# Investigating the Generalization of a Special Property of Cubic Polynomials to Higher Degree Polynomials

**David Miller**

**Abstract:** In this paper, the author extends earlier work (Miller, 2011; Miller and Moseley, 2012) relating roots of polynomial functions to tangent lines of their graphs to fourth and fifth degree polynomials. Using sliders, the author demonstrates how students can informally test the generalizability of relationships they uncover through dynamic capabilities of GeoGebra.

**Keywords:** GeoGebra, dynamic geometry software, polynomials, roots, tangent lines

## 1. INTRODUCTION

Miller (2011) discussed how to use a free interactive algebra-geometry program, GeoGebra, to investigate a special property of cubic polynomials. In that article, Miller (2011) demonstrates various ways to use GeoGebra to illustrate the special property that, given a cubic polynomial with three real zeros, a tangent drawn to the curve at the point at which the abscissa is the mean of two of those zeros will intersect the horizontal axis at the other zero. This property was extended by Miller and Moseley (2012) to higher degree polynomials. These two articles will provide a better foundation for readers that would like a better idea of the ideas in this article. We will use GeoGebra to extend how this special property of cubic polynomials can be generalized to higher degree polynomials. This will be done by discussing and illustrating in GeoGebra the cases for fourth and fifth degree polynomials, deriving formulas in terms of all but one of the zeros for the fourth and fifth degree polynomials, and stating the formula for higher degree polynomials. It is recommended that illustration in GeoGebra for the sixth degree polynomial be done by the reader to further familiarize themselves with the GeoGebra commands and the mathematics so that a complete understanding can be obtained.

## 2. FOURTH DEGREE POLYNOMIAL EXAMPLE

### 2.1. Defining a Polynomial with Sliders

Insert four sliders by clicking on the GeoGebra slider tool, labeling them $a$, $b$, $c$ and $d$. For each slider set the minimum to -10, the maximum to 10, and increment to 0.01. In the **Input** bar (by default at the bottom of the GeoGebra window) insert the command

```
f(x)=(x-a)*(x-b)*(x-c)*(x-d)
```

as suggested in Figure 1. This defines function $f$ with roots $a$, $b$, $c$ and $d$.

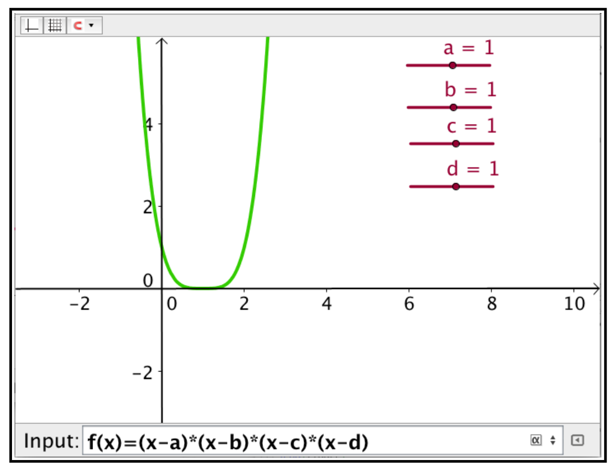


Fig 1: Graph of $f(x) = (x-a)*(x-b)*(x-c)*(x-d)$.

Next, type following commands into the **Input** bar one at a time: `a=-2`, `b=-1`, and `c=0`. Doing so will assign new values to the slider variables. Alternatively, you can adjust the sliders directly by dragging. After completing this step, your screen should look similar to that shown in Figure 2.

### 2.2. Constructing Roots

Next, we plot the $x$-intercepts of $f(x)$. In the input bar, start typing in `roots`. The GeoGebra autocomplete feature provides a drop down menu of different commands with `roots` in them. Select `Roots[<Function>, <Start x-value>, <End x-value>]`, then replace command parameters so that the command `Roots[f(x),-10,10]` appears in the input bar. Then press **Enter**. Points $A$, $B$, $C$, and $D$, the $x$-intercepts of the polynomial are generated. Your screen should look similar to Figure 3.

