RATE OF CHANGE: DEGREES VS. RADIANS

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Abstract

In this article, we will discuss using GeoGebra to present and connect multiple representations. The purpose of the activities shared is to help students reflect on the consequences of different units when graphing trigonometric functions. Specifically, the activity should help students compare the consequences, with respect to the rate of change, of using radians or degrees when graphing trigonometric functions.

Keywords: Degrees, Radians, Multiple representations, Trigonometry

1 INTRODUCTION

The teaching and learning of trigonometry leaves much to be desired. For many students, the study of trigonometry is a test of their memorization skills (e.g., what is $\sin 30°$, when is $\tan\theta = \frac{\sqrt{2}}{2}$) with limited conceptual understanding. Research suggests at least part of the struggle with learning trigonometry is a lack of foundational knowledge of angle and angle measure (Moore, 2013, 2014). Fi (2003) further explains that part of the struggle in understanding angle and angle measure stems from the fact that students anchor their understanding of angle measure in degrees and cannot reason about radians directly. In what follows, we will share an activity designed in GeoGebra to help students compare radians and degrees when graphing trigonometric functions.

2 FRAMING THE PROBLEM

2.1 Purpose

The focus of this activity is to consider the consequences of using radians or degrees on the graph and rate of change of a trigonometric function to help students further develop their understanding of each unit independent of the other. We used the sine function for this activity and considered several specific questions using the GGB file shared below.

2.2 The Role of Technology

Technology's role in this activity was to present and connect multiple representations (Cullen et al., 2020). In the GGB file, you will see two different representations of the sine function. Additionally, and more central to the discussion here, you will also see a connection between the radius of the unit circle and radians as the unit along the x-axis, which you can see when you click the check box labeled x-radius. You will also notice that we chose to label the x-axis using integer values of radians rather than the more common use of fractions of π along the x-axis. This choice was made to help students reflect on what a radian is directly rather than thinking in degrees and converting to radians Fi (2003). In his research, he found that some students considered π as the unit of measure for radians. He stated that in the interview with the participant, "π was misunderstood as the unit for the radian measure, and the interviewee argued that 1 radian equaled 180°" (Fi, 2003, p. 202). This finding encouraged us to label the x-axis with integer values of the radius rather than fractional values of π to help our preservice mathematics teachers avoid the misconception that π is a unit.

3 METHODOLOGY

The file linked below displays a geometric interpretation of the sine function Hertel and Cullen (2011); Cullen and Martin (2018) with a graph of the sine function and a line tangent to the sine curve at $x = 0$. (Note: We included the tangent line so students could reflect on ideas related to the rate of change without requiring knowledge of derivatives.) Students were instructed to compare the use of the different units given as options below. Specifically, we asked them to consider the slope of the tangent line at $x = 0$ and the max/min for each pairing of units. The file has check boxes that allow students to turn on units (i.e., radius or degrees) for either axis. We encourage you to explore the online app (`https://www.geogebra.org/m/gkspuhdg`) and answer the questions before reading ahead.

3.1 Procedure

Students started with radius on each axis with the maximum of 1 at approximately 1.6 and the minimum of -1 at approximately 4.7 (see figures 1 and 2). Because the tangent line at $x = 0$ appears to go through $(0, 0)$ and $(1, 1)$, students approximated the slope of the tangent line to be 1.

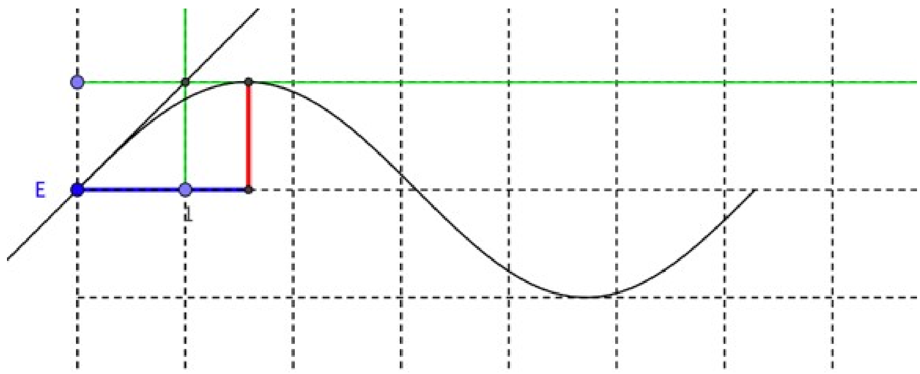

Figure 1. Max and slope for unit pairing 1.

