

BUILDING AN ANALOG CLOCK WITH COMPLEX NUMBERS

David Glassmeyer¹ and Binita Patel²

¹Kennesaw State University

²Innovation Academy, Fulton County Schools

Abstract

In this article, we describe a lesson that enabled 10th and 11th grade students to create an analog clock using GeoGebra. This self-directed, exploratory lesson is built on students' prior knowledge of complex numbers and vector transformation and relies on the technological features of GeoGebra.

Keywords: Complex numbers, modelling, programming, student exploration, vectors

1 INTRODUCTION

Analog clocks can motivate students to explore a surprising number of mathematical topics. Mathematics educators have detailed how analog clocks have successfully supported elementary students to problem solve and reason about several mathematical topics (Earnest, 2022, Earnest et al., 2017, Earnest et al., 2018). Asking students to simulate an analog clock has been used in computer science settings to demonstrate the value of mathematics to programming students (Wu, 2012). However, there are no examples of how analog clocks can support high school mathematical content knowledge and programming knowledge. We propose such an activity can be successful because of the strong basis of mathematics built into the logic of computer programming and how students can use the various features and settings of Web 2.0 tools to understand mathematical calculations and their associated graphical representations (Hegedus et al., 2017; Kallia et al., 2021).

This article details a lesson centered on creating an analog clock using GeoGebra. We implemented the lesson with a class of 10th and 11th-grade students who had selected Information Technology as their pathway at the school and had exposure to complex numbers in their mathematics course, although their experience was limited. All students that participated were enrolled in or had completed Geometry. Students selected the activity from a set of activities offered for the day, and 58% of those that chose the analog clock option were enrolled in the gifted program. To engage all learners, this activity established a differentiated classroom setting to allow students to be active explorers, work flexibly in groups, and apply their understanding of key concepts with open-ended responses (Conklin, 2015).

The implementation site was a STEM magnet school structured with pathway-specific activities that allowed students to explore a task that may not necessarily align with the standards or pacing of their

courses. Driven by topics of interest, teachers design a two-hour activity that utilizes students' skills and allows for student inquiry across multiple disciplines. During the activity, we presented the challenge of building an analog clock using GeoGebra that would use the relationships between the hour and minute hand to display a specified time. With the increased use of digital clocks in the modern world, many students do not understand how to interpret the direction of the clock hands. We were curious how the rotation of vectors around the origin on the coordinate plane would strengthen or complicate students' conceptual understanding of clock mechanics.

2 ACTIVATING QUESTIONS

The lesson began with the graphing and multiplication of complex numbers in the GeoGebra graphing calculator. Before introducing the animated clock on GeoGebra, we invited students to reflect on the properties of complex numbers, particularly for the multiplication of two complex numbers, using questions such as:

$$\text{Multiply } (2 + 3i)(4 + 5i)$$

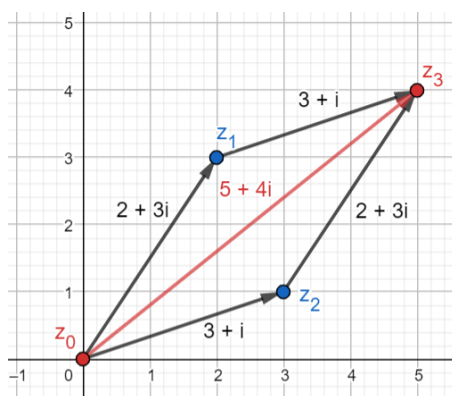
Since students had previously learned how to multiply imaginary numbers in mathematics class, the initial question offered them an opportunity to refresh their understanding.

Next, we encouraged students to think about the structure of complex conjugates and—whether from memory or conceptual understanding—determine the results of multiplying constants and imaginary numbers in different combinations. Questions included:

- If $(a + b)(a - b) = a^2 - b^2$, then what do you think $(a + bi)(a - bi)$ will be?
- If $(a + b) + (a - b) = 2a$, then what do you think $(a + b) + (a - bi)$ will be?

Using GeoGebra, students graphed vectors of their choice. Students then discussed the relationship between a single vector (z_1) and a secondary, resultant vector (z_3) determined from the addition of z_1 and a new vector (z_2). Identifying the geometric connection between complex numbers and vector addition (Figure 1) offered students the opportunity to brainstorm their approach to transforming the animated components of their product (Soto and Martinez-Cruz, 2018).

Figure 1. Students practiced addition and subtraction of complex numbers algebraically and geometrically to begin envisioning the direction and magnitude of the vectors that would represent their clock hands.