Next, type `g(x)=Derivative[f(x)]` into the **Input** bar as suggested in Figure 4. This plots $f'(x)$. Hide the plot by right clicking on the curve (on a Mac, press **control** and

Manuscript received Sept 11, 2012; revised Sept 20, 2012; accepted Oct 19, 2012.
David Miller is an associate professor at West Virginia University, Morgantown WV (email: damiller@mix.wvu.edu).


Fig 2: Graph of $f(x) = (x-a)*(x-b)*(x-c)*(x-d)$ for $a=-2$, $b=-1$, $c=0$, and $d=1$.

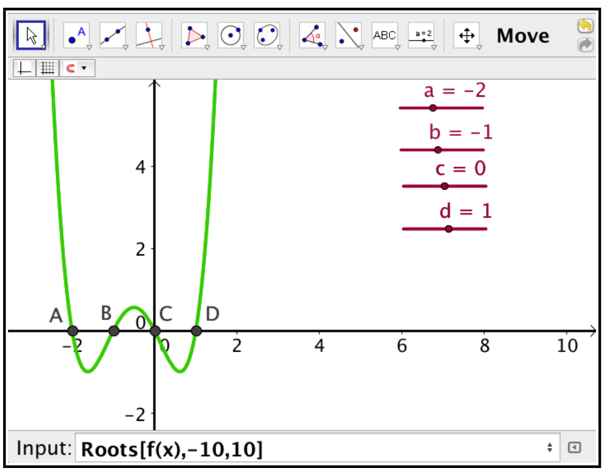


Fig 3: Graph of $f(x) = (x-a)*(x-b)*(x-c)*(x-d)$ for $a=-2$, $b=-1$, $c=0$, and $d=1$ with roots.

click) and unchecking the **Show Object** option. Alternatively, in Algebra view, click on the blue circle to the left of $g(x)$ to hide its plot. This is done to avoid overloading the applet with graphics; we only need the derivative to define the slope of the tangent line in later steps.

## 2.3. Constructing Tangent Lines Passing Through a Root

We need to provide some detail for the next formula that needs to be inputted into GeoGebra. We want to find the solutions to the quadratic equation obtained from the proof for higher degree polynomials. Miller and Moseley (2012) note that for polynomials of degree $n > 2$, there exist $n-2$ values $x_0$ such that

$$\sum_{i=1}^{n-1}\prod_{j=1}^{n-1}(x_0 - x_j) \quad = \quad 0 \tag{1}$$

where $i \neq j$ and $x_1$, $x_2$, $\ldots x_{n-1}$ are zeros of the polynomial of degree $n-1$. When a tangent is drawn to a graph of the polynomial of degree $n-1$ with abscissa equal to

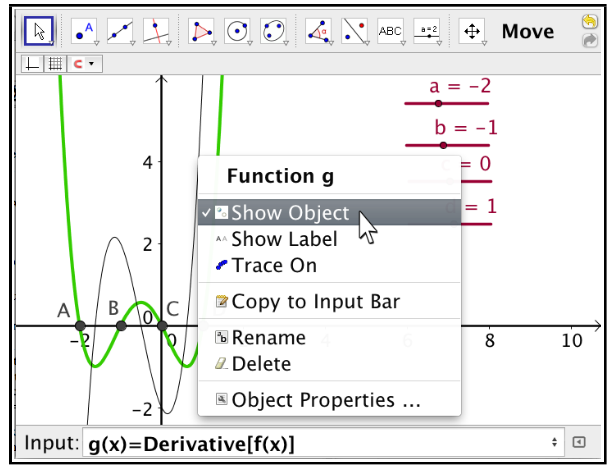


Fig 4: Constructing and hiding $g(x)$, the derivative of $f(x)$.

$x_0$, the line will intersect the horizontal axis at the nth zero.