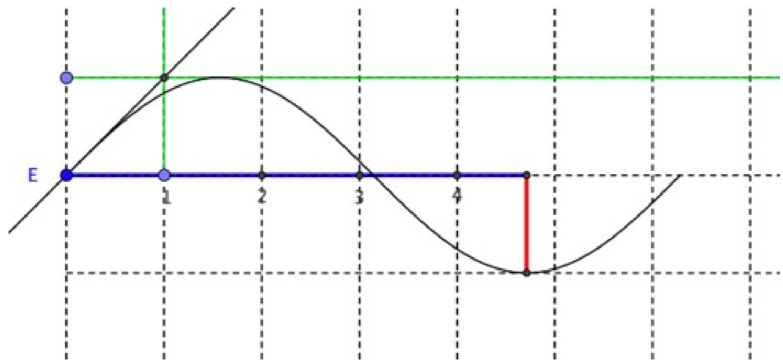

Figure 2. Min and slope for unit pairing 1.

Using the second pairing, degrees on the x-axis and radii on the y-axis, students observed the maximum of 1 at $90°$ and a minimum of -1 at $270°$. Approximating the slope of the tangent line could have been more enjoyable. Students approximated the slope as $\frac{1}{60°}$ and $\frac{1.6}{90°}$ from figures 3 and 4, respectively.

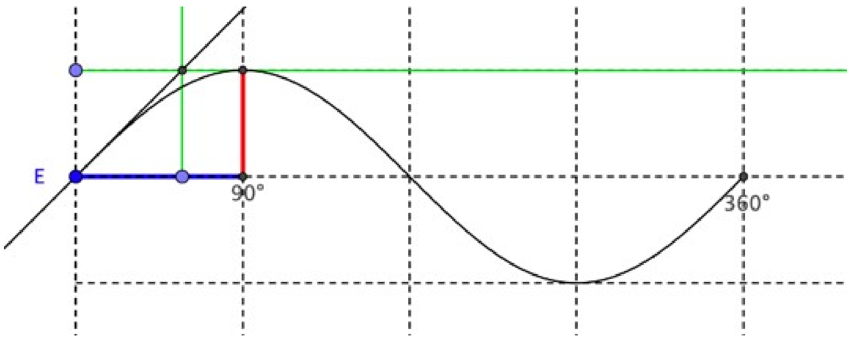

Figure 3. Max and slope for unit pairing 2.

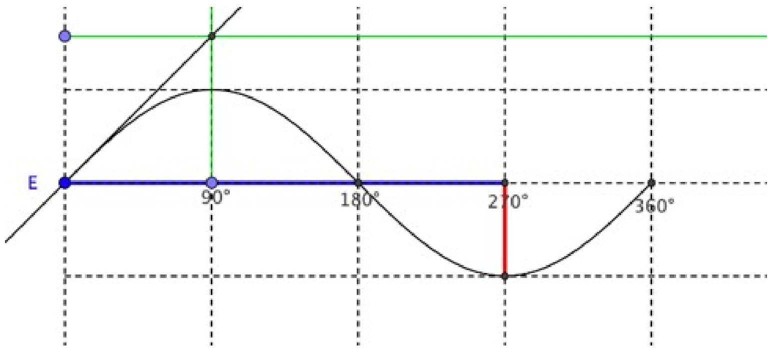

Figure 4. Min and slope for unit pairing 2.

The third and fourth pairings presented an opportunity to discuss what using degrees on the y-axis would mean. We posed this question and allowed students to discuss it in groups and share their thinking. Students suggested that you imagine the circle's circumference as unrolled and assign that length to be $360°$. You have associated a length with a degree measurement, thus establishing a linear scale. Then, they imagined aligning that scale with the y-axis.

Next, we asked students to look for the max/min and slope of the tangent line when the x-axis is in radii and the y-axis degrees. Our students found that the max/min appeared to be around $60°$ and $-60°$, occurring at about 1.6 and 4.7 radians. Additionally, looking at the slope of the tangent line, they produced a rough approximation of $\frac{60°}{1}$ or $60°$. Alternatively, a student could approximate this same slope as $\frac{90°}{1.6}$, which may connect more easily to the other case with mixed units (i.e., x-axis in radians and y-axis in degrees).

