A DYNAMIC MODEL TO ENGAGE STUDENTS IN THE TRIANGLE'S INHERENT SIMILARITY

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Abstract

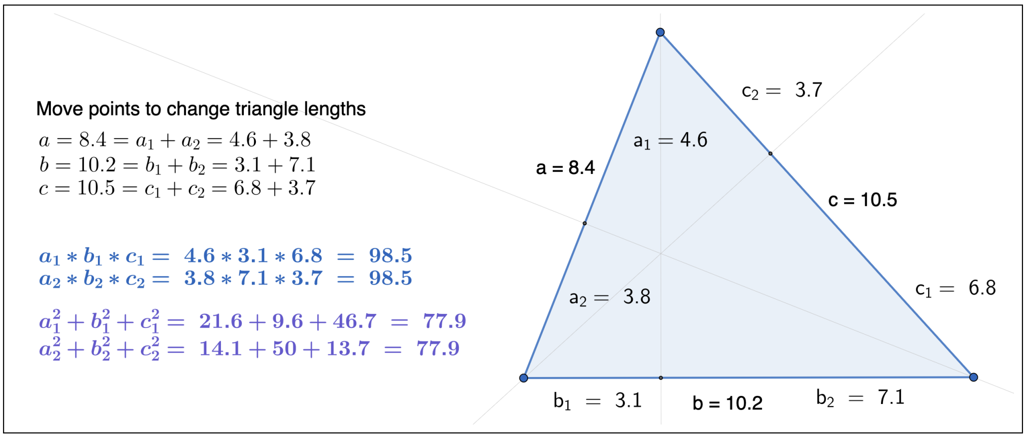
This paper presents a dynamic model created with GeoGebra to explore and demonstrate relationships inherent in triangles, particularly those involving side partitions formed by the triangle's altitudes. These relationships, while not commonly featured in high school geometry curricula, offer a unique opportunity for students to investigate mathematical theorems dynamically. The paper examines the relationships $a_1b_1c_1 = a_2b_2c_2$ *and* $a_1^2 + b_1^2 + c_1^2 = a_2^2 + b_2^2 + c_2^2$*, providing an interactive GeoGebra model, detailed formal proofs, and pedagogical resources to support classroom implementation. Connections to educational standards are also discussed, highlighting the versatility of this approach for both educators and students.*

Keywords: triangle similarity, dynamic geometry software, classroom implementation

THE DYNAMIC MODEL

In this section, we present a proof of the relationships involving the partial lengths of the sides partitioned by the three altitudes of a triangle. For readers who wish to explore these relationships dynamically before delving into the proofs, an interactive GeoGebra (2024) model is available at the following link (best viewed on a non-mobile device): `https://www.geogebra.org/calculator/dy2hsv8s`. As suggested in Figure 1, the GeoGebra model allows users to manipulate the vertices of a triangle dynamically.

Figure 1. GeoGebra model showing the relationships between side partitions of a triangle.