For our current example, we have

$$\sum_{i=1}^{n-1}\prod_{j=1}^{n-1}(x_0 - x_j) = (x_0-x_2)(x_0-x_3) + (x_0-x_1)(x_0-x_3) + (x_0-x_1)(x_0-x_2) \tag{2}$$

where $x_1 = a$, $x_2 = b$, and $x_3 = c$. From this it follows that

$$(x_0-b)(x_0-c) + (x_0-a)(x_0-c) + (x_0-a)(x_0-b) = 0 \tag{3}$$

Expanding (3) yields

$$3x_0^2 + 2(a+b+c)x_0 + ab + ac + bc = 0. \tag{4}$$

Using the quadratic equation, the solutions to this equation are $x_0 = \frac{a+b+c\pm\sqrt{a^2+b^2+c^2-ab-ac-bc}}{3}$, which gives us the two solutions which we'll refer to as $e$ and $h$. According to Miller and Moseley (2012), lines tangent to $f(x)$ at abscissa values $e$ and $h$ will intersect the horizontal axis at $x_4 = d$.

Next, enter $e$ and $h$ into GeoGebra by typing the following commands into the **Input** bar:

```
Input: e=(a+b+c+sqrt(a^2+b^2+c^2-(a*b+a*c+b*c)))/3
Input: h=(a+b+c-sqrt(a^2+b^2+c^2-(a*b+a*c+b*c)))/3
```

Next, as illustrated in Figure 5, type `y-f(e)=g(e)*(x-e)` into the input bar to get the tangent line at $(e, f(e))$ that has an $x$-intercept at the other zero, $D$. Repeat for tangent line at $h$, by cutting and pasting `y-f(h)=g(h)*(x-h)`. Notice this tangent line also has a $x$-intercept at point $D$.

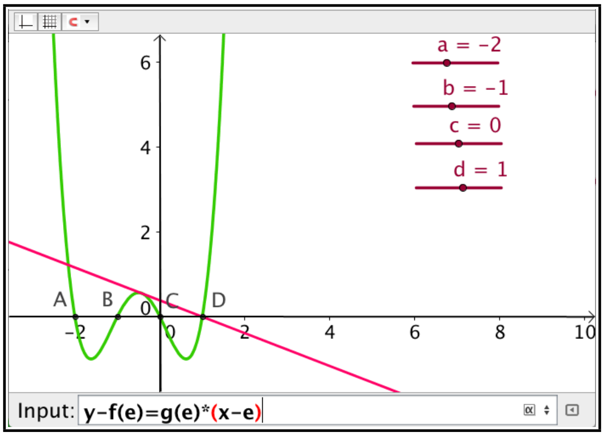


Fig 5: Screenshot of the GeoGebra graph of $f(x)$ with tangent line intersecting at point $D$.

## 2.4. Changing Object Properties

Note that the curves in the preceding figures have color. While this is not necessary, color helps students distinguish various objects on the screen. You can enhance the appearance of your sketches by modifying various object properties of your sketch. In your most recent sketch, right click (on a Mac, **control** click) on your tangent line and select **Object Properties**. You'll see a window similar to the one depicted in Figure 6.

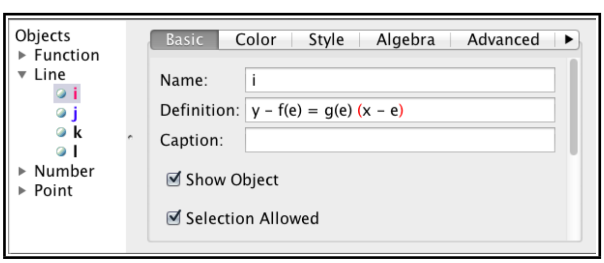


Fig 6: Object properties dialog box for tangent line $i$

As Figures 7 and 8 illustrate, you can modify the color and thickness of the line (in Figure 5, I chose Red with thickness of 6).

