FILLING VESSELS: AN EXCITING WAY TO INVESTIGATE FUNCTIONAL DEPENDENCIES

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Abstract

When discussing and analyzing functional dependencies, schoolbooks, and teachers often use different filling curves—asking students to match various vessels with their corresponding graphs. After presenting the three basic ideas of functional thinking, the authors demonstrate ways of determining functional equations from analyses of filling data and discuss the use of dynamic applets with students.

Keywords: Filling curves, modeling, mathematical inquiry, functional thinking

1 INTRODUCTION

When designing teaching materials and courses, it is essential to consider the mental representation of the mathematical concepts they will help form. The German concept "Grundvorstellungen" (basic ideas) covers three aspects of this issue. The first feature is the constitution of the meaning of a mathematical concept by linking it back to familiar knowledge. The second one is generating a corresponding mental representation of that concept; it enables operative action at the level of thought. Finally, the third quality of this didactical concept is the ability to apply a concept to real-life situations by recognizing a corresponding structure in subject-related contexts (Vom Hofe & Blum, 2016).

Especially concerning functional thinking, we know three basic ideas which cover the entire spectrum of this topic (Frey, Sproesser, & Veldhuis, 2022):

1. A function matches each element in a set precisely with one element of a target set. This is the basic idea of the assignment.

2. Functions are used to demonstrate how changes in one variable affect the second variable or how the second variable is influenced by the first. This basic idea is called covariation.

3. The third aspect focuses on a function as a mathematical object with its specific representations and properties. This is the object's basic idea of a function.

The following sections cover the first and second basic ideas outlined above.

2 Filling vessels – definition and examples

Filling curves are functions that indicate the liquid level in different vessels as a function of time, usually assuming that the vessels are filled at a constant inflow rate. To plot filling curves, a function $V : \mathbb{R}_0^+ \to \mathbb{R}_0^+, h \to V(h)$ must first be defined, which gives the volume of the liquid in the vessel as a function of the liquid level h. This can be done by using geometric aids, in the case of more complex vessels, by integrating the function of the cross-sectional area (as a function of height), or—when filling bodies of revolution—by calculating the respective volume of revolution. The inverse function of this function V then gives the liquid level as a function of the liquid's volume. To specify the change in liquid level as a function of time, the liquid volume V must be replaced by a function $V : \mathbb{R}_0^+ \to \mathbb{R}_0^+, t \to V(t)$ with $V(t)$ corresponding to the volume of liquid in the vessel after t time units. Since the filling rate is usually assumed to be constant, the function V is a linear function of the form $V(t) = ct$ where the scalar $c \in \mathbb{R}_0^+$. In the subsequent sections, filling curves of different bodies will be explored.

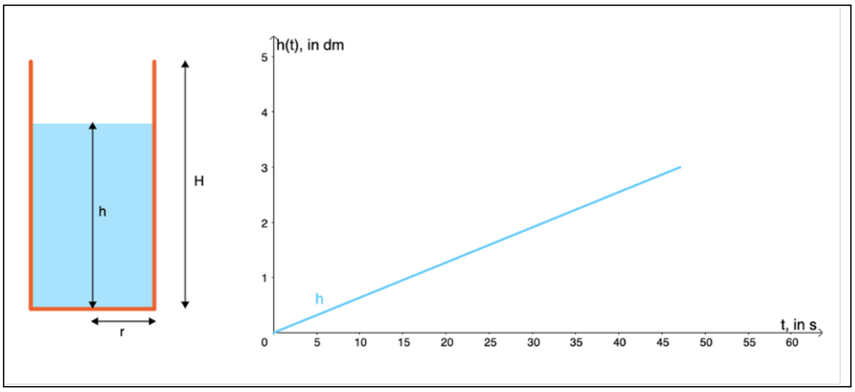
2.1 *Filling a cylinder*

First, we will look at calculating the function of the liquid level when filling up a cylinder with radius r and height H. If the vessel is filled up until the height h, the volume of the liquid in the container is given by $V(h) = r^2\pi h$. Its inverse function h, given by $h(V) = \frac{V}{r^2\pi}$, expresses the liquid level as a function of the volume in the vessel. Assuming a constant flow rate c, the liquid level h can be expressed as a function of time:

$$h(t) = \frac{ct}{r^2\pi}.$$

The connection between time and liquid level is thus linear, which also holds for all other prismatic bodies. When defining filling curves for specific containers, it is essential to adjust the units if necessary. If, for example, the velocity of the filling process is given in liters per second, the vessel's dimensions must be given in (or changed to) decimeters. Additionally, the definition and the value set must be specified since the vessel would overflow after a certain amount of time (i.e., $t_0 = \frac{r^2\pi H}{c}$). To look at a more concrete example, a cylinder with a radius $r = 1$ dm and a height of $H = 3$ dm must be filled at a rate of 0.2 liters per second. The process would end after about $15\pi \approx 47.12$ seconds. The liquid level (see Figure 1) as a function of time for this particular process can be expressed by $h : [0, 15\pi] \to [0, 3]$ for $t \to \frac{0.2t}{\pi}$ as plotted in Figure 1.