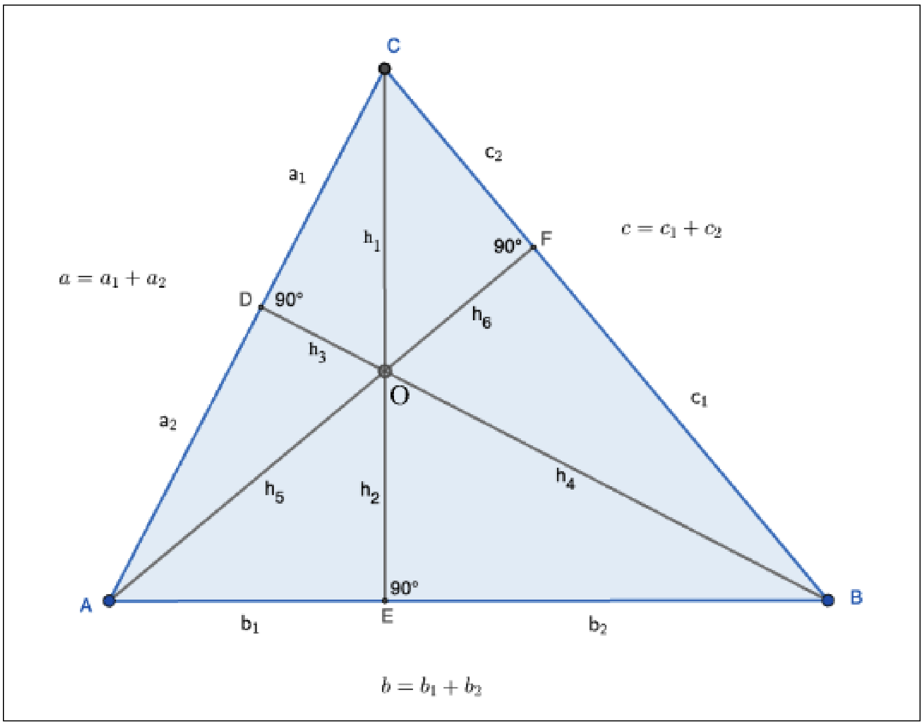


As the side lengths change, the corresponding formulas update automatically to demonstrate the relationships using partial lengths. This dynamic approach addresses common misconceptions about triangles by illustrating that these relationships persist across all triangle types—acute, right, and obtuse. It provides an engaging way for students to explore geometric relationships and encourages deeper mathematical thinking.

FORMAL EXPLORATION AND PROOF

This section presents the main results for a triangle, which were first detailed in a longer paper (Kaufman, 2012). Figure 2 illustrates the setup, including partitioning sides by the triangle's altitudes.

Figure 2. Diagram of a triangle with sides partitioned by altitudes.



Points D, E, and F separate sides BC, AC, and AB, respectively, as:

$$a = a_1 + a_2, \quad b = b_1 + b_2, \quad c = c_1 + c_2.$$

Altitudes are shown in gray so that:

$$AC \perp BD, \quad AB \perp CE, \quad BC \perp AF.$$

Triangles $\triangle AFC$ and $\triangle BCD$ are similar since they are right triangles sharing $\angle ACB$. From this similarity, we establish:

$$\frac{a}{c_2} = \frac{c}{a_1}, \quad \text{or equivalently,} \quad aa_1 = cc_2.$$

By similarity of other right triangles, we also find:

$$aa_2 = bb_1, \quad \text{and} \quad bb_2 = cc_1.$$

Additionally, by examining $\triangle DOC$ and $\triangle EOB$, which are similar, we obtain:

$$\frac{a_1}{h_1} = \frac{b_2}{h_4}.$$

From similar triangles involving other segments, we also have:

$$\frac{a_2}{h_5} = \frac{c_1}{h_4}, \quad \frac{b_1}{h_5} = \frac{c_2}{h_1}.$$

Step-by-Step Combination of Results

1. Multiplying the three equations above:

$$\frac{a_1}{h_1} \cdot \frac{b_1}{h_5} \cdot \frac{c_1}{h_4} = \frac{a_2}{h_5} \cdot \frac{b_2}{h_4} \cdot \frac{c_2}{h_1}.$$

   Since the denominators are identical, we conclude:

$$a_1 b_1 c_1 = a_2 b_2 c_2.$$

   Next, we observe that:

$$(h_1 + h_2)^2 = a^2 - b_1^2 = c^2 - b_2^2.$$

   Thus:

$$c^2 - a^2 = b_2^2 - b_1^2 = b(b_2 - b_1) = bb_2 - bb_1.$$

2. Using earlier results:

$$bb_1 = aa_2, \quad bb_2 = cc_1.$$

   We substitute to find:

$$b(b_2 - b_1) = cc_1 - aa_2 = cc_1 - cc_2 + cc_2 - aa_2.$$

   Since $cc_2 = aa_1$, this simplifies to:

$$b(b_2 - b_1) = c(c_1 - c_2) + a(a_1 - a_2).$$

3. Rewriting, we establish:

$$b_2^2 - b_1^2 = c_1^2 - c_2^2 + a_1^2 - a_2^2.$$

   Equivalently:

$$a_1^2 + b_1^2 + c_1^2 = a_2^2 + b_2^2 + c_2^2.$$

4. Finally, by examining the proportional relationships between altitudes, we see that:

$$h_1 h_2 = h_3 h_4 = h_5 h_6.$$

Interpretive Commentary for Educators

The relationships derived above reveal several invariants in triangle geometry. These results provide opportunities for classroom exploration, as students can verify the relationships dynamically using GeoGebra and validate them through algebraic reasoning. The invariants include:

1. **Product of Partial Lengths** ($a_1b_1c_1 = a_2b_2c_2$) demonstrates a consistent relationship between the side partitions formed by the altitudes.
2. **Sum of Squares of Partial Lengths** ($a_1^2 + b_1^2 + c_1^2 = a_2^2 + b_2^2 + c_2^2$): This equality highlights a geometric invariant applicable to all triangle types.
3. **Relationships Among Altitudes** ($h_1h_2 = h_3h_4 = h_5h_6$): This consistency further deepens students' understanding of triangle properties.