After reviewing addition and subtraction, students paired the paper-pencil approach with GeoGebra's graphing capabilities to briefly discover vector transformations in the complex plane through multiplication. They solved each calculation on paper, observing their written solution before graphing it in GeoGebra to illustrate the transformation.

- What happens when we multiply $a + bi$ by i ?
- What happens when we multiply $a + bi$ by i^2 ?

Students could quickly and confidently identify the resulting rotation from multiplying by powers of i . At this point in the lesson, differences in student understanding of complex numbers became more apparent. Individually, students explored the [GeoGebra applet](#) to visualize vector transformation in steps similar to those they followed with addition and subtraction. Some students replayed each animated step to understand the change in magnitude and direction of resultant vectors, and at their own pace, they adjusted the starting vector to create different examples. Most notably, after a few trials, students could adjust vector W in the applet to rotate vector WZ about the origin, much like a clock hand.

3 INTRODUCING AND SUPPORTING THE TASK

As a class, we then discussed the mechanics of an analog clock, ensuring students considered how the hour hand moved in relation to the minute hand (Figure 2).

Figure 2. Analog clock.



During a single second, (a) the second hand will make $\frac{1}{60}$ of a rotation, which on the clock shown would mean moving from 50 seconds to 51 seconds or equivalently from $\frac{50}{60}$ of a full rotation to $\frac{51}{60}$ of a full rotation; (b) the minute hand will move $\frac{1}{3600}$ of a full rotation, from $\frac{45 \cdot 60 + 50}{3600} = \frac{2750}{3600}$ of a full rotation to $\frac{45 \cdot 60 + 51}{3600} = \frac{2751}{3600}$ of a full rotation; and (c) the hour hand will move $\frac{1}{43200}$ of a rotation, moving from $\frac{11 \cdot 60 \cdot 60 + 45 \cdot 60 + 50}{43200} = \frac{42350}{43200}$ of a full rotation to $\frac{11 \cdot 60 \cdot 60 + 45 \cdot 60 + 51}{43200} = \frac{42351}{43200}$ of a full rotation.

Following this discussion of an analog clock, we asked students: *How can we multiply complex numbers in a way that controls the degree of rotation?*

4 STUDENT DISCOVERY—STUDENTS’ INTEGRATION OF TRIGONOMETRY

As we discussed the structure of an analog clock, students drew connections to the unit circle, particularly because of the stagnant center and the significant checkpoints consistently located around the circle’s perimeter. They began identifying specific time markings with angle measures, referencing 2:00 as having the hour hand at 30° and the minute hand at 90° . Listening to the shift in perspective, we proposed the option to work in polar coordinates (Figures 3 and 4).

Figure 3. Vector representation using Cartesian coordinates in rectangular form.

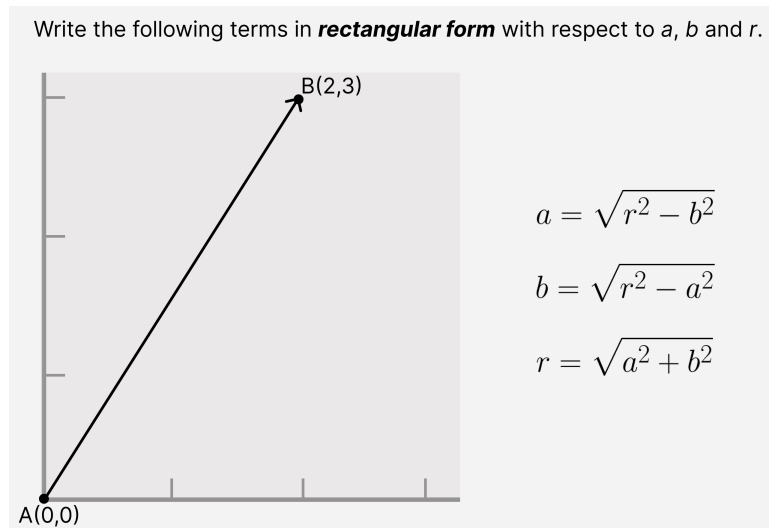
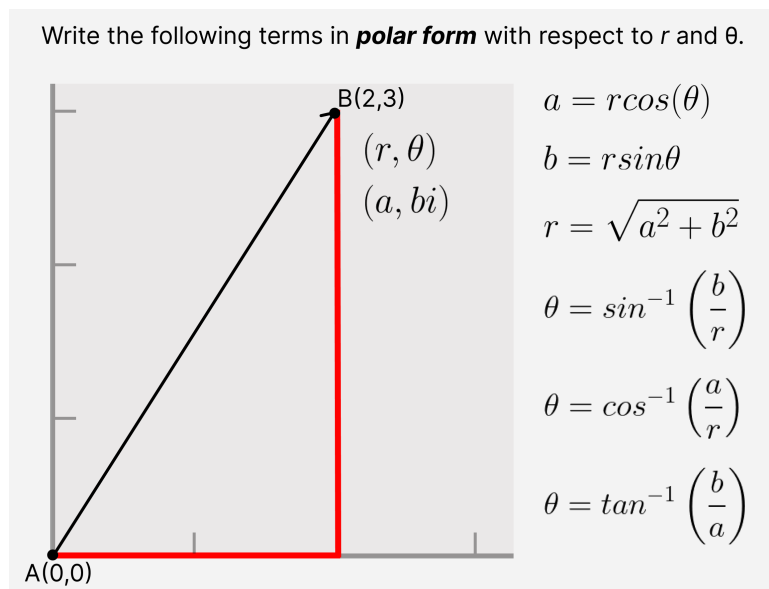


