A VISUAL APPROACH TO GROUP HOMOMORPHISMS USING GEOGEBRA

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Abstract

This article, excerpted from an ongoing doctoral research project, presents a visual approach to the study of group homomorphisms using GeoGebra as a tool for construction and exploration. Focusing on the classical homomorphism from the symmetric group S_3 *to the cyclic group* $\mathbb{Z}_2$*, the study analyzes two distinct visual representations—one in 2D and the other in 3D. These constructions highlight fundamental algebraic properties such as kernel, image, and cosets, while also illustrating the preservation of operations and offering an intuitive application of the First Isomorphism Theorem.*

Keywords: Group Homomorphisms, GeoGebra, Mathematics Visualization

1 INTRODUCTION

The study of group homomorphisms examines mappings that preserve the algebraic structure of groups, enabling an analysis of their properties and relationships (Gallian, 2016; Gonçalves, 1995). Despite their theoretical relevance, homomorphisms often present high levels of abstraction that hinder intuitive understanding, especially among students and novice researchers.

This study is motivated by the need to enhance didactic approaches for teaching group structures in abstract algebra, providing resources that support teachers in planning and conducting effective lessons. In this context, visual resources such as dynamic software environments have shown potential to simplify abstract concepts, establish bridges between theoretical and concrete representations, and support clear and systematic content exposition (Alves and Araújo, 2014; Alves, 2020; Sousa et al., 2024b,a; Carter, 2009).

Recent studies have highlighted GeoGebra's versatility in representing mathematical structures, including permutation groups, symmetries, and relationships between finite groups and other algebraic systems (Sousa et al., 2024b,a,c,d). Building on these contributions, this work proposes a didactic strategy that associates homomorphism theory with dynamic visual representations using GeoGebra.

The present article is intended for mathematics educators and teacher trainers seeking ways to make abstract algebraic topics more accessible. By leveraging the interplay between theory and visualization, we aim to enable the exploration of core concepts such as kernel, image, and cosets in a guided and interactive manner, promoting both conceptual understanding and practical application in the classroom.

2 Group Homomorphisms and their Properties

Definition 1 (Group Homomorphism)**.** *Let G and H be groups with binary operations $*$ and $\cdot$, respectively. A function $\phi : G \to H$ is called a **group homomorphism** if (Garcia and Lequain, 2015):*

$$\phi(a * b) = \phi(a) \cdot \phi(b) \quad \forall a, b \in G$$

This implies that the operation in G is compatible with the operation in H under the action of ϕ, preserving the algebraic structure. For a homomorphism between groups to exist, it must satisfy some particular properties, which are:

- Preservation of the identity element: For any homomorphism $\phi : G \to H$, the identity element of G, denoted here as e_G, is mapped to the identity element of H, denoted analogously as e_H, that is: $\phi(e_G) = e_H$.

- Preservation of inverses: For any $a \in G$, we have:

$$\phi(a^{-1}) = \phi(a)^{-1}.$$

- Kernel and image: The kernel of a homomorphism ϕ is defined as:

$$kef(\phi) = \{x \in G \mid \phi(x) = e_H\}$$

  where the kernel is a normal subgroup of G. The image of ϕ is:

$$Im(\phi) = \{\phi(x) \mid x \in G\}$$

  where the image is a subgroup of H.

The parity properties of a permutation and its signature are fundamental to understanding the structure of the symmetric groups $S_3 e S_4$ in the context of homomorphisms, as they allow the classification of permutations based on the number of transpositions (2-cycles) they comprise.

Definition 2 (Parity of a permutation)**.** *Let $\sigma \in S_n$ be a permutation in the symmetric group of n elements. We say that (Gallian, 2016):*

*(i) σ is an **even permutation** if it can be expressed as the product of an even number of transpositions. (ii) σ is an **odd permutation** if it can be expressed as the product of an odd number of transpositions.*

The classification of permutations as even or odd will be used to distinguish cosets in the study of the homomorphisms addressed.