Teachers can encourage students to explore these results step by step, bridging dynamic observations in GeoGebra with rigorous proofs.

PEDAGOGICAL CONSIDERATIONS

Though not found in standard high school geometry textbooks, the theorems and relationships presented in this paper provide unique opportunities to enhance curriculum content. Teachers can foster deeper understanding through exploration, critical thinking, and mathematical reasoning by integrating the dynamic GeoGebra model and formal proofs. These activities help counter the misconception that mathematics is static and unchanging, demonstrating its dynamic and exploratory nature.

Classroom Applications

The GeoGebra model is a versatile tool for classroom instruction, allowing students to visualize geometric relationships dynamically. Suggested classroom activities include:

- **Interactive Exploration**: Students can manipulate the vertices of the triangle in the GeoGebra model to observe how the relationships $a_1b_1c_1 = a_2b_2c_2$ and $a_1^2 + b_1^2 + c_1^2 = a_2^2 + b_2^2 + c_2^2$ hold for all triangle types—acute, right, and obtuse.
- **Guided Proof Exercises**: Teachers can guide students through key steps of the proof, encouraging them to replicate or extend the reasoning provided. This process reinforces triangle similarity, proportionality, and algebraic manipulation.
- **Student Conjectures**: Encourage students to hypothesize and test additional relationships in triangles based on observations from the dynamic model. This approach nurtures inquiry and reasoning skills.

Alignment with Educational Standards

The activities described align with several *Common Core Standards for Mathematics* (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010), including:

- **HSG-CO.C.10**: Prove theorems about triangles. For example, the relationships explored in this paper extend students' understanding of similarity and proportionality.

- **HSG-SRT.B.5**: Use congruence and similarity criteria for triangles to solve problems and prove relationships in geometric figures.

These connections make the material particularly suitable for honors-level or advanced geometry courses. The dynamic model and proofs extend traditional curriculum content, engaging students in rich mathematical discovery.

Future Exploration and Final Remarks

To extend these ideas, teachers could challenge students to create their own GeoGebra models or investigate additional triangular relationships. For example, students might explore how altitudes and medians relate to side partitions in different triangles. These activities help bridge the gap between theoretical mathematics and practical problem-solving, encouraging students to see themselves as active participants in the discovery process.

To engage students, GeoGebra was used to create a dynamic model to demonstrate mathematical results found for any triangle. One of the authors created the model in an afternoon without previous knowledge of GeoGebra. Teachers who are new to GeoGebra will find it intuitive to use. This ease of use allows teachers to adopt dynamic models in their classroom activities quickly. By encouraging students to explore mathematical concepts visually, GeoGebra fosters a more engaging learning environment.

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APPENDIX A: POTENTIAL LESSON PLAN

This appendix provides a structured lesson plan to help students explore and prove the relationships described in this paper. The lesson plan includes two main components: a dynamic exploration using the GeoGebra model and formal proof development. Part 1 (observing a potential relationship) can be used alone or with Part 2 (Proving the relationships). Part 2 could be used as a full-class participatory activity, or the instructor could omit certain elements of the proof and allow students to discuss and fill them in groups.

Objective

Students will explore and prove a relatively new theorem about triangles, focusing on the relationships $a_1b_1c_1 = a_2b_2c_2$ and $a_1^2 + b_1^2 + c_1^2 = a_2^2 + b_2^2 + c_2^2$. Students will strengthen their understanding of triangle similarity, proportionality, and algebraic reasoning through this process.

Standards Alignment

This lesson aligns with the following Common Core Standards for Mathematics:

- **HSG-CO.C.10**: *Prove theorems about triangles. Theorems include measures of interior angles of a triangle sum to* $180°$*; base angles of isosceles triangles are congruent; the segment joining midpoints of two sides of a triangle is parallel to the third side and half the length; the medians of a triangle meet at a point.*
- **HSG-SRT.B.5**: *Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.*

Part 1: Observing Relationships Dynamically

1. Open the GeoGebra model: `https://www.geogebra.org/calculator/dy2hsv8s`.
2. Manipulate the vertices of the triangle and observe how the relationships $a_1b_1c_1 = a_2b_2c_2$ and $a_1^2 + b_1^2 + c_1^2 = a_2^2 + b_2^2 + c_2^2$ hold for all types of triangles (acute, right, and obtuse).
3. Discuss:
   - What do you notice about the products $a_1b_1c_1$ and $a_2b_2c_2$ as the triangle changes shape?
   - How do the sums $a_1^2 + b_1^2 + c_1^2$ and $a_2^2 + b_2^2 + c_2^2$ compare?
   - What happens when the triangle becomes obtuse? Do any of the side lengths appear to become negative?