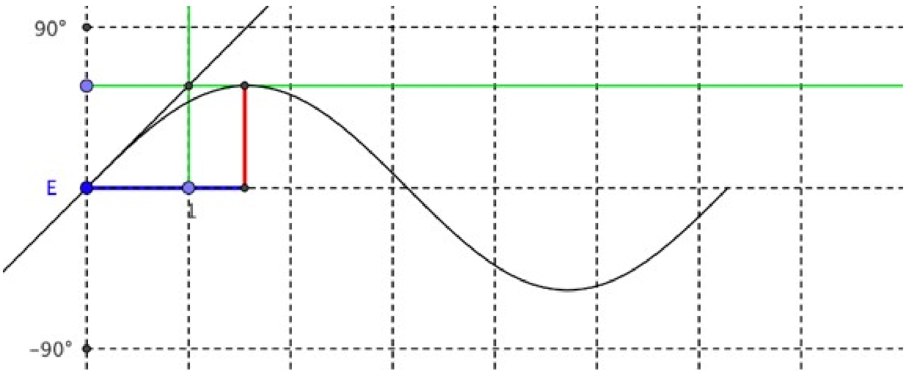

Figure 5. Max and slope for pairing 3.

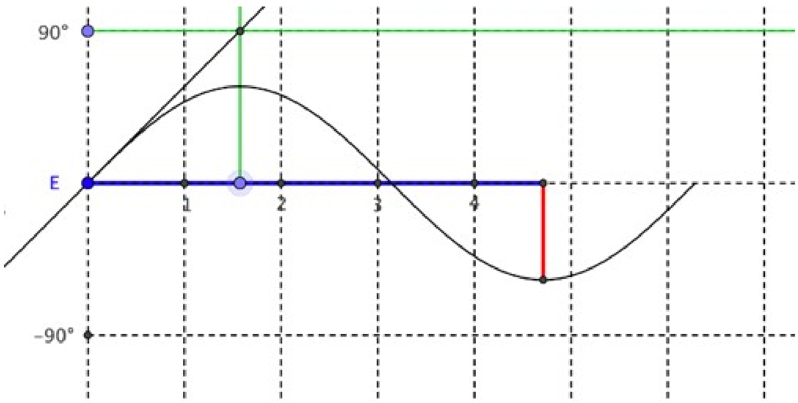

Figure 6. Min and slope for pairing 2.

With the final pairing, x-axis degrees y-axis degrees, our students reported that the max/min are about $60°$ and $-60°$ and occurring at $90°$ and $270°$. Additionally, the slope of the tangent line is approximated to be 1 in this case.

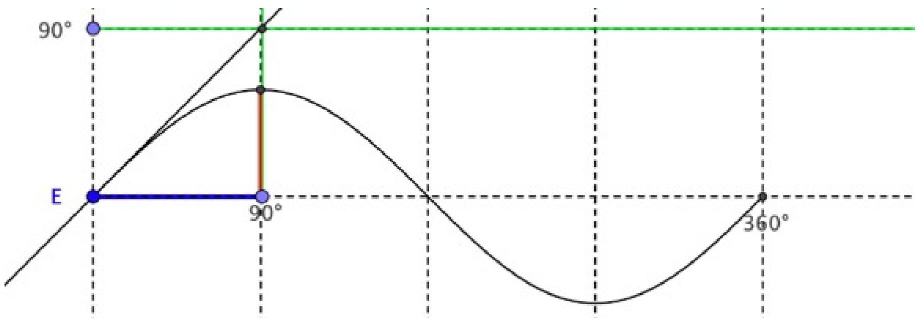

Figure 7. Max and slope for pairing 4.

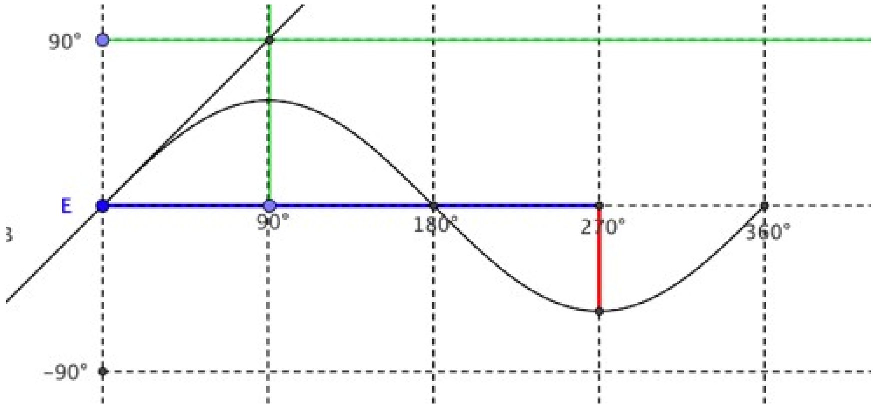

Figure 8. Min and slope for pairing 4.