Definition 3 (Signature of a permutation). *Let $\sigma \in S_n$ be a permutation represented by the correspondence between the sets $\{1, 2, \dots, n\}$ and $\{b_1, b_2, \dots, b_n\}$. The signature of σ, denoted by sgn(σ), is given by (see proof in Domingues and Iezzi, 2003):*

$$sgn(\sigma) = \prod_{i<j} \frac{a_i - a_j}{b_i - b_j}$$

where the product is calculated for all pairs (i, j) with $i < j$. This definition implies that: (i) The signature of the identity permutation is 1. (ii) The signature of a transposition is -1.

The fundamental property of this definition is that it does not depend on the order of the columns in the matrix representing σ. Moreover, the signature of a permutation is related to the parity of the number of transpositions in its decomposition. If σ is an even permutation, $\text{sgn}(\sigma) = 1$; if σ is odd, $\text{sgn}(\sigma) = -1$.

Theorem 4 (First Homomorphism Theorem). *If $\phi : G \to H$ is a group homomorphism, then the quotient group $G/ker(\phi)$ is isomorphic to the image of ϕ, that is:*

$$G/\mathit{ker}(\phi) \cong \mathit{Im}(\phi)$$

The theorem establishes a fundamental relationship between the quotient group $G/\ker(\phi)$, the kernel $\ker(\phi)$ and the image $\text{Im}(\phi)$, allowing us to understand the structure of G in relation to H (see the proof in Gonçalves, 1995).

The First Isomorphism Theorem states that for a group homomorphism $\phi : G \to H$, the quotient group $G/\ker(\phi)$ is isomorphic to the image $\text{Im}(\phi)$. This means that the structure of the image can be understood entirely in terms of the cosets of the kernel. This result plays a central role in our visual constructions, as it allows us to interpret the geometric grouping of elements (e.g., in concentric circles) as representatives of cosets under this quotient structure.

The definition of group homomorphism and its properties underpins the visual construction and analysis of the examples in the following sections. The kernel, $\ker(\phi)$, is defined as the set of domain elements mapped to the codomain's identity, while the image $(Im(\phi))$ consists of elements reached by the homomorphism. Structural operation preservation by homomorphisms has also been formally established.

Parity and permutation signature concepts clarify the classification of elements in S_3 and S_4. Parity, based on the number of transpositions, and the signature, represented by the homomorphism $\text{sgn} : S_n \to \mathbb{Z}_2$, distinguish even and odd permutations, organizing cosets relative to the kernel.

3 Construction in GeoGebra: The case $\phi : S_3 \to \mathbb{Z}_2$

In this mapping, the symmetric group S_3, of order 6, is mapped onto the cyclic group $\mathbb{Z}_2$, of order 2. The parity of permutations defines the homomorphism:

- Even permutations $\{e, (123), (132)\}$ are mapped to $0 \in \mathbb{Z}_2$.
- Odd permutations $\{(12), (13), (23)\}$ are mapped to $1 \in \mathbb{Z}_2$.

Regarding the mathematical interpretation of this homomorphism, we have:

(a) Parity of permutations: Even permutations have a signature of $+1$, while odd permutations have a signature of -1. The homomorphism ϕ checks the parity and assigns:

$$\phi(\sigma) = \begin{cases} 0 & \text{if } \operatorname{sgn}(\sigma) = +1 \\ 1 & \text{if } \operatorname{sgn}(\sigma) = -1 \end{cases}$$

(b) Preserved properties:

- The identity element e is mapped to the identity $0 \in \mathbb{Z}_2$.
- The composition of two permutations preserves parity, reflected as addition in $\mathbb{Z}_2 \pmod 2$.

The construction was implemented in GeoGebra using algebraic commands to assign colors and labels to each permutation in S_3, according to its parity. Functions such as `Mod`, `If`, `SetColor`, and `Text` were applied to dynamically display the image of each element under the homomorphism ϕ, allowing interactive visualization of kernel elements and cosets in the domain.

The visual representation of this mapping is presented in Figure 1.

