Volume and area ratios with GeoGebra

Libuse Samkova

Abstract: This article was motivated by student work in a biological laboratory where a question was asked regarding the liquid level corresponding to $\frac{1}{5}$ of the volume of a laboratory flask. In this article, we use GeoGebra to explore such questions, analyzing volume and area ratios for containers of various shapes. We present various dynamic GeoGebra models including one of the experiment which initiated the article.

Keywords: GeoGebra, modeling, algebraic proof, physics

1. INTRODUCTION

This article was motivated by student work in a biological laboratory during the demonstration of a candle experiment similar to that suggested illustrated in Fig. 1.

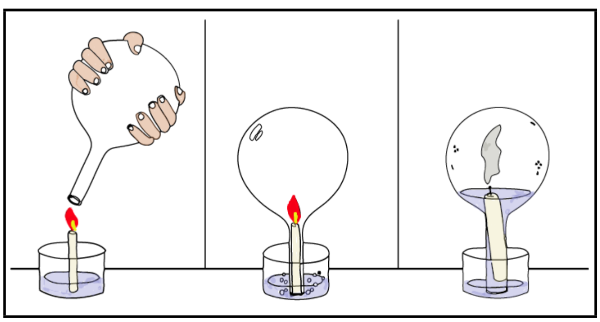

Fig 1: An illustration of a candle experiment (Gallástegui 2011).

The student experimenter ended the presentation by stating that "the liquid in the flask occupies 20% of the flask volume," and the audience nodded in agreement. Is it really so easy to determine $\frac{1}{5}$ of the flask volume? Consider the following steps required to complete the calculation.

- We have to determine $\frac{1}{5}$ of the volume of the flask whose shape consists of a ball and a cylinder linked together.
- We have to take into account the volume of the submerged part of the candle.

2. BALL CASE

Let's start with a simplified version of the problem by calculating $\frac{1}{5}$ of the volume of a ball (and ignoring, for the moment, the volume of the liquid in the cylindrical portion of the flask). Note that $\frac{1}{5}$ of the ball volume does not correspond to $\frac{1}{5}$ of the ball height.

The volume of a ball with radius r is given by a formula $\frac{4}{3}\pi r^3$. The liquid in the ball reaches an unknown height h, and its volume equals the volume of the shaded region shown in Fig. 2 (which we refer to as a spherical cap).

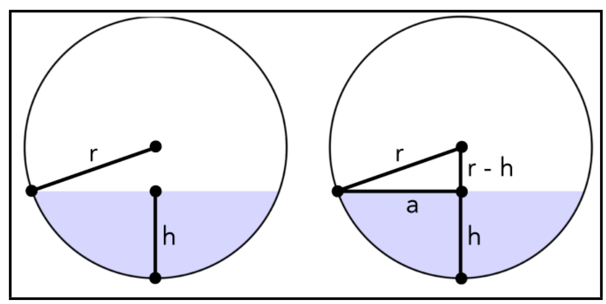

Fig 2: Front views of the liquid in the ball (left), and of the spherical cap (right).

A basic volume formula for a spherical cap with height h and base radius a is given by

$$V_{cap} = \frac{1}{6}\pi h(3a^2 + h^2) \quad \text{(Weisstein, 2012a)} \tag{1}$$

From Pythagorean theorem we have

$$a^2 = r^2 - (r-h)^2 = r^2 - r^2 + 2rh - h^2 = 2rh - h^2 \tag{2}$$

which implies that

$$\begin{aligned} V_{cap} &= \frac{1}{6}\pi h(6rh - 3h^2 + h^2) \\ &= \frac{1}{6}\pi h(6rh - 2h^2) \\ &= \frac{1}{3}\pi h^2(3r - h) \quad \text{(Weisstein, 2012a)} \end{aligned} \tag{3}$$

For clarity in subsequent calculations, we express the height of the liquid as a k-multiple of the height of the ball, i.e., $h = k \cdot 2r$. Then,

$$V_{cap} = \frac{1}{3}\pi(k\cdot 2r)^2(3r - k\cdot 2r) = \frac{4}{3}\pi k^2 r^3(3-2k) \tag{4}$$

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Next, we solve the following equation to find a liquid level corresponding to $\frac{1}{5}$ of the ball volume.

$$
\begin{aligned}
V_{cap} &= \frac{1}{5}\cdot V_{ball} && (5)\\
\frac{4}{3}\pi k^2 r^3(3-2k) &= \frac{1}{5}\cdot\frac{4}{3}\pi r^3 && (6)\\
k^2(3-2k) &= \frac{1}{5} && (7)\\
2k^3-3k^2+\frac{1}{5} &= 0 && (8)
\end{aligned}
$$

Note that the ratio $\frac{1}{5}$ appears as a constant term in the resulting cubic equation, as shown in (8).