Figure 4. Vector representation using polar coordinates in polar form.



We noticed that they became more engaged as the students delved into the trigonometric view and understanding of their clock. Though all students in the classroom had experience with imaginary numbers, a select few had not been exposed to trigonometry and thus could not integrate the unit circle into their clock’s functionality.

5 FEATURES OF ANIMATION

While students had found unique paths to the construction of a clock, we were able to talk as a class with common terminology about the magnitude and direction of the hour hand and minute hand as vectors. Even though not all students understood polar coordinates comprehensively, they were willing to consider an alternative view of their clock, especially when they had difficulty animating it to have automatic movement. With the syntax of polar coordinates still projected on the board, we invited students to consider an understanding of the placement of the two hands within the space of the circle given a minute hand that is twice the magnitude of the hour hand.

- *Can you tell what time it is?* $(1;0) \wedge (2;90) = 3:00$
- *Can you tell what the coordinates are?* $3:10 = (1; 5) \wedge (2;30)$

Collectively, we commented on the counter-intuitive depiction of a time a few minutes after the hour as having a negative value. A few minutes later, as students continued building their programs, one student commented that his clock would animate independently, but it would only move backward in time. We were able to reference our conversation about polar coordinates for him to have a specific measurement to seek while adjusting the settings.

The clock's speed, particularly with the relationship in speeds of each hand, seemed to be the most advanced feature once students had acclimated themselves with the sliders within GeoGebra. One student, in particular, ended the activity with [one version of the clock](#), noting that it had “the incorrect ratio of the hours – the second hand moves at [a speed of] 1, the minute hand at .02, and the hour hand at :01.” Once he was able to control the animation, he set each hand to move seemingly slower than its more precise counterpart. The following week, he created an [updated version](#) with “all the latest bells and whistles; it goes in the correct direction and obeys how time works.” He reflected that though calculating the ratio for each speed was fairly straightforward, it was difficult to set the speed for each clock hand in GeoGebra and save them individually.

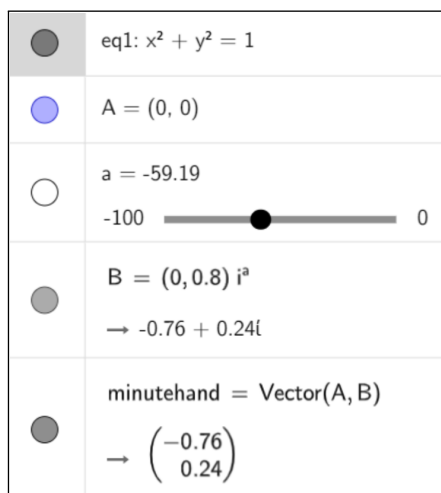
As students began exploring the sliders and settings within GeoGebra, even those sitting with classmates they knew well were more engrossed in their product, clicking into various settings to find the constraint that would make the adjustment to match the digital clock to the one they envisioned. The naturally collaborative environment of student-centered learning methods for a common project with multiple outcomes invites students to take on the responsibility to cooperate effectively and value different perspectives and approaches during the problem-solving process. Regarding the slider settings, we found getting the speed correct on the second hand challenging. The default speed for a slider is 1, which can be altered using the [slider commands](#) to reflect time more accurately by using a speed of 10 to [approximately model a 1Hz timer](#), meaning 1 cycle per second. More complicated programming is needed to model the passage of time precisely, and we found such methods to include using [GeoGebra scripts](#), [additional sliders](#), or the [get command feature](#). We consider accurately modeling the second hand as an opportunity for extension and challenge for our students rather than an expectation or goal of the activity.

When the teacher noted similarities between the structure or animation of two students that did not know one another, the students used their programs as a common ground to spark conversation. They even rearranged their seating to help one another program an additional feature.

