ON THE BROCARD CIRCLE*

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Abstract

The Brocard circle (or seven-point circle) is a circle derived from a given triangle. There have been several discoveries regarding the Brocard circle. In this study, a compilation of existing properties of the Brocard Circle has been presented, along with properties not currently found. These properties were discovered using GeoGebra.

Keywords: Brocard circle, Triangle geometry, GeoGebra

1 Introduction

During the 1881 meeting of the French Association for the Advancement of Science, the mathematician Henri Brocard presented a paper entitled *Études d'un nouveau cercle du plan du triangle*. This paper described something we now call the "Brocard circle."

Definition 1. *The Brocard circle of a triangle $\triangle ABC$ is the circle whose diameter is the segment that connects its circumcenter O and its Lemoine point L. Where, the Lemoine point is the intersection of the three symmedians of the triangle (see Brocard (1881)).*

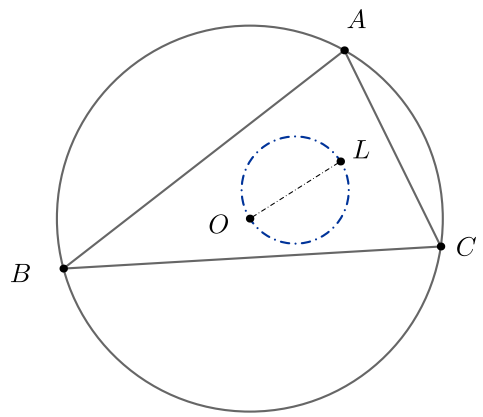

Figure 1. Diagram of the Brocard circle (dotted in blue)

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In the studies described in the presentation, Brocard (1881) presents the definition of Brocard points and the property of those points in the Brocard circle. Furthermore, it is within this presentation that the first and second Brocard triangles are formally defined. The following are details of these findings:

Definition 2. *For each $\triangle ABC$ there are exactly two points Ω_1 and Ω_2 that satisfy $\angle\Omega_1 AB = \angle\Omega_1 BC = \angle\Omega_1 CA = \omega$ and $\angle\Omega_2 AC = \angle\Omega_2 CB = \angle\Omega_2 BA = \omega$. These points are called the first and the second Brocard points and ω is called the Brocard angle of the triangle*

Theorem 3. *The Brocard points Ω_1 and Ω_2 lie on the Brocard circle (the circle whose diameter is OL).*

Definition 4. *Let O be the circumcenter, and L the Lemoine point of $\triangle ABC$. Draw $\overline{OX}$, $\overline{OY}$ and $\overline{OZ}$ perpendicular to the sides BC, CA and AB, and let them meet the Brocard circle in the points A', B', C'. Therefore, it can be considered that $\triangle A'B'C'$ is the first Brocard triangle*

Definition 5. *Let L be the Lemoine point of $\triangle ABC$. It is satisfied that $\overline{AL}$, $\overline{BL}$ and $\overline{CL}$ meet the Brocard circle in the points A'', B'', C''. As a result, it can be stated that $\triangle A''B''C''$ is the second Brocard triangle*

2 Mathematical Preliminaries

After Brocard's presentation, Brocard's circle began to be studied. At first, Emmerich (1891) and Lachlan (1893) prepared translations of Brocard's original work in German and English, which were used by mathematicians of the time to carry out their investigations. As one of the pioneers in this field stands out Johnson (1929) who listed more properties than those mentioned in Brocard's original work. Three of these properties are shown below.

Corollary 6. *The quadrilateral $L\Omega_1 O\Omega_2$ is a cyclic kite (a quadrilateral with reflection symmetry across a diagonal and two right angles).*

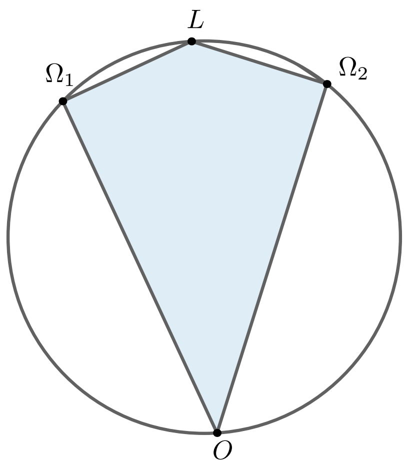


Figure 2. cyclic kite (shaded) in the brocard circle

Theorem 7. *In the Brocard circle, the Brocard points Ω_1 and Ω_2 are symmetrical with respect to the diameter OL.*

