# GEOMETRY OF EQUILATERAL TRIANGLE SYMMETRIES: A MATHEMATICAL PERSPECTIVE FOR TEACHER TRAINING

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**Abstract**

*This article presents a didactic proposal for teaching the symmetries of the equilateral triangle, grounded in Group Theory and the methodology of the Fedathi Sequence. The focus is on exploring the algebraic properties of rotations and reflections mediated by the GeoGebra software, aiming at both the continuing education of mathematics teachers and the initial training of undergraduate students. The proposal seeks to foster visualization, conceptual understanding, and the articulation between Geometry and Algebra. The approach is also based on the Theory of Objectification, emphasizing the active role of the teacher in constructing knowledge. It is expected that the described experience will contribute to strengthening abstract reasoning through visual and interactive representations.*

**Keywords**: Symmetry Groups, GeoGebra, Visualization, Teacher Education.

## 1 INTRODUCTION

Symmetry groups describe the invariances of a geometric figure under transformations that preserve its structure, such as rotations and reflections. These groups have relevant applications in several fields of knowledge, including Physics, Chemistry, Computer Science, and Art (Armstrong, 1988; Cotton, 1990). The formalization of these properties occurs through Group Theory, a branch of Abstract Algebra that provides a framework for the study of algebraic structures. In the field of Geometry, the symmetry groups of polygons and regular solids are noteworthy for enabling connections between geometric and algebraic representations (Coxeter, 1973).

Despite their theoretical and didactic importance, symmetry groups remain underexplored in proposals aimed at teacher education, especially in continuing education programs. There is a scarcity of approaches that articulate algebraic foundations with visual representations mediated by digital technology. In this context, GeoGebra stands out as an interactive tool that allows for dynamic exploration of isometric transformations, facilitating conceptual understanding of symmetries through manipulable visual constructions (Hohenwarter et al., 2008).

This work presents a didactic proposal based on the Fedathi Sequence methodology (Borges Neto, 2018) and the Theory of Objectification (Radford, 2014), aimed at the continuing education of Mathematics teachers for the final years of elementary education, as well as at the initial training of undergraduate students in teacher education programs. The proposal consists of the construction and analysis of the symmetries of the equilateral triangle using GeoGebra, promoting the understanding of group algebraic structure through guided exploratory activities.

The Theory of Objectification guides the conception of teacher education as a collective process, in which teachers reframe their practice by actively engaging with mathematical concepts in collaborative learning situations. Meanwhile, the Fedathi Sequence offers a didactic structure based on four phases (Position Taking, Maturation, Proof, and Solution), fostering teacher autonomy and protagonism in knowledge construction.

The proposal integrates visual and manipulative elements, such as the use of cut-out polygons and dynamic applets, to facilitate the understanding of concepts such as identity, inverse, and non-commutativity within the context of the symmetry group of the equilateral triangle. Furthermore, it aims to promote the articulation between the geometric language and the algebraic structure of groups, providing didactic resources for the application of these concepts in classroom settings.

This article is part of a doctoral research project affiliated with the Northeast Teaching Network Graduate Program (RENOEN/UFC), focusing on the design of didactic sessions mediated by digital technologies. The following section presents the methodological framework and discusses the results of the proposal's development.

It is important to note that this work presents a didactic proposal that has not yet been implemented. The proposal consists of the construction and analysis of the symmetries of the equilateral triangle using GeoGebra, aiming to promote an understanding of group algebraic structure through guided exploratory activities for teacher education purposes.

## 2 Theoretical and Methodological Framework

The didactic proposal presented in this article is structured based on the Fedathi Sequence (Borges Neto, 2018), a methodology aimed at organizing teaching through four phases: Position Taking, Maturation, Proof, and Solution. This approach seeks to promote student protagonism through problem-solving and teacher mediation centered on investigative processes.

In the context of teacher education, this methodology fosters the autonomous construction of mathematical knowledge, especially when combined with technological resources such as GeoGebra, which enables experimentation and interactive visualization of abstract concepts, such as the symmetries of the equilateral triangle. The software allows the manipulation of geometric transformations, connecting visual representations to the algebraic structure of groups.

The research is also grounded in the Theory of Objectification (Radford, 2014), which understands learning as a social process in which subjects assign meaning to concepts through collaborative activity. In this proposal, the participating teacher engages in a didactic session where they interact with dynamic constructions, formulate conjectures, and validate properties of symmetry groups.