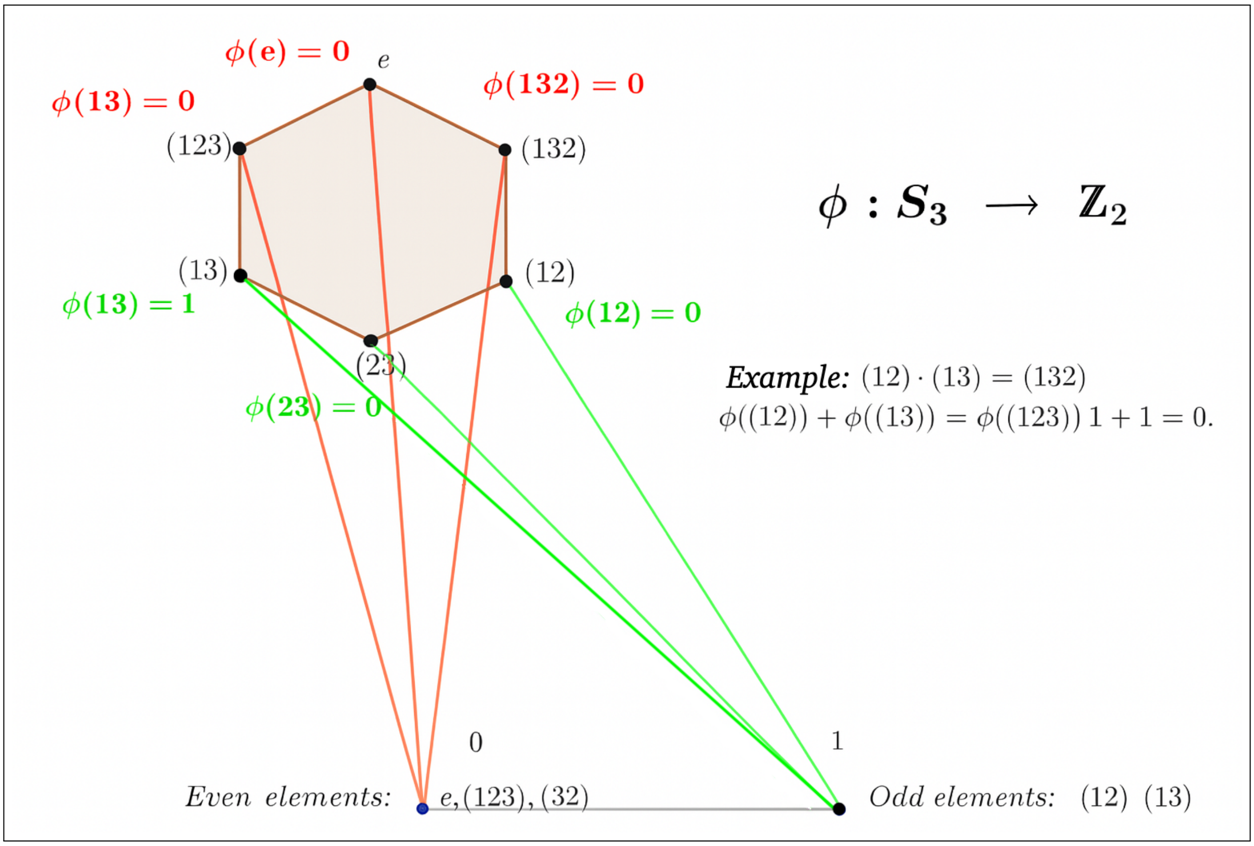

Figure 1. Homomorphism $\phi : S_3 \to \mathbb{Z}_2$.

In Figure 1, which shows the graphical representation in GeoGebra, it can be observed that colors are used as follows: (a) red for elements mapped to 0, and (b) green for elements mapped to 1. Additionally, arrows connect the elements of S_3 to their respective values in $\mathbb{Z}_2$.

To construct this visual representation in GeoGebra, the following steps can be followed:

1. Use the polygon tool to create a regular hexagon, representing the 6 elements of S_3.
2. Label the vertices with the elements $e, (123), (132), (12), (13), (23)$.

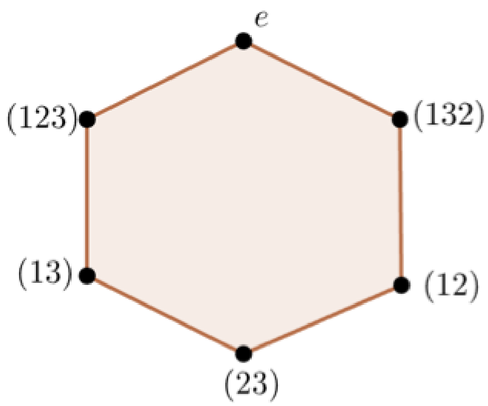

Figure 2. S_3 group.