Theorem 8. *In an scalene triangle, the Lemoine point lies on the arc of the Brocard circle, which is between the largest and the smallest of the angles of the first Brocard triangle.*

Theorem 9. *Let Ω_1 and Ω_2 be the Brocard points, ω the Brocard angle, and O the circumcenter, then $\triangle O\Omega_1\Omega_2$ are isosceles. More specifically*

$$\Omega_1 O = \Omega_2 O$$

and

$$\angle \Omega_2 O \Omega_1 = 2\omega$$

Four decades after Johnson (1929)'s publication, new research was done. At first, Carr (1970) provided foundational definitions and equations for the circle, followed by Coolidge (1971) introducing 65 new theorems in Brocardian geometry, but without advancing the understanding of the Brocard circle. It was not until 1995, with Honsberger (1995) work, that significant advances occurred, introducing novel discoveries in this circle. Among these discoveries, the following stands out:

Corollary 10. *Consider Ω_1 and Ω_2 as the Brocard points of $\triangle ABC$. The points $P = \overrightarrow{B\Omega_1} \cap \overrightarrow{C\Omega_2}$, $Q = \overrightarrow{C\Omega_1} \cap \overrightarrow{A\Omega_2}$, and $R = \overrightarrow{A\Omega_1} \cap \overrightarrow{B\Omega_2}$ are assumed to be in the Brocard circle. Therefore, $\triangle PQR$ is the first Brocard triangle.*

A few years after this discovery, C. Kimberling Kimberling (n.d. a) published what we know today as "The Encyclopedia of Triangle Centers (ETC)", which contains definitions of all of the discovered points that can be created from a triangle. In that article, he introduces a new notation system for each point where every triangle center should be written in the form X_n. His description of the midpoint of Brocard's diameter (X_{182}) is as follows:

Definition 11. *The point X_{182} is the midpoint of Brocard diameter (the distance OL in the Brocard circle) and therefore the center of the Brocard circle.*

After a while, it was shown that more points lie on Brocard's circle. At first, Yiu (2002) identified one of these points in 2003, but that information was never published until it was retrieved in 2019 and consequently included in the ETC in Kimberling (n.d. b) as the point X_{1083}. Here is a formal definition of that point and its properties within Brocard's circle

Definition 12. *Let L the Lemoine point and O the circumcenter. Define X_{105} as the isogonal conjugate of the point where line OL meets the line at infinity and X_{644} the midpoint of the trillinear multiplier for Yff parabola. The point X_{1083} is the midpoint of the segment $X_{105}X_{644}$*

Theorem 13. *The point X_{1083} lies on the Brocard circle.*

A year later, Dergiades and Yiu (2004) show another point passing through the Brocard circle; to be more specific, the point X_{1316} is defined as follows:

Definition 14. *The point X_{1316} well-know as the 5th Moses interception is the orthogonal projection of the Lemoine point L on the Euler line.*

Definition 15. *The 5th Moses interception lies on the Brocard circle.*

In the last place, during the 18th International Conference on Geometry and Graphics in Italy, held in 2018, V. Oxman and others presented what to the author's knowledge is the latest on the Brocard circle (see Oxman et al. (2018)). These results are summarized below.

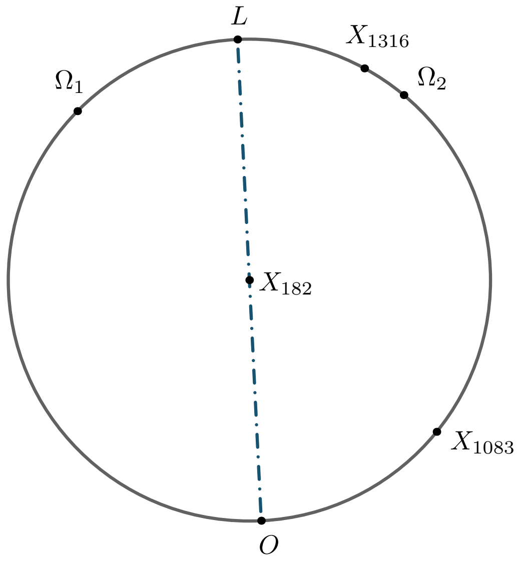


Figure 3. An overview of the Brocard circle in 2004 (up to the present)