4 RESULTS

After our students finished collecting the information, we asked them to summarize their findings in a table (see Table 1).

	x-axis radians	**x-axis degrees**
y-axis radians	max/min $(1.6, 1)/(4.7, -1)$ tangent slope: 1	max/min $(90°, 1)/(270°, -1)$ tangent slope: $\frac{1.6}{90°}$
y-axis degrees	max/min $(1.6, 60°)/(4.7, -60°)$ tangent slope: $60°$ (or $\approx \frac{90°}{1.6}$)	max/min $(90°, 60°)/(270°, -60°)$ tangent slope: 1

Table 1. Summary of Findings.

We asked our students to tell us what they saw in this table with respect to the effects of the units on each axis on the max/min and the slope of the tangent line to the sine curve at $x = 0$. As they shared with the class, we anticipated students pointing out that the units affect both the slope and the max/min. Probing further, they pointed out that the location of the max/min was determined by the units on the x-axis, 1.6 and 4.7 for radii, and $90°$ and $270°$. Similarly, the max/min value was controlled by the units on the y-axis, 1 and -1 in radii, and 1.6 and 4.7 in degrees. They also noted three things about the slope of the tangent line.

1. If the units are the same on the x- and y-axis then the slope is 1.
2. If the units are mixed, the slope is not 1.
3. When the mixed units are swapped on the axis, the slope is the reciprocal.

As stated earlier, we chose not to label the axis with fractions of π because we felt we would lose the connection to the unit being the circle's radius. However, students may notice that 1.6 and 4.7 are approximations for $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, respectively. Additionally, they might recognize that each slope approximation for the tangent line is simply a pairing of conversions between the two units.

5 Discussion

So why should anyone care about this discussion? When graphing trigonometric functions, we typically use the radius of the unit circle for our scale on the y-axis (i.e., radians). Still, we sometimes use radians and degrees for the x-axis. For precalculus students (and perhaps me as their instructor), the introduction of radians seems unmotivated. Why not just continue with degrees? By reflecting on the discussion in the activity, we can point out to our students that in calculus, we will be very interested in the rate of change of functions. If we use degrees for the x-axis but radians for the y-axis, the rate of change of the graph will be affected. The slope of the tangent line, in our example, was designed to be a stand-in for $\lim_{x\to 0} \frac{\sin x}{x}$.

This is a way to lay the foundation for our students to appreciate the consequences of units on the derivative of a function. Specifically, considering the derivative of the sine function, we have given them a start to think about the fact that if we use degrees for the x-axis but radians for the y-axis, the derivative of the sine function will not be the cosine function but rather the cosine function times a conversion factor of approximately $\frac{1.6}{90°}$, or more precisely $\frac{\pi}{180}$.

So, one last time, our students compared radians and degrees. We chose to look only at radians for the y-axis but considered both radians and degrees for the x-axis, the two standard pairings of units for graphing trigonometric functions. If we decide the slope of that tangent line is important to us, it is beneficial to use the same units on the two axes; it makes the slope 1.

Additionally, if we want the max/min of the sine function to be 1 and -1, respectively, we should make sure that we use radii on the y-axis. We used this activity with students who had already taken calculus; thus, we could reflect on the connections to calculus with them. However, suppose you use this activity when introducing radians in a trigonometric unit. In that case, you may focus on the consequences of the different units to lay the foundation for what comes ahead. Namely, in calculus, we are very interested in the rate of change, and thus, using radians on both axes is the only unit that would keep that rate of change 1 and the max and min 1 and -1.

Radians	**Degrees**
x-axis: radians	x-axis: degrees
y-axis: radians	y-axis: radians
max/min: $(1.6, 1)/(4.7, -1)$	max/min: $(90°, 1)/(270°, -1)$
Slope of tangent line: 1	Slope of tangent line: $\frac{1.6}{90°}$
$\lim_{x\to 0} \frac{\sin x}{x} = 1$	$\lim_{x\to 0} \frac{\sin x}{x} \approx \frac{1.6}{90°}$
$\frac{d}{dx} \sin x = \cos x$	$\frac{d}{dx} \sin x \approx \frac{1.6}{90°} \cos x$

Table 2. Features of the graph of sine.

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