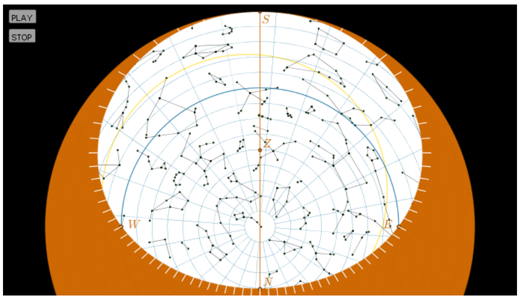
  There are many other results that we can corroborate in relation to the different latitudes of the observer. For example, we can check the number of daylight hours at different latitudes and measure the height of the Sun in different seasons. It is an interesting experiment to set the planisphere at a latitude close to the Arctic Circle and follow the path of the Sun. We can appreciate that in summer (Figure 7d) the Sun is very low and that at noon it is very close to the north cardinal point. The animation shows that in summer, the Sun does not set during the day. The opposite situation occurs in winter: the Sun does not rise on any day of the winter season.



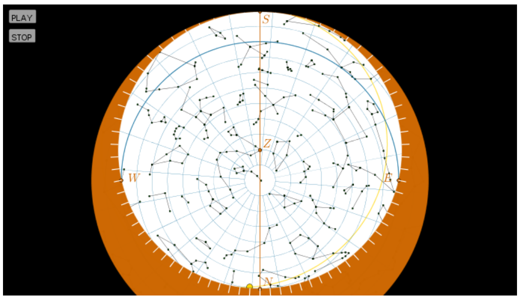
(a) Latitude 10°, July 21*st* 0:00



(b) La Palma 28°, July 21*st* 0:00



(c) Madrid 40°, July 21*st* 0:00



(d) Latitude 70°, July 21*st* 0:00

Figure 7. Skies at different latitudes of the observer

7 Conclusions

The geometry of projections, the necessary mathematics, and the versatility of GeoGebra allow us to develop a digital planisphere that fulfills our main objective: to access a digital tool that replicates physical models of paper planispheres. It is a valuable resource that enriches the study of astronomy, bridging the gap between paper models and advanced simulation software, and complements training in important skills such as spatial reasoning, coordinates learning, basic scientific concepts, and the differences that depend on the observer's viewpoint. Currently, the tool is used as part of an "Introductory Astronomy Course for Teachers" offered annually in Madrid. We have received suggestions for slight improvements to the application and others for creating alternative planispheres, such as those for the southern hemisphere and the zenithal planisphere, which are our current focus.

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