Theorem 16. *The symmedian from vertex A of $\triangle ABC$ is tangent to the Brocard circle if and only if* $a^2 = \dfrac{b^2 + c^2}{2}$

Lemma 17. *Let L be the Lemoine point of right triangle $\triangle ABC$ with $m(\angle BAC) = 90°$ and let $\overline{AD}$ be the altitude of $\triangle ABC$. Then L is the midpoint of $\overline{AD}$. It follows that L, the circumcenter O of $\triangle ABC$, and the foot of the altitude, D, are all on the Brocard circle*

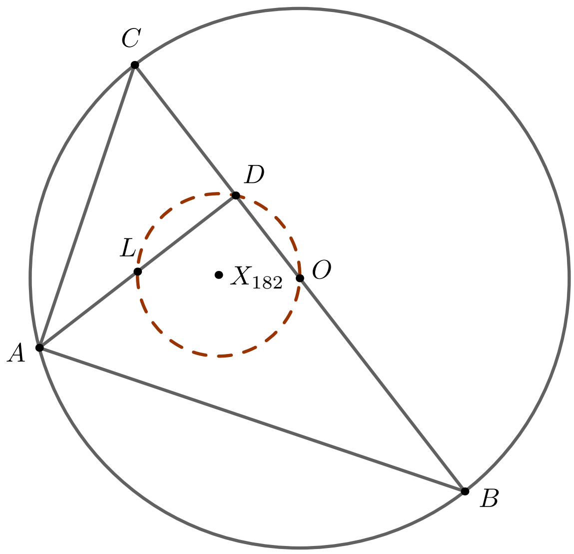


Figure 4. Diagram of Lemma 17 showing the foot of altitude D in the Brocard circle

3 Main Results

This section presents findings in the Brocard circle. All properties that have been discovered and have not been previously documented have been added.

Theorem 18 (Brocard–Lemoine Theorem)**.** *The Brocard circle of an isosceles right-angle triangle is always tangent to the unequal side of the triangle.*

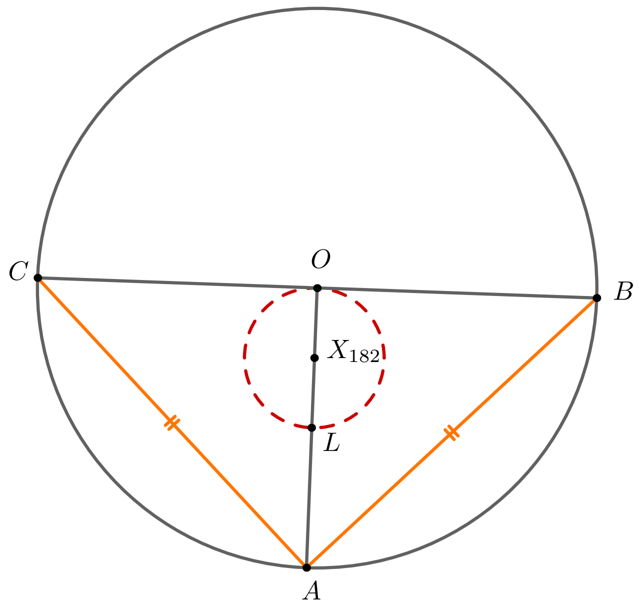


Figure 5. The hypotenuse of an isosceles right-angle triangle is always tangent to the Brocard circle

Proof. Since O is the circumcenter of (ABC), it follows that $OA = OB = OC$. Thus, O is the mid-point of $\overline{AC}$, and because in an isosceles right triangle the median to the unequal side is perpendicular, we obtain

$$\overline{AO} \perp \overline{BC}. \tag{1}$$

Considering the Lemoine point L described in Definition 1, for an isosceles right triangle it is well known that $\overline{AO}$ is the A-symmedian. Thus, L lies on $\overline{AO}$, and consequently X_{182} also lies on $\overline{AO}$. Since X_{182} lies on $\overline{AO}$, combining this with Aguilera (2023), we deduce

$$\overline{X_{182}O} \perp \overline{CB}. \tag{2}$$

Finally, by the tangent theorem, the segment $\overline{CB}$ is tangent to Brocard's circle. □

Theorem 19 (Brocard-Midpoint Theorem)**.** *In the Brocard circle of any triangle, if Ω_1 and Ω_2 are the Brocard points, L is the Lemoine point, O is the circumcenter and X_{182} the midpoint of the Brocard diameter, it will be fulfilled that $\triangle\Omega_1X_{182}L$, $\triangle\Omega_2X_{182}O$, $\triangle\Omega_1X_{182}O$, and $\triangle\Omega_2X_{182}L$ will all have the same area.*