6 STUDENT DISCOVERY AND INTERACTION WITH GEOGEBRA

Most students had previously used an online graphing calculator platform to quickly visualize the graph of numerous functions, though they had not used GeoGebra to explore new concepts. We observed students adjusting settings such as sliders to control animations from a perspective more relevant to computer science. Most students used multiplication by changing powers as we explored in the introductory activity to continue the clock hands' rotation; as they altered the settings of the slider representing the exponent, they found that the value needed to continuously decrease to create a clockwise rotation, opening an opportunity for us to extend our discussion of multiplication by powers of from the mathematical perspective (Figure 5). Students' engagement in the cooperative learning experience guided them to discover the functionality of their own individual programs, and during this iteration of the lesson's implementation, students did not discover the Rotate function available in GeoGebra as demonstrated in [other applications](#).

Figure 5. The slider, a , represents the changing exponent that rotates the vector representing the clock's minute hand.



After creating an hour hand, one student reflected on the creation of a minute hand and shared that “it was something you said about the powers of” that motivated them to explore the construction of another vector and its relation to the first. Across the different groups, we noticed that all students set the clock hands as two individual vectors with a relationship in the animation between the two, making it difficult to program the clock to a specific time. Moreover, only a select few students labeled the hours and minutes on the clock's perimeter for it to be legible under its functional purpose, likely because our initial instruction focused on vector transformation.

This activity had Project-Based Learning (PBL) elements, allowing students to approach their learning experience more authentically. They could easily glance at their classmates' working clocks and seek help, identifying the circle surrounding the analog hands, the graphic design and colors, or the animation of the hands themselves as an opportunity for peer guidance. Through the focused, concrete task with multiple options for solution and customization, students were able to “see themselves as creating with and learning through technology in collaboration” (DeGennaro and Brown, 2009, p. 15). They were empowered to be active leaders in developing both the process and the product, which formatted the lesson as an integration of theory and practice. The teacher's role was to serve as a facilitator, supporting and celebrating student success and outcomes (Escobar and Qazi, 2020).

7 STUDENT AVENUES AND OPPORTUNITY FOR DIFFERENTIATION

This activity has several opportunities for extensions. For example, a subsequent lesson might further explore the depiction of the clock hands with polar coordinates to delve into setting the clock hands to specified angles. Representing specified times could introduce an additional computer science activity that might incorporate a sync of the GeoGebra application with the computer's time setting or converting a user-input digital time to the analog clock equivalent. As mentioned previously, other features could be added to the clock, such as ensuring the second hand automatically models the passage of time.

As students began their initial clock display, there were various outlets for creating functionality within their project. Many students—particularly those with limited prior exposure to advanced mathematics beyond Geometry—based the relationship of vectors on complex numbers, rotating the minute hand with respect to the hour hand by multiplying by variations of i . However, nearly all students that had practice with trigonometry preferred to utilize polar coordinates to manipulate the movement of vectors in the 360° or 2π space. As a result, the outputs of each student were different. Some students produced stagnant clocks that could represent different times by adjusting the settings, while others generated clocks with automated animation even if the clock did not tick in the same tempo as standard second, minute, or hour intervals. The proactive, student-centered approach to the clock creation allowed students various opportunities to access the content, process, and product and receive feedback that could serve as an informal formative assessment (Davis and Autin, 2020).

8 SUGGESTIONS FOR CLASSROOM IMPLEMENTATION

The concept of a functional clock was open to interpretation as there were no formal requirements for how students would relate the two vectors representing clock hands or how their project would involve sliders for animation. The flexibility of the project output was made possible by the format of the school's designated activity-based session, though it highlighted the value of complex tasks to help students make sense of problems and persevere in solving them (National Council of Teachers of Mathematics, 2014).

We noticed students were intrinsically motivated to create what they considered a functional clock or became interested in adjusting their clock after witnessing the work of their peers. We also observed students learning about GeoGebra through peer assistance, discovering the extensive settings and capabilities of the platform. Students became less concentrated on the mathematical procedures and instead used conversations with the teacher to develop an understanding of how changes to certain program components affected outputs.

This activity could be used in a classroom with structured feature requirements to match the course and unit standards. Students begin the task with a common entry point, though inquiry-based learning can allow students to explore different representations of their solutions. The authenticity and challenge of the analog clock activity include many indicators for engaged learning that promote students as responsible for their learning through their roles, teacher roles, learning tasks and goals, and assessments (Jones et al., 1995). Offering the opportunity in class for students to have explorative task-based activities on a particular day of the week, similar to the Friday implementation shared here, can encourage empowerment and a deeper understanding of mathematical concepts.

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