PROJECTIVE GEOMETRY: A HISTORICAL OVERVIEW AND PERSPECTIVES FOR EDUCATION

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Abstract

This article aims to bring a discussion focused on the historical evolution of projective geometry and perspectives that aim at its application in the educational context, especially with the use of GeoGebra for the presentation of its concepts. The historical bias of this geometry field was analyzed in the light of the possibilities of its approach in basic education. The research methodology of this work is of a qualitative nature, bibliographic in nature, where we relate materials that bring the evolution of projective geometry from its genesis to the present day. In the meantime, we bring a perspective of transition between the Euclidean-projective geometries, as a way to elucidate some concepts of projective geometry and facilitate its understanding from the visualization with the contribution of GeoGebra.

Keywords: projective geometry, history of mathematics, visualization.

1 INTRODUCTION

Projective Geometry is a field of Mathematics that deals with the relationships between the real object and its projected image. In this sense, we can call it the geometry of what we see, and, as such, it shares a non-Euclidean character intrinsic to the geometries that describe the physical world. Therefore, Projective Geometry is a two-dimensional geometry without parallel lines or Simple Elliptic Geometry (Barros and Andrade, 2010).

In this article, we discuss the historical evolution of Projective Geometry and perspectives that aim at its application in the educational context, especially using GeoGebra to present its concepts. In this sense, we bring a theoretical model for its approach and application of one of its axioms in a visual perspective, supported by the GeoGebra software. The course of this research continues, firstly, bringing aspects of the origin of Projective Geometry and its evolution, the perspective of several authors to understand the obtained advances and their influences.

At first, the evolution of Projective Geometry is linked to art, and its genesis is associated with visualization. Later, its mathematical conceptualization starts to be considered, integrating the abstract theoretical field to be deepened and formalized, culminating in theories that underlie and define it until the present day.

After a historical analysis, we bring a perspective on the implementation of Projective Geometry in the educational context, in compliance with the characteristics that foster the need for its study, as well as the different possibilities of its approach. Based on this, we investigated which landmarks in the history of Geometry teaching allowed the insertion of Projective Geometry in the classroom.

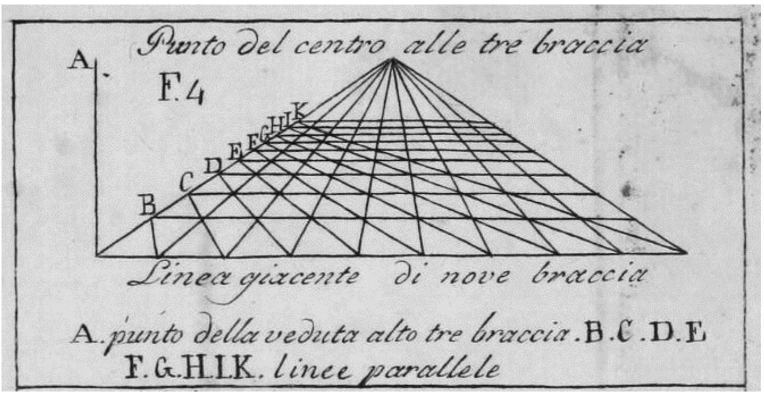
Finally, we present the importance of using software in teaching Geometry and, in particular, Projective Geometry, where we exemplify one of its axioms with the contribution of GeoGebra, which, in our view, favors its understanding.

2 A HISTORICAL OVERVIEW OF PROJECTIVE GEOMETRY

According to Auffinger and Valentim (2003), the origin of Projective Geometry dates back to the Renaissance, with the emergence of concepts such as vanishing point (or point of leakage) and perspectivity, which belong to the conic projection, introduced years later.

Alberti (1991), in his work De Pictura (Figure 1), already presented the concept of projection from an observer that, although not supported by his work, consisted of a primitive concept of Projective Geometry, since it used a fictional observer for the creation of projected images and the idea of depth perspective in his paintings:

Figure 1. The vanishing point.



In Figure 1, we have the idea of perspective generated on what would be a tiled floor. The parallel lines meet in the vanishing plane, giving the impression of depth in the image. An observer, external to the image, projects lines that meet the base points of the parallels, creating lateral sections determined by the points B, C, D, E, F, G, H, I, and K. This made it possible to create lines parallel to the horizontal, defining the proportional tiles according to the perspective. In the words of the author, we have:

> I have set out the general context briefly and, I believe, in a not entirely obscure manner, but I perceive that the content is such that, while I cannot demand praise for eloquence in exposition, the reader who does not understand it at first acquaintance will probably never understand it, no matter how hard you try (Alberti, 1991, pp.58-59, free translation).