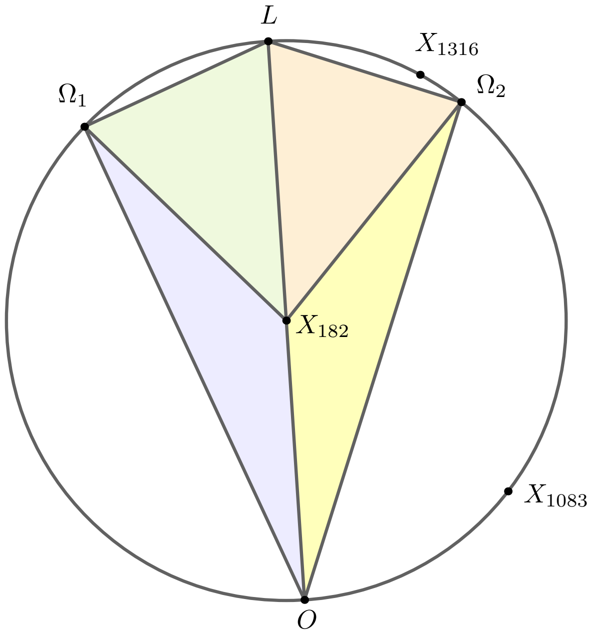


Figure 6. Ilustration of the Brocard-Midpoint Theorem in the Brocard Circle

Proof. According to Theorem 7, in the Brocard circle it holds that

$$L\Omega_1 = L\Omega_2. \tag{1}$$

Since X_{182} is the midpoint of the Brocard diameter, the distances from X_{182} to the relevant points are equal:

$$OX_{182} = \Omega_1X_{182} = \Omega_2X_{182} = LX_{182}. \tag{2}$$

From Aguilera (2023) and (2), triangles $\triangle\Omega_1X_{182}O$ and $\triangle\Omega_2X_{182}O$ are congruent by SSS. Similarly, $\triangle\Omega_1X_{182}L$ and $\triangle\Omega_2X_{182}L$ are congruent. Let LX_{182} and OX_{182} be the bases of $\triangle\Omega_1X_{182}L$ and $\triangle\Omega_1X_{182}O$, respectively. Their heights lie on the Brocard diameter; hence they have equal areas. The same argument applies to $\triangle\Omega_2X_{182}L$ and $\triangle\Omega_2X_{182}O$. Thus,

$$(\Omega_1X_{182}L) = (\Omega_2X_{182}L) = (\Omega_2X_{182}O) = (\Omega_1X_{182}O). \tag{3}$$

□

Corollary 20. *$\triangle\Omega_1X_{182}L$, $\triangle\Omega_2X_{182}O$, $\triangle\Omega_1X_{182}O$ and $\triangle\Omega_2X_{182}L$ are some of the Aguilera-Brocard triangles (see Aguilera (2023) and Kimberling (n.d. c)).*

Theorem 21 (Brocard−G Theorem)**.** *The centroid G of a triangle can always be found in the interior of the brocard circle.*

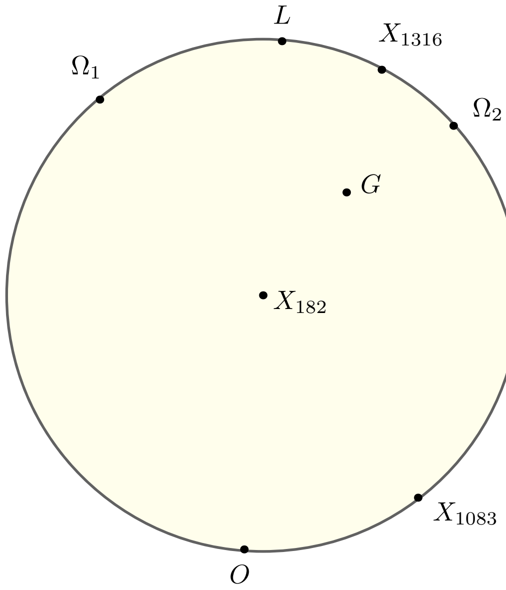


Figure 7. An illustration of the Brocard circle containing the Centroid G within it