This equation does not have an easy manual solution, but we can solve it graphically with help of GeoGebra. We generate the graph of a function $f(x) = 2x^3 - 3x^2 + \frac{1}{5}$ and construct the x-coordinate of the intersection of the graph and the x-axis. This coordinate is the required solution to the cubic equation (8).

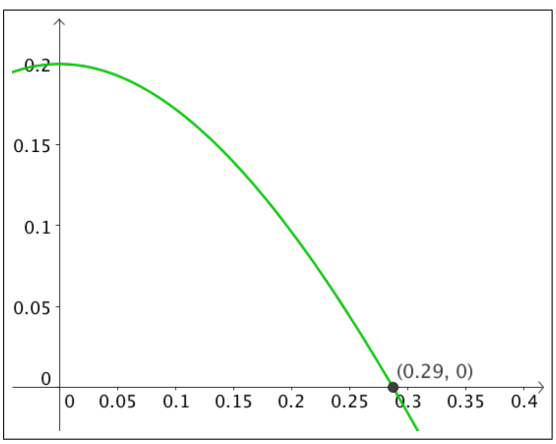

Fig 3: Graphical solution to $2x^3 - 3x^2 + \frac{1}{5} = 0$.

As shown in Fig. 3, the solution is approximately 0.29. What does it mean? If we have a ball with diameter 10 cm, then the height of the ball is also 10 cm, and the liquid occupying $\frac{1}{5}$ of the ball volume reaches a height of $0.29\cdot 10$ cm ≈ 2.9 cm.

2.1. Generalizing the ball case

Next, we generalize the problem for arbitrary ratio $\frac{m}{n}$ of ball volume. The corresponding k now satisfies the equation $2k^3 - 3k^2 + \frac{m}{n} = 0$. As shown in Fig. 4, we use sliders m and n to generate function $f(x) = 2x^3 - 3x^2 + \frac{m}{n}$, then - as before - we approximate the solution using the intersection tool.

Alternatively, we can create a dynamic sketch modeling the liquid in the ball, as shown in Fig. 5. (*Editor's note: A copy of the sketch, Fig5.ggb, is provided alongside the pdf version of this paper.*)

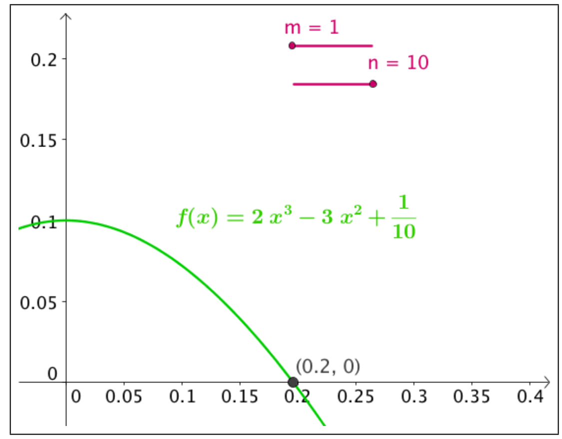

Fig 4: Graphical solution using sliders.

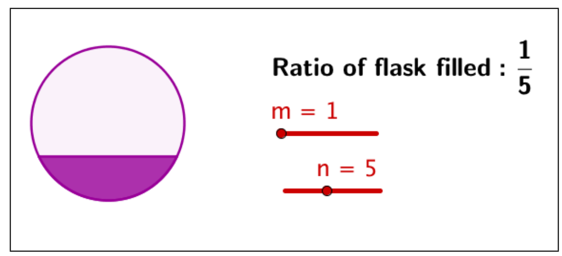

Fig 5: Dynamic model illustrating a solution for $\frac{1}{5}$ of ball volume.

The model is based on the graphical solution from Fig. 4. First we create a circle with radius r and sliders for m and n. For given m and n, we define height ratio k as the solution to the equation $2x^3 - 3x^2 + \frac{m}{n} = 0$. Then we create a circumcular arc with a height $k\cdot 2r$, and adjust its opacity to 80 percent to simulate liquid.