Alberti (1991) expounded his notions of painting, believing that it would be simpler for the reader to be able to reproduce the technique, even if, as previously mentioned, he did not clearly understand the reasons behind the position of the observer (who should be at the same height of the vanishing point or leak point) and the drawing of parallels. According to the author, we understood that a vanishing point (or leak point) is the reference on the horizon to draw lines in a drawing and build a perspective. The word perspective comes from Latin and means "to see through". It is a two-dimensional representation of something three-dimensional. His work, in this regard, can be considered much more instrumental than educational.

Almost two centuries later, Girard Desargues (1591-1661) formalized these concepts, however, without wide acceptance. Granger (1974) states that the works of René Descartes (1596-1650) overshadowed the works of Desargues, given their rapid acceptance, thus making him one of the greatest names in Mathematics.

Desargues presented in his works the theorem known by his name – the Desargues' Theorem – one of the main foundations of Projective Geometry, which addresses the projective property between two triangles. It was only at the beginning of the 19th century, following the works of Jean-Victor Poncelet (1788-1867), that the works of Desargues began to be widely studied. In the work of Poncelet, known as the father of Projective Geometry, the emergence of this branch of geometry was established with the work called Traité des Proprietés Projectives des Figures (1822).

Chaves and Grimberg (2020) point to Poncelet's academic evolution, highlighting the stages in the production of his treatise and how, for 15 years (1807-1822), Poncelet worked in a compartmentalized way to consolidate his ideas. Still, according to the authors, in the years he studied at the École Polytechnique, Poncelet took classes with Hachette and had contact with concepts defined by Gaspard Monge (1746-1818), which would base his notions concerning Projective Geometry. In the work of Monge (1795), some of these concepts can be mentioned, such as:

> The first is to give methods for representing on a drawing sheet with only two dimensions, that is, length and width, all bodies in nature that have three dimensions: length, breadth, and depth, provided these could be rigorously defined. The second object is to give a way of recognizing the forms of bodies through an exact description and to deduce all the truths that result from their forms and their respective positions (Monge, 1795, p. 5).

Such concepts portrayed (i) the idea of projecting a three-dimensional image onto a plane and (ii) representing an image of a plane as a three-dimensional figure (Chaves and Grimberg, 2020). These concepts guided Poncelet in formalizing some definitions in Projective Geometry, understanding it as a geometry that is based not on the world we live in but on the world we see: that is, on the image projected by our eyes (Auffinger and Valentim, 2003).

Still at the end of the 19th century, Felix Klein (1849-1925), based on the studies of Arthur Cayley (1863-1895), began the development of the theory of invariants, which in turn had a direct connection to the principles of Projective Geometry Dias and Grimberg (2015). In Cayley's studies, he concludes that "the most elementary geometry is projective, which broke the paradigm of Euclidean Geometry being the most fundamental, on which Projective Geometry was based until then" (Dias and Grimberg, 2019, p. 217).

Based on its origins, it is possible to analyze modern studies on Projective Geometry, seeking to understand its main advances and relate them to the educational field, as discussed in the next section.

3 PROJECTIVE GEOMETRY IN EDUCATION

At the beginning of the 20th century, the mathematician Oswald Veblen (1880-1960) presented a study on Projective Geometry, bringing into detail the axioms that underlie it and carrying out the demonstrations that he thought possible. In the author's conception, his work summarizes an overview of the works of Felix Klein (Veblen, 1904).

Years later, in 1967, Achilles Bassi (1907-1973) published the book Elements of Projective Geometry, which consisted of a collection of handouts from the course in Projective Geometry taught by him at the San Carlos School of Engineering (Brazil), as well as other courses taught posteriorly. Bassi (1967) presented in his book, in detail, an approach to Projective Geometry aimed at the school environment, however, suppressing some topics that he considered unnecessary for Engineering students.

Bassi (1967) believed that it was more important to introduce the basic concepts of Projective Geometry as a continuity of concepts already established in Elementary Geometry than to present them in a completely abstract way. Thus, he believed knowledge would emerge naturally, making it possible to understand its evolution and build mathematical thinking. About Bassi's work, Silva and Táboas (2013) points out that:

> The introduction of themes or concepts through a historical method, à la Achille Bassi, was also advocated by several authors from the turn of the 19th to the 20th century. They believed that the study of scientific disciplines should follow a historically oriented method (Silva and Táboas, 2013, p.4022).