Proof. By Leibniz' formula, valid for any point X in the plane,

$$XA^2 + XB^2 + XC^2 = 3XG^2 + GA^2 + GB^2 + GC^2.$$

Evaluating at $X = O$ (circumcenter) and $X = K$ (Brocard point) and subtracting yields

$$3R^2 - (KA^2 + KB^2 + KC^2) = 3(OG^2 - KG^2). \tag{1}$$

From Aguilera (2023) and (1) we obtain

$$OK^2 = R^2 - \frac{1}{3}(KA^2 + KB^2 + KC^2) + KG^2,$$

hence

$$\text{Pow}_{(OK)}(G) = OG^2 + KG^2 - OK^2 = \frac{1}{3}[GA^2 + GB^2 + GC^2 - KA^2 - KB^2 - KC^2]. \tag{2}$$

But $GA^2 + GB^2 + GC^2$ is the minimum of $XA^2 + XB^2 + XC^2$ over all points X in the plane, with equality only at $X = G$. Therefore the right–hand side of (2) is strictly negative unless the triangle is equilateral. Thus

$$\text{Pow}_{(OK)}(G) < 0,$$

which means that G lies strictly inside the Brocard circle (and on it in the equilateral case). □

Definition 22. *The Euler line intersects the brocard circle at two triangle centers which are the circumcenter O and the 5th moses intersection (X_{1316}).*

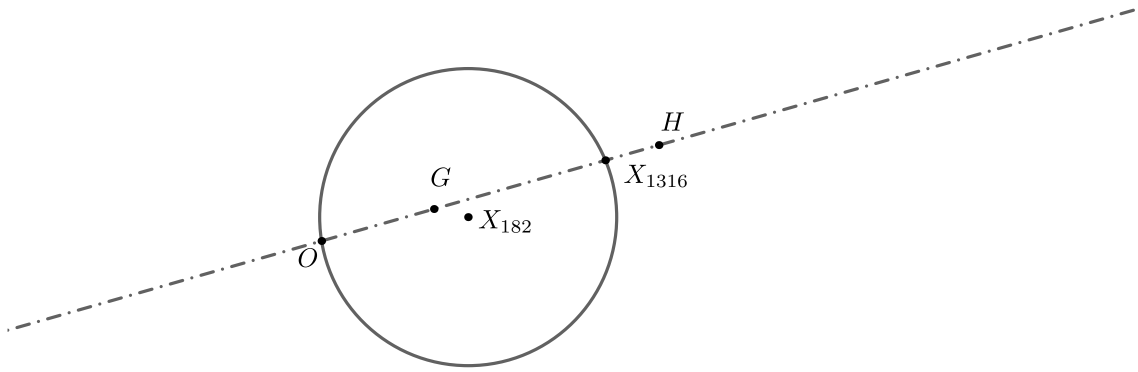


Figure 8. A diagram of the 5th Moses intersection in the Brocard circle

Construction of the 5th moses intersection. Select a point on your sheet of paper and name it O, this will be the circumcenter. Use this to create a circle Γ using a compass. On that circle, make three points so that they form $\triangle ABC$, subsequently draw the Lemoine point L and the point X_{182} using the Definition 11 for create the Brocard circle, determine the centroid and orthocenter of $\triangle ABC$, and then draw the Euler line. As stated in the Definition 22, it is clear to see that the Euler line cuts the Brocard circle at O but it also does it at another point, that point is the 5th moses interception.

Definition 23. *The point X_{1083} is the point that passes through the line $\overleftrightarrow{X_1X_6}$ and touches the Brocard circle.*

Construction of the point X_{1083}. Use the same indications for the construction of Definition 22 to construct the Brocard circle. In the same way, the Lemoine point will be constructed as well, and it will be called X_6, as will the incenter which will be called X_1. Finally, we will map out the line $\overleftrightarrow{X_1X_6}$, as a consequence and based on the work of Kimberling (n.d. b) the point that touches $\overleftrightarrow{X_1X_6}$ on the Brocard circle will be X_{1083}.