With this dynamic model we can investigate various volume ratios. Note that the liquid level for $\frac{1}{10}$ of the volume reaches approximately $\frac{1}{5}$ of the ball height. This corresponds to the numerical solution in Fig. 4.

3. BACK TO THE EXPERIMENT

3.1. Building the flask with liquid

With some work, a dynamic sketch modeling the liquid in the flask from the original candle experiment can be constructed. We do so first without the candle, answering the question "What does $\frac{1}{5}$ of the flask volume look like?" Again, the model is constructed dynamically, with sliders used to modify flask proportions (ρ controls the radius of the cylinder; r, the radius of the ball; h, the height of the cylinder). These features are illustrated in Fig. 6. (*Editor's note: A copy of the sketch, Fig6.ggb, is provided alongside the pdf version of this paper.*)

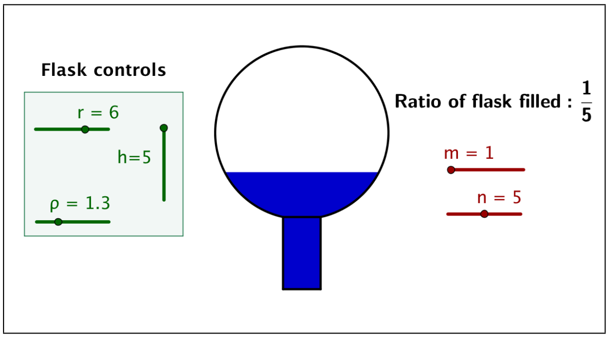

Fig 6: A dynamic model of the flask volume

3.2. Adding the candle

Building on the previous sketch, we create a model of the experiment with a candle inside the flask. The construction requires knowledge of the Archimedes' principle of buoyancy which states that the buoyant force exerted on a solid floating in the liquid is equal to the weight of the volume of the liquid which is displaced by the solid, i.e.,

$$V_{solid} \cdot g \cdot \rho_{solid} = V_{submerged\ part} \cdot g \cdot \rho_{liquid} \tag{9}$$

$$V_{submerged\ part} = V_{solid} \cdot \frac{\rho_{solid}}{\rho_{liquid}} \tag{10}$$

see (Weisstein 2012c and Burley, et al. 1997). The liquid in our experiment is the water, with density 1000 kg/m^3. The density of the candle is determined by an auxiliary experiment as 950 kg/m^3. This means that

$$V_{submerged\ part} = V_{candle} \cdot \frac{950}{1000} \tag{11}$$

$$V_{submerged\ part} = 0.95 \cdot V_{candle} \tag{12}$$

Note that (12) implies the following:

- If 95% of the candle height is less than the depth of the water, then the candle floats in the water, and the height of the submerged part of the candle equals the 95% of the candle height.
- Otherwise the candle stands on the bottom.

We use sliders to model changes in candle proportions (c_h for the height of the candle, c_r for base radius). See the final illustration in Fig. 7. (*Editor's note: A copy of the sketch, Fig7.ggb, is provided alongside the pdf version of this paper.*)

4. FURTHER MODELS OF VOLUME AND AREA RATIOS

A cone provides an interesting, and more challenging, context for exploring volume ratios. Unlike the ball, a cone does not satisfy the rule that half of the volume corresponds to half of the height.

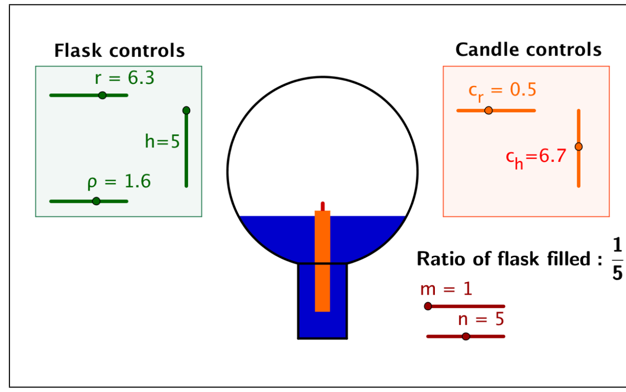

Fig 7: A dynamic model to the candle experiment.