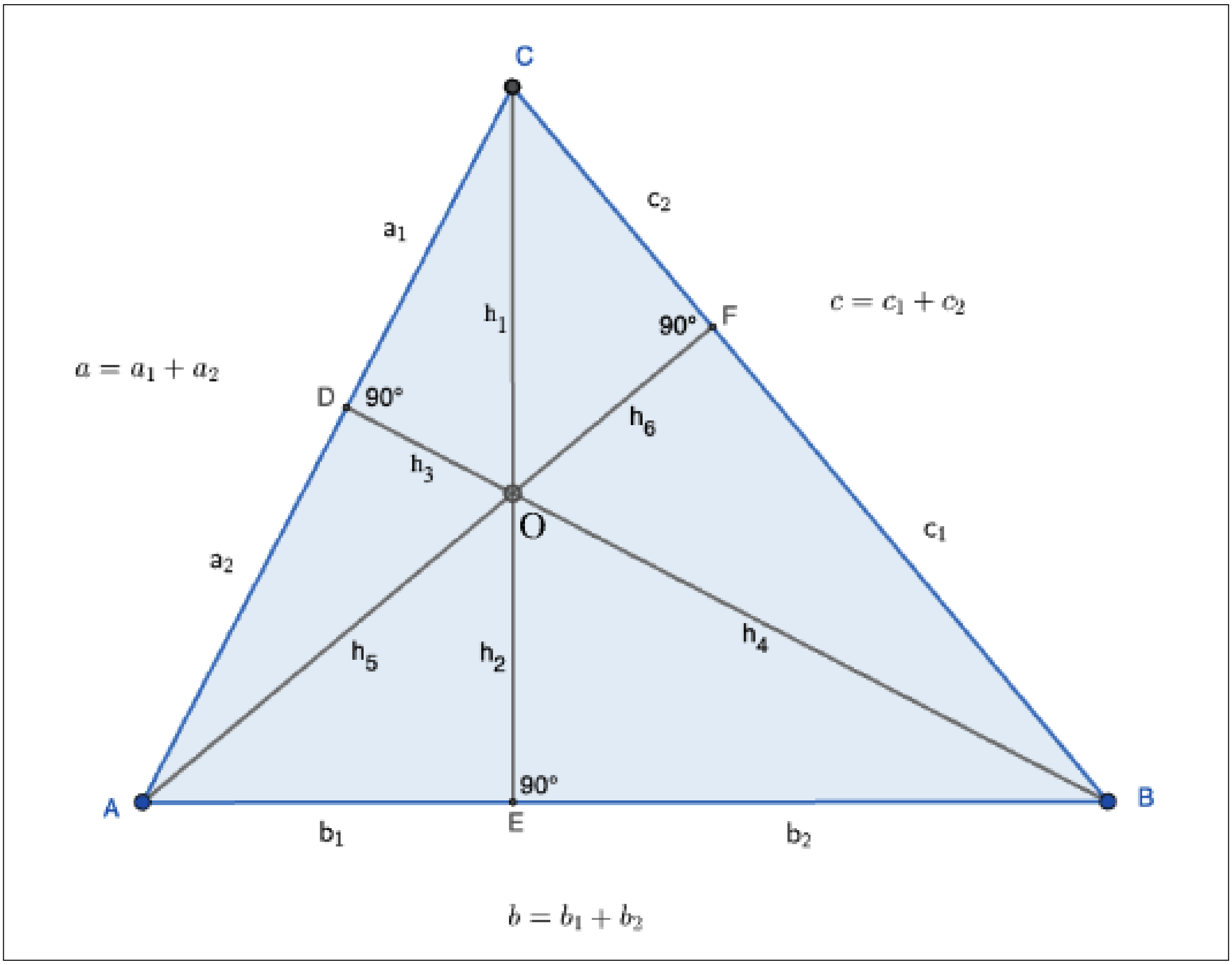
Part 2: Proving the Relationships

Students will work in groups or individually to prove the relationships observed in Part 1. Teachers can provide the following steps as scaffolding:

- Use similarity of triangles to establish proportional relationships between sides.
- Derive $a_1b_1c_1 = a_2b_2c_2$ by combining proportional relationships.
- Show that $a_1^2 + b_1^2 + c_1^2 = a_2^2 + b_2^2 + c_2^2$ using algebraic manipulations and substitutions.

GeoGebra Model

Figure 3. Screenshot of the GeoGebra model illustrating the relationships in a triangle.