Based on Silva and Táboas (2013) analysis, it is possible to perceive the transition in the study of Projective Geometry, migrating from a purely mathematical and abstract view to a historical approach focused on the educational context. Bassi (1967) considered it essential to adapt mathematical knowledge to the context of the classroom and to understand the nature of the subject.

Regarding the history of Geometry and its study in Brazil, as well as its characteristics, we have the genesis of its teaching focused on military practical application, which refers to the periods of wars. The first math books were devoted to knowledge of calculus and geometry needed by artillerymen and firefighters (Oliveira et al., 2020). Thus, the first studies on Geometry started with Technical and Geometric Drawing, still in reproduction character, without considering relations and geometric properties that guided these techniques.

Currently, the National Curricular Common Base (Brasil, 2018), the guiding document of the Basic Education curriculum in Brazil, points out that the study of Geometry is mainly focused on Euclidean Geometry, with emphasis on plane and spatial figures. Only in one of the grade levels (9th grade), which is the transition year between elementary and high school in Brazil, does the student have contact with the study of orthogonal figures and the concept of perspective, which is the only characteristic of Projective Geometry addressed, given its applicability in a more intuitive way.

When approaching Geometry in the classroom, using plane representations in both Plane and Spatial Geometry is still very common. Although access to software is open and free, and there are possibilities for its use through devices such as computers and cell phones, its use in teaching is still restricted, especially concerning the relationship of two and three-dimensional models (Sousa et al., 2021).

Based on the above and considering the applicability of Geometry in different contexts, we understand the importance of the visual component. In this sense, we point out the relevance of using Dynamic Geometry software, such as GeoGebra, for understanding the relationship between elements and properties in Projective Geometry.

4 PROJECTIVE GEOMETRY AND GEOGEBRA

The study of Projective Geometry in Basic Education from the use of Dynamic Geometry software is essential to the understanding of its concepts and its learning Silva (2022). The use of GeoGebra, in particular, brings a vast contribution to the teaching of these concepts in a visual perspective, with possibilities of construction and manipulation of certain elements (Sousa et al., 2021).

However, it is known that in the Brazilian context, the lack of professional training regarding the use of software for teaching Geometry is still recurrent due to several factors, such as limited time for planning, lack of skill, and sufficient technological resources in public educational institutions (Alves et al., 2022). In addition, the concepts of Projective Geometry, particularly, are not widespread in teacher training Lopes et al. (2022). In this sense, we understand that it is necessary to present Geometry – especially Projective – dynamically, relating it to Euclidean Geometry and establishing a propitious scenario for the Euclidean-Projective Transition.

In this scenario, GeoGebra can enable the understanding from the visual perspective, the manipulation, and the construction of geometric elements of Projective Geometry, including the three-dimensional scenario with its 3D window. With the software, we can present to the student the relationship between the forms of representation of the Projective Plane, where the axioms of Projective Geometry apply (Figure 2 and Figure 3), observing the representation of a pair of parallel lines in the projective plane and its composition three-dimensional:

Figure 2. Parallel lines ($\overline{AB}$) and ($\overline{AC}$) meeting at the vanishing point.

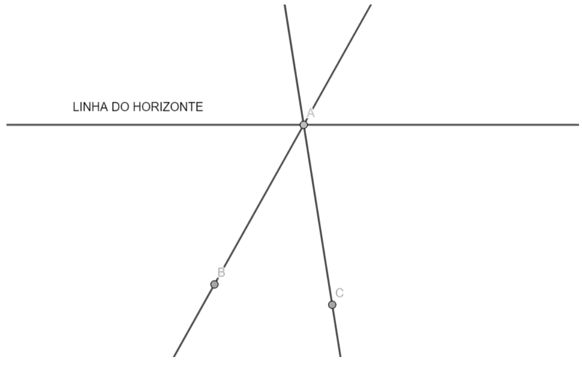
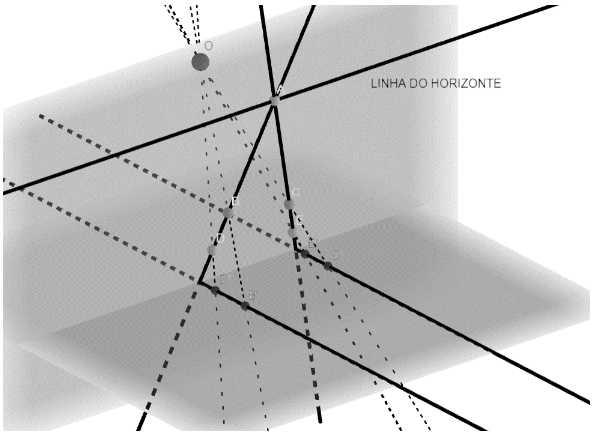


Figure 3. Three-dimensional composition originating from Figure 2.