The volume of a cone with base radius r and height h is given by a formula $\frac{1}{3}\pi r^2 h$. The liquid in the cone reaches an unknown height h_c, and its volume equals the volume of a conical frustum with height h_c, and base radius r. The top radius of the conical frustum is not obvious, but it can be derived from h_c, r, and h. Let's express h_c as a k-multiple of the height of the cone, i.e., $h_c = k \cdot h$. Then the similarity of triangles shown in Fig. 8 gives the top radius of the conical frustum equal $r(1-k)$.

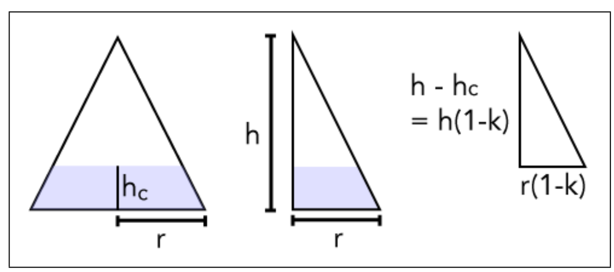

Fig 8: A front view of the liquid in the cone (left), and the two similar triangles (middle and right).

Weisstein (2012b) notes that the volume of a conical frustum can be expressed as

$$V_{frust} = \frac{1}{3}\pi h \left(r_{base}^2 + r_{base} r_{top} + r_{top}^2\right) \tag{13}$$

Thus, in our particular case:

$$\begin{aligned} V_{frust} &= \frac{1}{3}\pi kh \left(r^2 + r^2(1-k) + r^2(1-k)^2\right) \\ &= \frac{1}{3}\pi r^2 kh(3 - 3k + k^2) \end{aligned} \tag{14}$$

We seek a liquid level corresponding to $\frac{m}{n}$ of the cone volume. We accomplish this in the following manner.

$$V_{frust} = \frac{m}{n} \cdot V_{cone} \quad (15)$$

$$\frac{1}{3}\pi r^2 kh(3-3k+k^2) = \frac{m}{n} \cdot \frac{1}{3}\pi r^2 h \quad (16)$$

$$k(3-3k+k^2) = \frac{m}{n} \quad (17)$$

$$k^3-3k^2+3k-\frac{m}{n} = 0 \quad (18)$$

Again, the ratio $\frac{m}{n}$ appears as a constant term in the cubic equation, but with a negative sign.

A GeoGebra model of the liquid in the cone can be constructed using an auxiliary function $f(x) = x^3-3x^2+3x-\frac{m}{n}$. It is worth noting that the level of the liquid does not depend on the base radius r, shown using a slider for r. This is illustrated in Fig. 9. (*Editor's note: A copy of the sketch, Fig9.ggb, is provided alongside the pdf version of this paper.*)

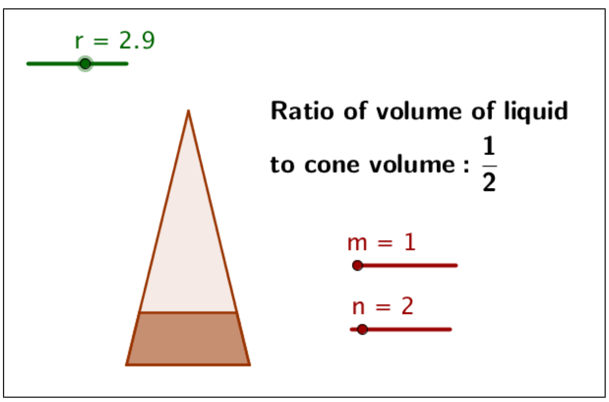

Fig 9: A dynamic model of the cone volume.

Similar dynamic models can be obtained for the area of plane figures, e.g., the area of a circle. In this case the auxiliary function is given by a formula

$$f(x) = \arcsin(2\sqrt{x-x^2}) - 2\sqrt{x-x^2}(1-2x) - \pi\frac{m}{n} \quad (19)$$

We encourage the interested reader to use this function to conduct further explorations.

5. CONCLUSION

The topics of volume and area ratios have not been a common part of mathematics lessons, due to the fact that equations (8), (18), and (19) do not have an easy manual solution. However, with the assistance of GeoGebra, such equations are now accessible to students.

The GeoGebra models presented here are restricted to two dimensions. There are skilled GeoGebra developers working on a 3D version of GeoGebra, so in the near future it will be possible to create similar 3-dimensional dynamic models for balls, cones, and other solids.

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