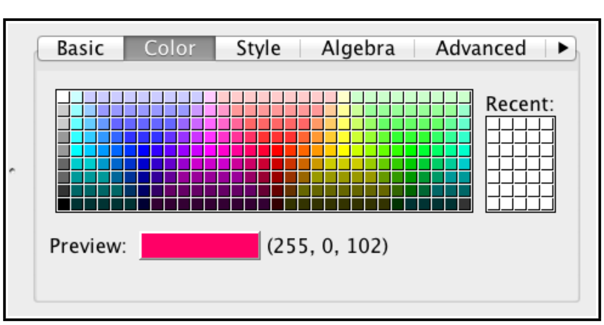


Fig 7: Color property for tangent line $i$

Repeat for other objects (e.g., $f(x)$ and sliders) in your sketch. Under object properties basic, you can label $f(x)$ and each tangent line by selecting the box for show label and clicking on the down arrow in the selection box to choosing name and value option. Your screen should look similar to Figure 9.

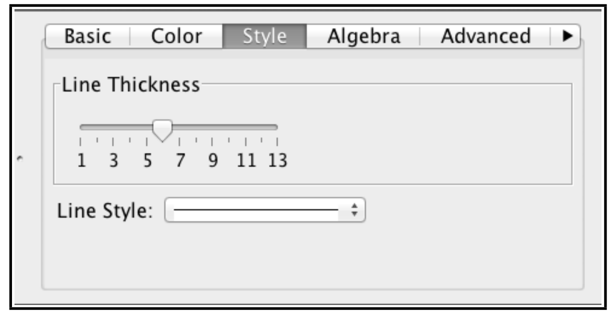


Fig 8: Thickness property for tangent line $i$

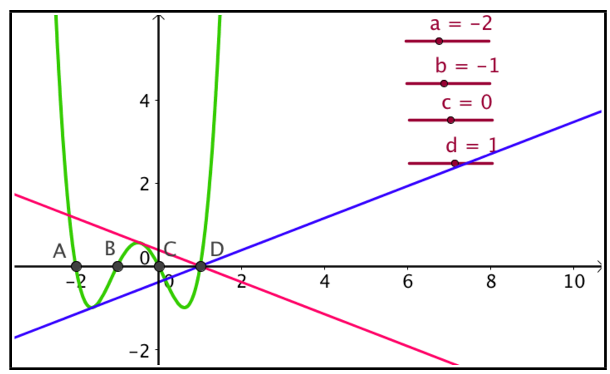


Fig 9: Screenshot of the GeoGebra graph polished with color.

## 2.5. Generalizing with Sliders

Before we start our next example, take a moment to informally confirm that our earlier property works for any value of $a$, $b$, $c$, and $d$. Choose any one of the sliders and select it. Use the keyboard right and left arrows to move the slider to different values (or simply drag with your computer mouse). Choose other sliders and vary them. We can see illustrations of cubic that has distinct real zeros and repeated real zeros. Miller and Moseley (2012) show in the general proof that the property holds with all numbers including complex zeros.

# 3. FIFTH DEGREE POLYNOMIAL EXAMPLE

Let's extend our earlier observation with a polynomial of degree 4 to polynomials of degree 5. We'll modify our existing GeoGebra sketch to see if our observations appear to hold in the new case.

Begin by adding a new slider to your sketch with minimum -10, maximum 10, and increment of 0.01. Next, redefine $f(x)$ by typing the following command into the **Input** bar. (Alternatively, one may redefine the function by double clicking on it in the **Algebra** view).

```
f(x)=(x-a)*(x-b)*(x-c)*(x-d)*(x-e)
```

Next, type `a=-2`, `b=-1`, `c=0`, `d=1`, and `e=2` in the **Input** bar. This generates a curve with distinct roots that displays well in the **Graphics View**. GeoGebra automatically updates roots to $A$, $B$, $C$, $D$, and $E$. After you complete these steps, your sketch should look similar to the one depicted in Figure 10 (note that we have deleted the tangent lines from our previous example).