3. Draw a line segment with endpoints at 0 and 1, representing the 2 elements of the set $\mathbb{Z}_2$.

Figure 3. $\mathbb{Z}_2$ group.

It is noteworthy that the representation of $\mathbb{Z}_2$ is a didactic and intuitive approach, widely used, although not always explicitly stated in formal textual sources.

The elements of S_3 are as follows: e: identity permutation; (12): transposition; (13): transposition; (23): transposition; (123): cyclic permutation; (132): cyclic permutation. Table 1 provides a detailed summary of the parity and mapping of each element:

Element of S_3	Parity	$\phi(\sigma) \in \mathbb{Z}_2$
e	even	0
(12)	odd	1
(13)	odd	1
(23)	odd	1
(123)	even	0
(132)	even	0

Table 1. Mapping summary.

4. Assign different colors to the vertices according to their image in $\mathbb{Z}_2$.

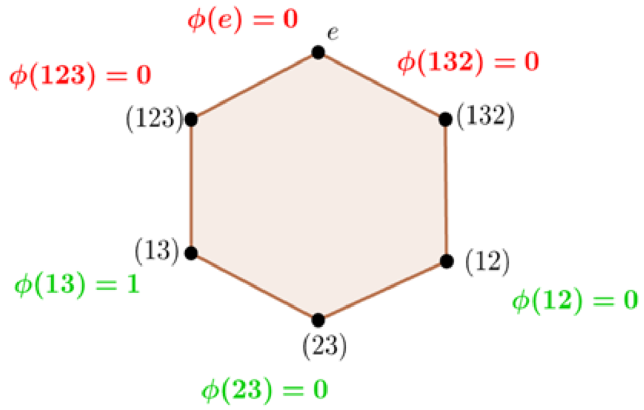

Figure 4. Definition of colors.

5. Add directional arrows connecting each element of S_3 to its mapping (0 or 1).

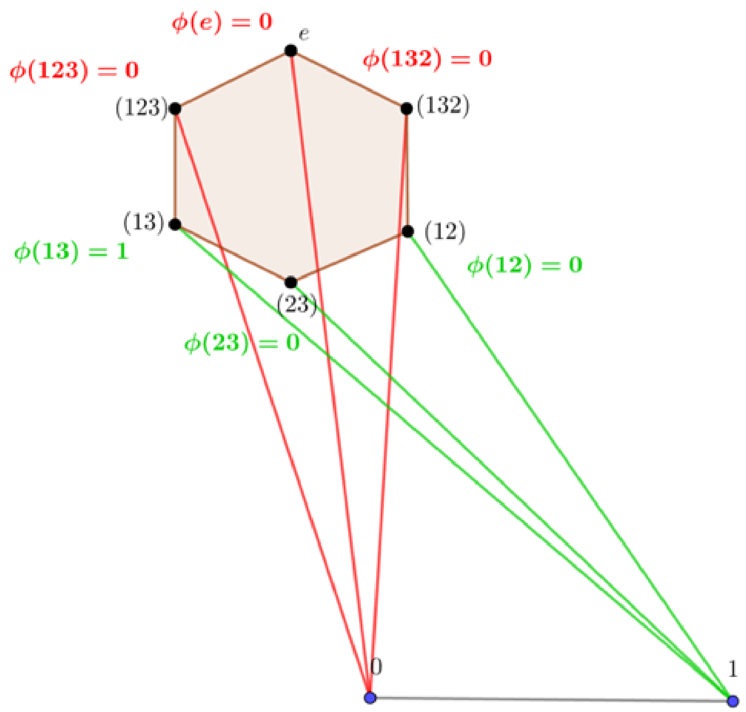

Figure 5. Addition of arrows.

In Figure 6, we present a visual representation of the homomorphism $\phi : S_3 \rightarrow \mathbb{Z}_2$ in GeoGebra, organized in side-by-side 2D and 3D windows. In the mapping $\phi : S_3 \rightarrow \mathbb{Z}_2$, colors were used to distinguish the elements according to their image in the cyclic group $\mathbb{Z}_2$. Elements mapped to 0 (even parity) are in red, while elements mapped to 1 (odd parity) are in green.