The focus is on the symmetric group of the equilateral triangle, a finite non-abelian group. Using GeoGebra, participants explore its six isometries (i.e., identity, two rotations, and three reflections), examining composition, identity, and inverses to visualize the group concept before formalizing it symbolically.
The choice of this approach is based on studies that highlight the difficulties faced by teachers and undergraduate students when dealing with topics in Abstract Algebra (Sousa et al., 2024b,a; Eves, 2002). The use of GeoGebra contributes to overcoming these challenges by making the concepts tangible through simulation and manipulation.

Thus, the theoretical and methodological framework of this study combines the didactic structure of the Fedathi Sequence with technological mediation and the sociocultural approach of the Theory of Objectification, composing a formative scenario where the teacher becomes an active agent in the construction of mathematical knowledge.

## 3 Symmetries of the Equilateral Triangle

The symmetries of the equilateral triangle constitute a classic and pedagogically rich example for introducing concepts from Group Theory. This triangle has exactly six symmetries that preserve its shape and structure: one identity transformation, two rotations, and three reflections. These transformations, when considered under the operation of composition, form the symmetry group of the equilateral triangle, known as the dihedral group $D_3$, which is isomorphic to the permutation group $S_3$.

The identity transformation keeps all vertices fixed. The two rotations are performed around the centroid of the triangle: a $120°$ counterclockwise rotation (denoted by ) and a $240°$ counterclockwise rotation (denoted by $\theta^2$). The three reflections occur about the medians of the triangle, which also serve as its altitudes, angle bisectors, and perpendicular bisectors, reflecting two vertices and keeping the third fixed. Figure 1 presents these six symmetries along with the corresponding permutations of the vertices.

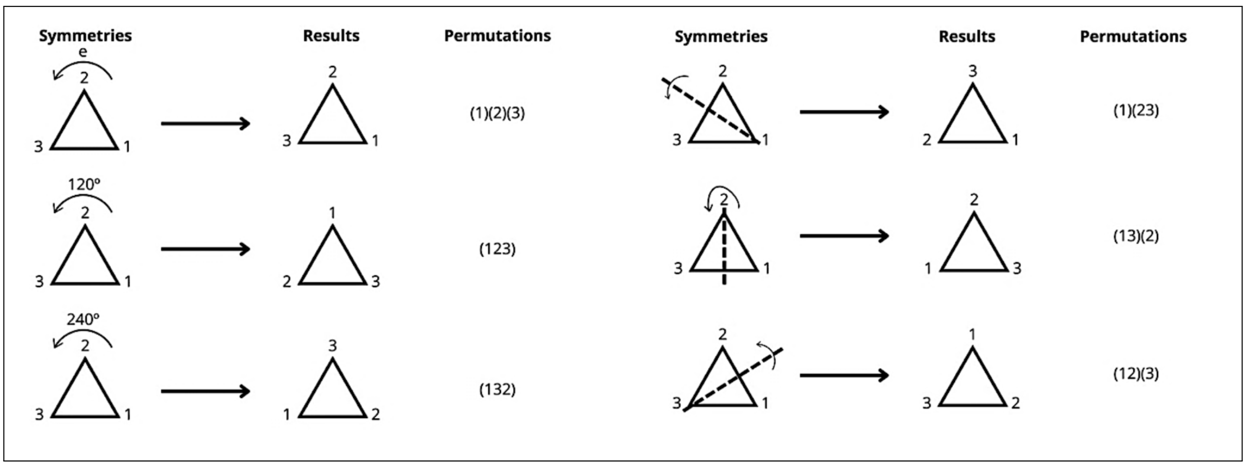



**Figure 1.** Symmetries of the Equilateral Triangle. *Note.* The figure illustrates the six isometries of the equilateral triangle. Each transformation corresponds to a permutation of the vertices, as indicated.

These symmetries satisfy the four properties that characterize a group: closure, associativity, existence of an identity element, and existence of inverses. Furthermore, the group $D_3$ is non-commutative: for instance, composing a reflection followed by a rotation may yield a different result than reversing the order of operations.

The formalization of these transformations within the context of Abstract Algebra offers students and teachers a concrete opportunity to understand group structure visually. The association between geometric symmetries and permutations allows the elements of the group to be represented by bijective functions on the triangle's vertices, highlighting the isomorphism with $S_3$.

By exploring these transformations using GeoGebra, participants were able to visualize and manipulate the effects of each symmetry, reinforcing the link between geometric representation and the underlying algebraic structure. This approach fosters the transition from visual reasoning to mathematical abstraction, an essential element for the meaningful learning of Group Theory concepts.