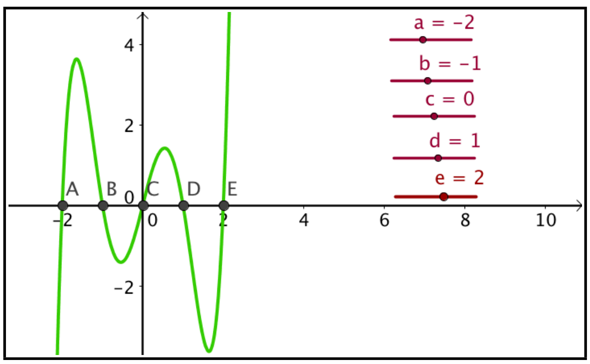


Fig 10: Graph of $f(x) = (x-a)*(x-b)*(x-c)*(x-d)*(x-e)$.

Using an approach analogous to our previous example, we'll use 4 of the 5 roots of $f(x)$ to generate a cubic equation. The 3 solutions to this cubic will be used as abscissa values of points of tangency for the graph of $f(x)$. If all works as expected, the three tangent lines that we construct at these points of tangency should pass through the remaining root of $f(x)$.

From Miller (2012), it is apparent that for higher degree polynomials

$$\sum_{i=1}^{n-1}\prod_{j=1}^{n-1}(x_0-x_j) = 0 \tag{5}$$

where $i \neq j$ and $x_1$, $x_2$, $x_3$, and $x_4$ are four roots of the fifth degree polynomial.

Expanding this out we have

$$\begin{aligned}(x_0-x_2)(x_0-x_3)(x_0-x_4)+&\\ (x_0-x_1)(x_0-x_3)(x_0-x_4)+&\\ (x_0-x_1)(x_0-x_2)(x_0-x_4)+&\\ (x_0-x_1)(x_0-x_2)(x_0-x_3)=0&\end{aligned} \tag{6}$$

When $x_1 = a$, $x_2 = b$, $x_3 = c$, and $x_4 = d$,

$$\begin{aligned}(x_0-b)(x_0-c)(x_0-d)+&\\ (x_0-a)(x_0-c)(x_0-d)+&\\ (x_0-a)(x_0-b)(x_0-d)+&\\ (x_0-a)(x_0-b)(x_0-c)=0(*)&\end{aligned} \tag{7}$$

Since this is a cubic equation, we use GeoGebra to solve it (rather than solving by hand). In Geogebra, type the following definition into the **Input** bar:

$$\begin{aligned}m(x) = (x-b)(x-c)(x-d)+&\\ (x-a)(x-c)(x-d)+&\\ (x-a)(x-b)(x-d)+&\\ (x-a)(x-b)(x-c)&\end{aligned}$$

Change the appearance of $m(x)$ to be a dotted line by right clicking on its graph, selecting **Object properties**, and from the **Style** tab changing the **Line Style** to dotted. This somewhat hides $m(x)$, but illustrates the three important points that are solutions of this cubic equation. After completing these steps, your sketch should look similar to the one depicted in Figure 11.

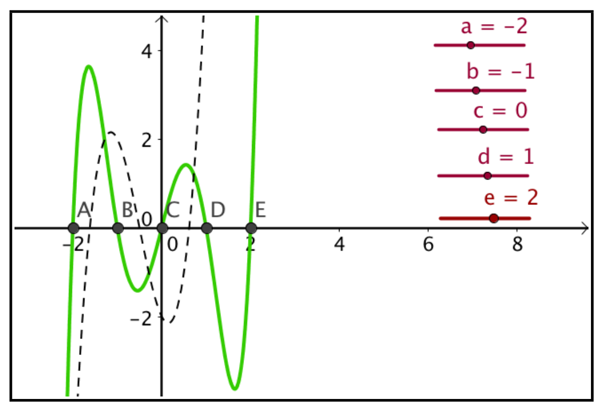


Fig 11: Graph of $f(x)$ and $m(x)$.

To find the zeros of $m(x)$ (i.e. solving the equation (*)), cut and paste into the **Input** bar, `Root[m(x)]`. You should see the points $F$, $G$, and $H$ appear on the graph and in **Algebra** view. These are the three points at which we anticipate the tangent lines have $x$-intercepts at $E$. To confirm this, we first obtain the $x$-coordinates of $F$, $G$, and $H$ by typing the following commands separately into the input bar: `n=x(F)`, `p=x(G)`, and `q=x(H)`.