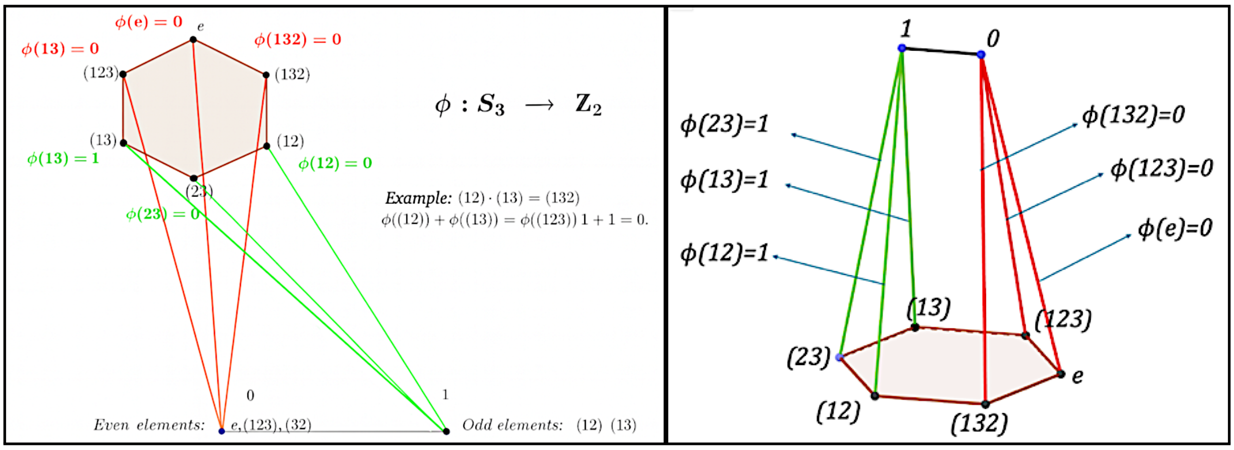

Figure 6. Visual representation of the homomorphism $\phi : S_3 \to \mathbb{Z}_2$ in 2D and 3D.

In the 2D window on the left, the elements are organized around a hexagon, while the arrows connect the elements to the corresponding images in the codomain represented by the points 0 and 1 at the bottom. In the 3D model (on the right), the 2D plane is elevated to create a three-dimensional relationship, where the group $\mathbb{Z}_2$ is represented by two connected vertices, illustrating the structural simplicity of the codomain. The colors used in the diagram are categorized in Table 2:

Color	Element in S_3	Image in $\mathbb{Z}_2$	Description
Red	$e, (123), (132)$	0	Even Permutation
Green	$(12), (13), (23)$	1	Odd Permutation

Table 2. Color categorization in $\phi : S_3 \to \mathbb{Z}_2$

The complete construction in GeoGebra enabled a detailed and structured visualization of the homomorphism $\phi : S_3 \to \mathbb{Z}_2$, reinforcing essential algebraic concepts such as the kernel, the image, and the coset structure. The use of color coding and spatial organization made it possible to represent these abstract notions in a tangible and pedagogically meaningful way.

4 Final Considerations

We presented a visual and interactive approach to explore group homomorphisms using GeoGebra constructions, focusing on the specific case: the homomorphism $\phi : S_3 \to \mathbb{Z}_2$. The goal was to illustrate fundamental concepts of Group Theory, such as the kernel, cosets, image, and the preservation of operations.

The construction demonstrated how visual tools can simplify the understanding of structures in Abstract Algebra. Through diagrammatic organization, the elements of the group S_3 were visually structured to highlight the properties of the homomorphism and the relationships between the involved groups. Additionally, the diagrams emphasized the practical application of the First Isomorphism Theorem in a visual manner.

From a pedagogical perspective, the proposed constructions serve not only to illustrate homomorphism concepts but also to support teaching strategies. By promoting visualization and interaction, the models help students grasp abstract algebraic ideas such as operation preservation, kernel, image, and quotient structure. These visual resources can be integrated into lessons to stimulate exploration, discussion, and conceptual consolidation in undergraduate algebra courses.

Although not yet implemented in a real classroom setting, this construction can be readily adapted for teacher education workshops or undergraduate algebra courses. The present study focuses on a classical homomorphism case; future works could explore more complex or non-abelian structures, including challenges in constructing visual representations for such mappings.

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