Summary of Observed Relationships

Table 1. Step-by-Step Proof of Relationships

Step	Statement	Reason
1	Let points D, E, and F separate sides a, b, and c, respectively, such that: $$a = a_1 + a_2,\ b = b_1 + b_2,\ c = c_1 + c_2$$ Altitudes are shown in gray so that: $$BD \perp AC,\ CE \perp AB,\ AF \perp BC$$	Given
2	$\triangle AFC$ is similar to $\triangle BCD$.	AAA Similarity Theorem
3	$$\frac{a}{c_2} = \frac{c}{a_1}$$	Similar triangles have proportional sides
4	$a \cdot a_1 = c \cdot c_2$.	Multiplication Property of Equality
5	Similarly to Step 2: $$\triangle ACE \sim \triangle DBA, \quad \triangle CEB \sim \triangle AFB$$	AAA Similarity Theorem
6	$$\frac{a}{b_1} = \frac{b}{a_2}, \quad \frac{c}{b_2} = \frac{b}{c_1}$$	Similar triangles have proportional sides
7	$a \cdot a_2 = b \cdot b_1, \quad b \cdot b_2 = c \cdot c_1$.	Multiplication Property of Equality
8	$\angle DOC \cong \angle EOB$.	Vertical angles
9	$\angle ODC \cong \angle OEB$.	Both are right angles
10	$\angle DCO \cong \angle EBO$.	Supplementary angles
11	$\triangle DOC \sim \triangle EOB$.	AAA Similarity Theorem
12	So, $$\frac{a_1}{h_1} = \frac{b_2}{h_4}$$	Similar triangles have proportional sides
13	Similarly: $$\triangle DOA \sim \triangle FOB, \quad \triangle EOA \sim \triangle FOC$$	Same reasoning as Steps 8–11

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Step	**Statement**	**Reason**
14	$$\frac{a_2}{h_5} = \frac{c_1}{h_4}, \quad \frac{b_1}{h_5} = \frac{c_2}{h_1}$$	Similar triangles have proportional sides
15	Combining the equations in Steps 12 and 14, we have: $$\frac{a_1}{h_1} \cdot \frac{b_1}{h_5} \cdot \frac{c_1}{h_4} = \frac{a_2}{h_5} \cdot \frac{b_2}{h_4} \cdot \frac{c_2}{h_1}$$	Multiplication
16	Since the denominators are the same: $$a_1 b_1 c_1 = a_2 b_2 c_2$$	Definition of Equivalent Fractions
17	Next, observe that: $$(h_1 + h_2)^2 + b_1^2 = a^2$$	Pythagorean Theorem
18	So: $$(h_1 + h_2)^2 = a^2 - b_1^2$$	Addition
19	Likewise: $$(h_1 + h_2)^2 + b_2^2 = c^2, \quad (h_1 + h_2)^2 = c^2 - b_2^2$$	Same reasons as (17) and (18)
20	$$a^2 - b_1^2 = c^2 - b_2^2$$	Transitive Property (18 and 19)
21	So: $$\begin{aligned} c^2 - a^2 &= b_2^2 - b_1^2 \\ &= (b_2 + b_1)(b_2 - b_1) \\ &= b(b_2 - b_1) \end{aligned}$$	Subtraction; Factoring; Substitution from (1)
22	Expanding terms: $$\begin{aligned} b(b_2 - b_1) &= bb_2 - bb_1 \\ &= cc_1 - aa_2 \\ &= cc_1 - cc_2 + cc_2 - aa_2 \end{aligned}$$	Distributive Property; Substitution (7); Additive Identity Property

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Step	Statement	Reason
23	And since $cc_2 = aa_1$, then: $$\begin{aligned} cc_1 - cc_2 + cc_2 - aa_2 &= cc_1 - cc_2 + aa_1 - aa_2 \\ &= c(c_1 - c_2) + a(a_1 - a_2) \end{aligned}$$	Substitution; Factoring
24	This can be rewritten as: $$b(b_2 - b_1) = c(c_1 - c_2) + a(a_1 - a_2)$$ $$(b_2+b_1)(b_2-b_1) = (c_1+c_2)(c_1-c_2)+(a_1+a_2)(a_1-a_2)$$ $$b_2^2 - b_1^2 = c_1^2 - c_2^2 + a_1^2 - a_2^2$$	Transitive Property (22 and 23); Three substitutions from (1); Difference of squares
25	Thus: $$a_1^2 + b_1^2 + c_1^2 = a_2^2 + b_2^2 + c_2^2$$	Use addition property to get all subscript 1 terms on the left side and subscript 2 terms on the right side

If of interest, another exercise could be to show that $h_1h_2 = h_3h_4 = h_5h_6$.