## 4 Didactic Proposal with GeoGebra

The didactic proposal presented here was developed within the context of continuing education for teachers of the final years of elementary school, based on the methodology of the Fedathi Sequence and grounded in the exploration of the symmetries of the equilateral triangle using GeoGebra. The research is characterized as qualitative and employed the technique of participant observation (Gil, 2008), since the researcher acted directly as the trainer during the sessions.

The formative activity was structured around the following problem-situation, presented during the Position Taking phase of the Fedathi Sequence:

> How can the implementation of interactive buttons in GeoGebra, corresponding to rotations and reflections, facilitate the visual exploration of the symmetries of the equilateral triangle, enabling the identification of the elements of the symmetry group and their compositional relationships?

During the Maturation phase, teachers began exploring the equilateral triangle using GeoGebra 3D. The constructions were carried out with buttons that activated successive $120°$ rotations (denoted $R_1$, $R_2$, and $R_3$) and reflections (denoted $S_1$, $S_2$ and $S_3$). Each of these transformations was applied and observed interactively, allowing participants to explore the symmetries and test different compositions visually.

Figures 2 to 5 illustrate the constructions of the symmetries in GeoGebra (Sousa et al., 2024), representing the rotations $R_1$, $R_2$, and $R_3$ of the equilateral triangle, enabling the observation of their actions on the vertices.