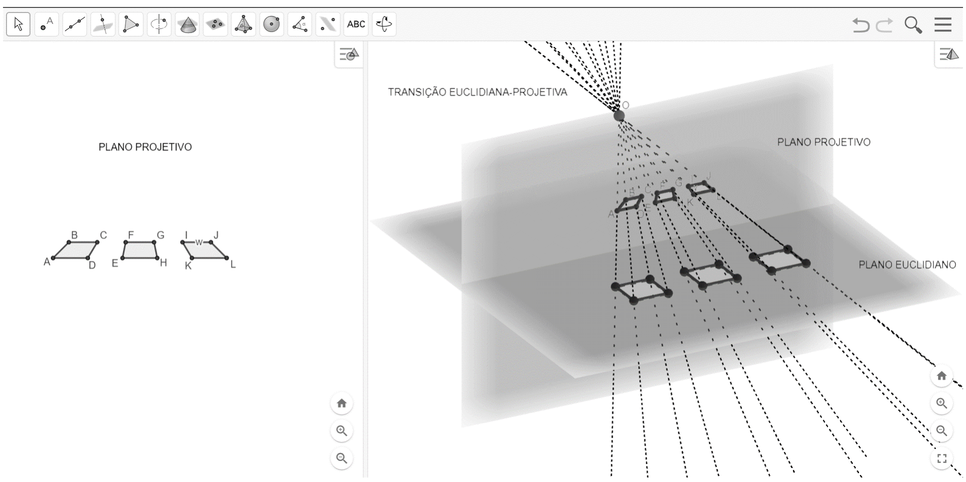


In Figure 3, it is not possible to take Point A (the meeting point of the parallels on the Horizon Line) to reconstruct the original straight lines, as this point is found at infinity in relation to the Euclidean Plane, making such a tracing impossible. In this construction, the vertical plane is the three-dimensional representation of the plane observed in Figure 2. The horizontal plane would be the original plane (Euclidean), which, when projected based on the observer (Point O), generates the proposed situation. In Figure 3, we have a Euclidean-Projective Transition, our object of study.

Projective Geometry uses two distinct planes and a point outside these called the observer (Auffinger and Valentim, 2003). In this way, a spatial scenario is created, whose visualization, when transposed to paper in a two-dimensional perspective, loses some of its characteristics and can make its understanding difficult. In this sense, we consider an approach that advocates the construction and manipulation of these elements in the three-dimensional plane, such as the 3D window of GeoGebra.

With GeoGebra 3D, it is possible to elaborate more complex constructions, but more understandably, expanding the geometric perception in the scenario of the Euclidean-Projective Transition. In Figure 4, for example, we present a sequence of projected squares in which, in the projective plane, the measurement criterion is disregarded, prioritizing only the shape:

Figure 4. Congruent squares projected in the Euclidean-Projective Transition.



In Figure 4, the point of congruence of the projection (the Observer) causes the deformation of the projected polygons, apparently losing their characteristics. However, in isolation from measurement-dependent properties, geometric properties are preserved, such as the central point, which can be demarcated by the meeting of diagonals, allowing the exploring of concepts such as the midpoint.

Thus, using GeoGebra and its three-dimensional component, we can approach projective geometry in an accessible way, in terms of visualization, and provide didactic resources for teachers and students interested in the area.

5 Final Comments

Based on the research and discussion presented, we can infer that using GeoGebra allows a better understanding of geometric concepts that depend on the visualization component for their learning, as is the case of some axioms of Projective Geometry.

Such concepts can be explored by the student in this tool in which, while manipulating its elements, the geometric behavior can be analyzed and its properties understood. In this sense, the axiom of parallels presented in this article shows that when we move the lines in the Projective Plane, it is possible to see in real-time how the lines in the Euclidean Plane behave. This is one of the characteristics added in the Euclidean-Projective Transition.

Regarding the historical panorama, Projective Geometry is a recent area when compared to the existence of Euclidean Geometry. We understand that this fact influences the limited dissemination of this field of research compared to the classical-theoretical analysis. However, when using a three-dimensional and correlated approach to Euclidean Geometry, we characterize the Euclidean-Projective Transition, enabling a better understanding of its properties.

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