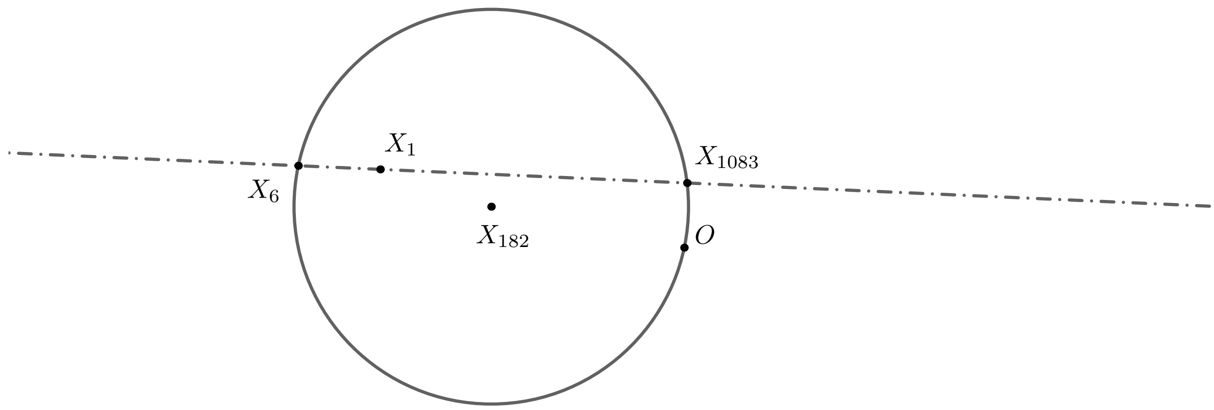
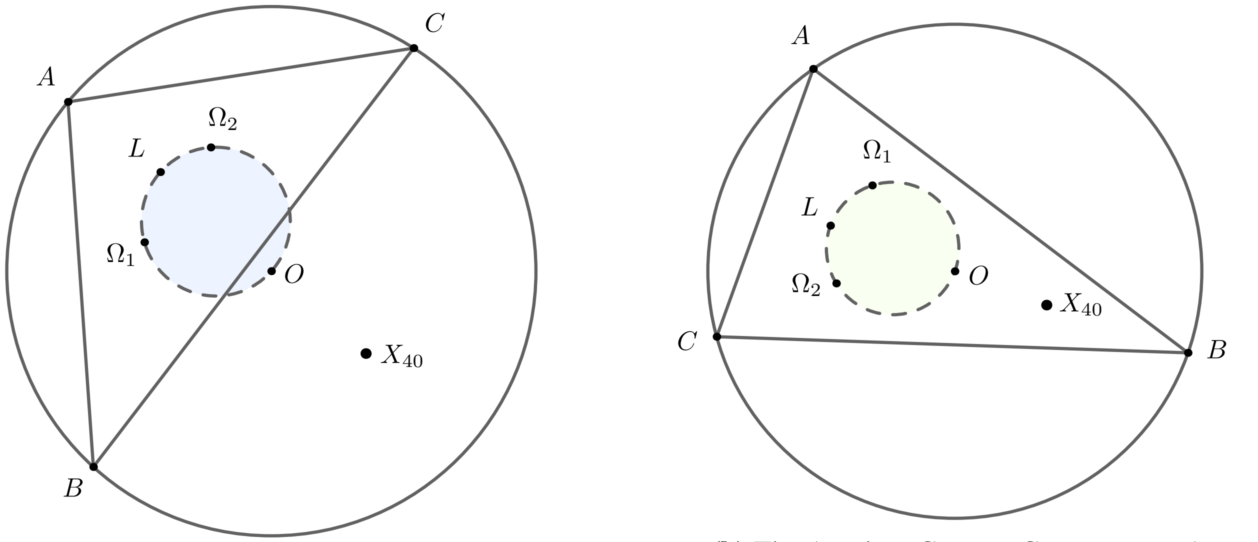


Figure 9. Diagram depicting point X_{1083} on the Brocard circle

Conjecture 24 (Aguilera-Camana Conjecture)**.** *The Bevan point* X_{40} *of a triangle is always situated outside the Brocard circle.*



(a) The Aguilera-Camana Conjecture for Obtuse Triangles.

(b) The Aguilera-Camana Conjecture in Acute Triangles

Figure 10. Specific Cases of the Aguilera-Camana Conjecture for Which the Statement Holds

Appendix-A

The following GeoGebra constructions correspond to the results presented in this article:
Brocard-Lemoine Theorem: `https://www.geogebra.org/m/aphfxssv`
Brocard-Midpoint Theorem: `https://www.geogebra.org/m/wggnus7x`
Brocard−G Theorem: `https://www.geogebra.org/m/rs2shfen`
Aguilera-Camana Conjecture: `https://www.geogebra.org/m/ukuttdxz`

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