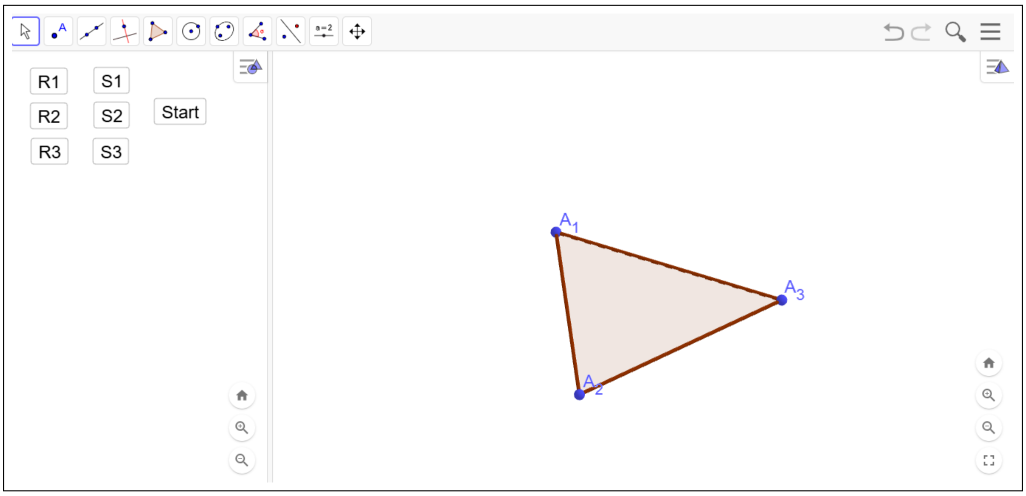



**Figure 2.** Symmetry of the equilateral triangle for $n = 3$.

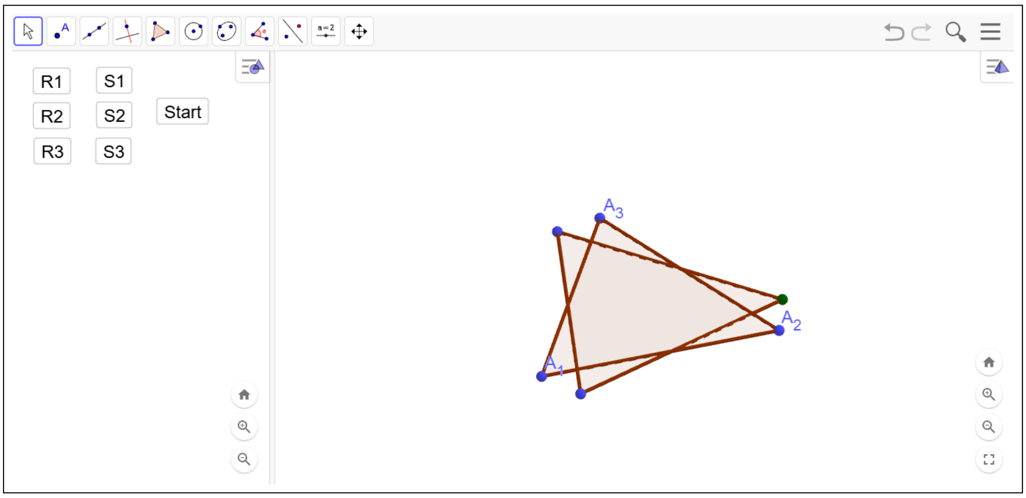



**Figure 3.** Construction of the equilateral triangle in GeoGebra for $R_1$.

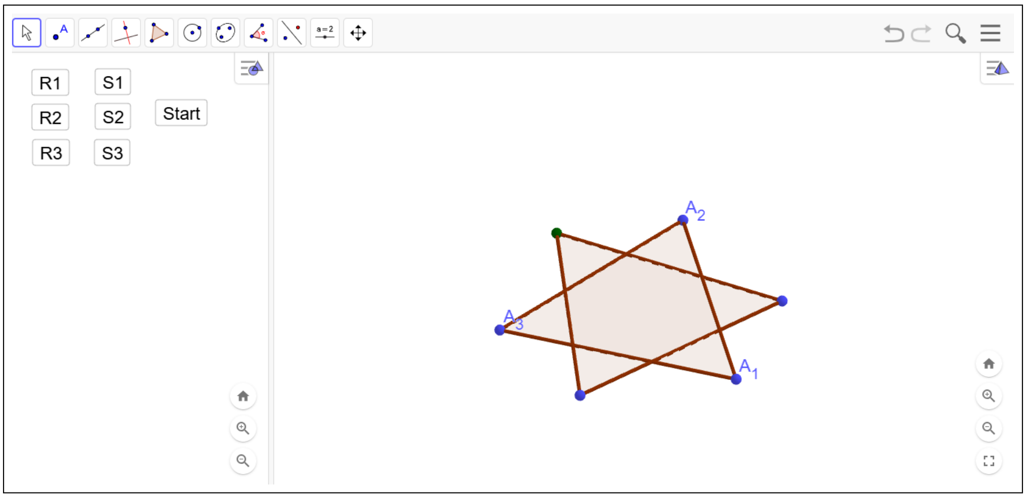



**Figure 4.** Symmetry in GeoGebra for $R_2$.

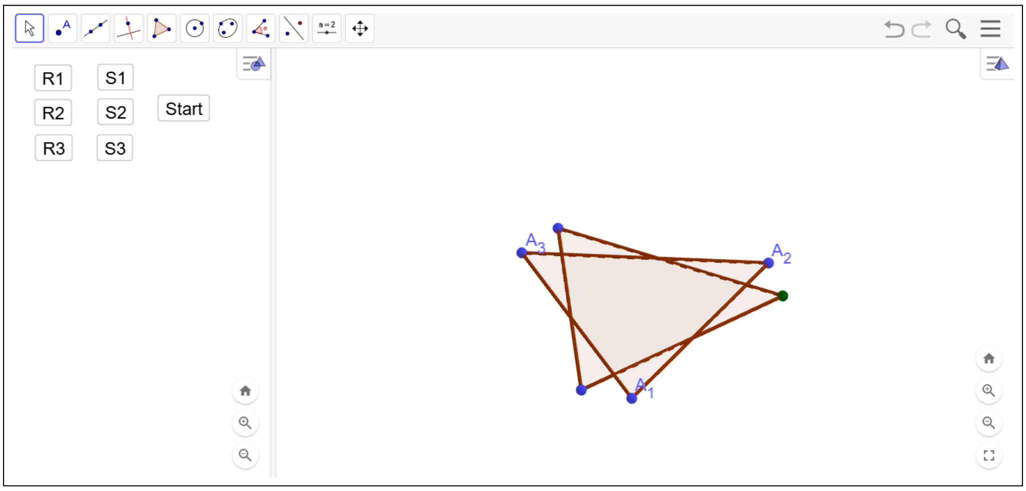
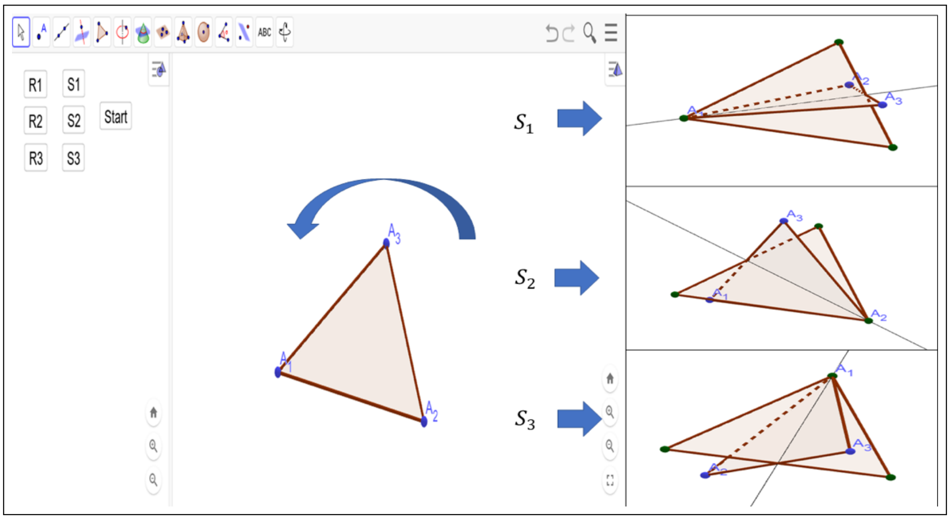