Next, enter `j(x)=Derivative[f(x)]` in the **Input** bar and hide it (again to avoid the screen to get to cluttered). We now insert the three tangent lines that we anticipate will intersect at the fifth zero, $E$.

Now type separately the following commands:

```
y-f(n)=j(n)*(x-n)
y-f(p)=j(p)*(x-p)
y-f(q)=j(q)*(x-q)
```

Since two of the tangent lines are the same for these specific parameters $a$, $b$, $c$, and $d$ type in `d=1.2` into the **Input** bar. This make three distinct tangent lines visible. Optionally, add color, line thickness and labeling to enhance your sketch. Ultimately, your sketch should appear similar to the one depicted in Figure 12.

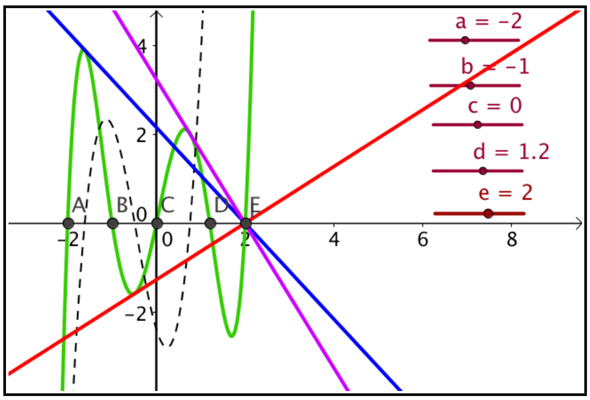


Fig 12: Graph of $f(x)$ and $m(x)$.

### 3.1. Generalizing with Sliders

Again do a drag test by varying the sliders to different values to informally confirm that the property holds for all real zeros $a$, $b$, $c$, $d$, and $e$.

## 4. CONCLUSION

This article has shown how to use GeoGebra to extend a mysterious property of cubic polynomials to more general properties for fourth and fifth degree polynomials. One can illustrate this for an $nth$ degree polynomial to find the $n-2$ points on the $x$-axis in terms of $n-1$ zeros, that is given by the solutions to the equation below, where $x_1$, $x_2$, $\ldots$, $x_{n-1}$ are the $n-1$ zeros, $\sum_{i=1}^{n-1}\prod_{j=1}^{n-1}(x_0-x_j)=0$, where $i\neq j$, such that the tangent line to the $nth$ degree polynomial at each $(s_i, f(s_i))$ for $i=1,2,\ldots,n-1$ intersects the $x$-axis at the other zero $x_n$. Here $s_i$ for $i=1,2,\ldots,n-1$ are the zeros of the resulting $n-2$ degree polynomial derived from the formula above. See the article Miller and Moseley (2012) for more details and the underlying calculus concept behind the general property for polynomials. The reader should illustrate this for the $6th$ degree polynomial and think about the proof before referencing the articles. This article shows some basic functions of GeoGebra so that the readers may familiarize themselves with the program. This is a good tool for students to discover some mathematics about polynomials in which they can see some specific examples and work on a more general proof via paper and pencil.

## REFERENCES

Hamilton, P. (2005). Mathematics IB Higher Portfolio: Zeros of Cubic Functions. Retrieved from `http://www.scribd.com/doc/18118175/Zeros-of-Cubic-Functions`.

Miller, D. (2011). Investigating Zeros of Cubics through GeoGebra. *Mathematics Teacher*, 105(2).

Miller D. & Moseley, J. (2012). Extending a Property of Cubic Polynomials to Higher-Degree Polynomials. *MathAMATYC Educator*, 3(3).

Stewart, J. (2007). *Essential Calculus: Early Transcendentals*. Thomson Brooks/Cole.

Sullivan & Sullivan (2006). *Algebra and Trigonometry: Enhanced with Graphing Utilities (4th ed.)*. Pearson Prentice Hall.