**Figure 5.** Construction of the equilateral triangle in GeoGebra for $R_3$.

During the activity, teachers identified the fundamental properties of the symmetry group: the existence of the identity element, closure under the composition operation, the presence of inverses, and associativity. The reflections were beneficial for highlighting the non-commutative behavior of the group, as shown in Figure 6.

**Figure 6.** Reflections in the equilateral triangle.




The symmetries $S_1$, $S_2$, and $S_3$ correspond to reflections across the medians of the triangle. The composition of these reflections with rotations reveals the non-abelian structure of the group $D_3$.

The Proof phase involved the systematization of the results by the participants, who discussed the characteristics of the symmetries and formalized, with the trainer's mediation, the concepts of group and its properties. The verification of the algebraic structure was supported by a composition table (Table 1), developed based on observations made using the applet.

| | | | | | | |
|---|---|---|---|---|---|---|
| ° | Id | $\theta$ | $\theta^2$ | F2 | F2 | F2 |
| Id | Id | $\theta$ | $\theta^2$ | F2 | F2 | F2 |
| $\theta$ | $\theta$ | $\theta^2$ | Id | F2 | F2 | F2 |
| $\theta^2$ | $\theta^2$ | Id | $\theta$ | F2 | F2 | F2 |
| F2 | F2 | F2 | F2 | Id | $\theta$ | $\theta^2$ |
| F2 | F2 | F2 | F2 | $\theta^2$ | Id | $\theta$ |
| F2 | F2 | F2 | F2 | $\theta$ | $\theta^2$ | Id |

**Table 1.** Composition Table of Symmetries of the Equilateral Triangle.

The analysis of these compositions allowed teachers to recognize, for example, that performing a 120-degree rotation three times results in the identity, and that the composition of two reflections can result in a rotation. These observations led to the identification of cyclic subgroups and confirmed that the set of symmetries forms a finite non-abelian group.

The proposal demonstrated that the use of GeoGebra not only facilitates the visualization of symmetries but also supports the construction of the group concept in an intuitive and grounded manner. The transition from concrete exploration to mathematical abstraction was mediated through experimentation, collective discussion, and formalization, enabling teachers to understand and apply fundamental concepts of Abstract Algebra in their pedagogical practices.

## 5 Final Considerations

The proposal presented in this article explored the symmetries of the equilateral triangle through an approach grounded in Group Theory, using GeoGebra as a didactic mediation tool. Based on the methodology of the Fedathi Sequence and the Theory of Objectification, the formative activity promoted the interactive visualization of rotations and reflections, thereby facilitating an understanding of the concepts of identity, inverse, non-commutativity, and closure.

The construction and analysis of geometric transformations in the digital environment provided in-service teachers the opportunity to concretely experience the properties of the dihedral group $D_3$, which is isomorphic to the permutation group. The direct manipulation of symmetries allowed for the connection between visual intuition and algebraic formalization, facilitating the transition from geometric reasoning to the abstract language of Algebra.

By integrating visualization, experimentation, and collective discussion, the proposal contributed to the professional development of teachers, offering didactic pathways for teaching concepts traditionally regarded as abstract. It is hoped that the experience described here may serve as a foundation for more investigative pedagogical practices connected to the technological resources available, broadening the possibilities for teaching and learning Algebra in teacher education.

It is essential to note that the proposed approach is theoretical and has not yet been implemented in real-world formative activities. It is expected that this study may serve as a foundation for future practical applications, supporting the development of innovative pedagogical practices in the teaching of Algebra and Geometry using digital technologies.

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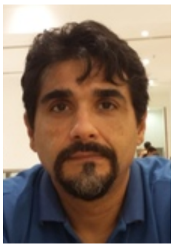

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