Figure 1. Filling a cylinder.



2.2 Filling compound bodies

It is also possible to define the filling curves of vessels composed of two bodies—for example, a vase consisting of two cylinders with different radii. We must define a piecewise function, the first section describing the filling process of the first part of the vessel and the next section expressing the adding body's filling process. Given a body composed of two cylinders with the radii $r_1 = 4$ dm and $r_2 = 2$ dm, both of height $H = 3$ dm, the volume V as a function of height would be defined as:

$$V(h) = \begin{cases} 16\pi h & \text{if } \ 0 \le h \le 3 \\ 48\pi + 4\pi(h-3) & \text{if } \ 3 < h \le 6 \end{cases}.$$

Thus, the inverse function h, expressing the liquid level as a function of the volume, is defined by

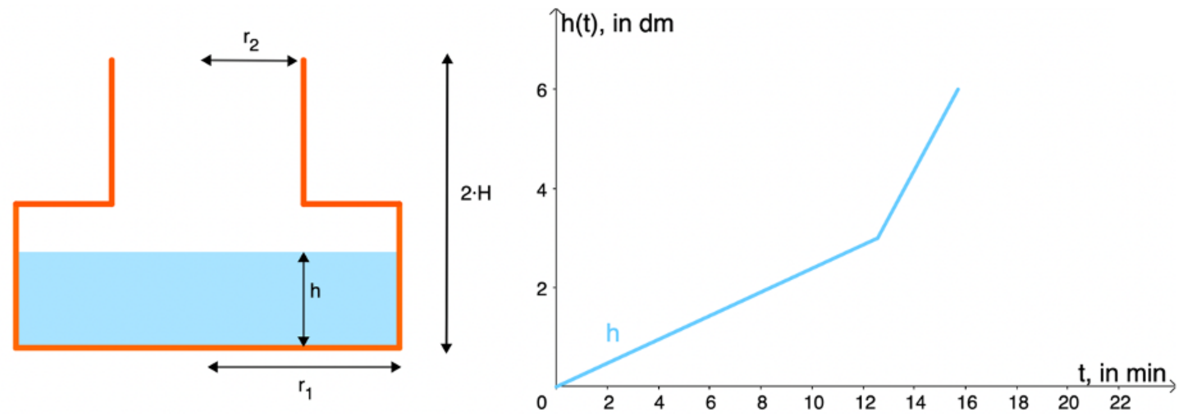
$$h(V) = \begin{cases} \frac{V}{16\pi} & \text{if } \ 0 \le V \le 48\pi \\ \frac{V-48\pi}{4\pi} + 3 & \text{if } \ 48\pi < V \le 60\pi \end{cases}.$$

Assuming an inflow rate of 12 liters per minute (0.2 liters per second), the liquid level as a function of time (in minutes) is given by

$$h(t) = \begin{cases} \frac{12t}{16\pi} & \text{if } \ 0 \le t \le 4\pi \\ \frac{12t-48\pi}{4\pi} + 3 & \text{if } \ 4\pi < t \le 5\pi \end{cases}$$

and plotted in Figure 2.

Figure 2. Filling a compound body.



Lambert and Hilgers (n.d.) also note that students often understand the relationship between the structure of the vessel and the filling curve even better when they have the opportunity to look at compound bodies. Intuitively, we recognize that the filling curve's slope changes precisely where the water level reaches the upper cylinder. At this point, the filling curve shows a buckle. Therefore, the function is continuous at this point but not differentiable.

Furthermore, one can see that the water level rises faster for vessels with a smaller radius. Finally, students will examine the vessels and the filling curve more closely to understand the exact relationship between the curve's slope and the cylinders' radii: The lower cylinder has a radius twice as large as the upper cylinder. Since both cylinders have the same height and the volume of the lower cylinder is four times as large as the volume of the upper cylinder ($V(2r) = 4r^2\pi h = 4V(r)$), the time required to fill the lower cylinder is four times longer than the time necessary to fill the upper half of the vessel. Therefore, the water level must rise four times as fast when the upper part of the vessel has been reached. This relationship is reflected in the functional equation and the graphical representation of the filling function.

2.3 Filling a paraboloid

Noting that a cylinder can also be considered as a rotational body, we explore filling curves of rotational bodies using the parabola given by the equation $p : y = ax^2$, where $a \in \mathbb{R}^+$, rotating around the y-axis. First, the equation must be solved for x^2, resulting in $x^2 = y/a$. The volume V of liquid in the vessel as a function of the liquid level h is then given by

$$V(h) = \pi \int_0^h x^2\,dy = \pi \int_0^h \frac{y}{a}\,dy = \frac{\pi h^2}{2a}.$$

The inverse function of V expresses the liquid level h as a function of the liquid's volume and is given by

$$h(V) = \sqrt{\frac{2Va}{\pi}}.$$

Assuming a constant inflow rate c (in liters per second), we can again replace the term V by ct where $c \in \mathbb{R}$ and t in seconds, resulting in a function that expresses the liquid level in the vessel as a function of time—a root function with the functional equation

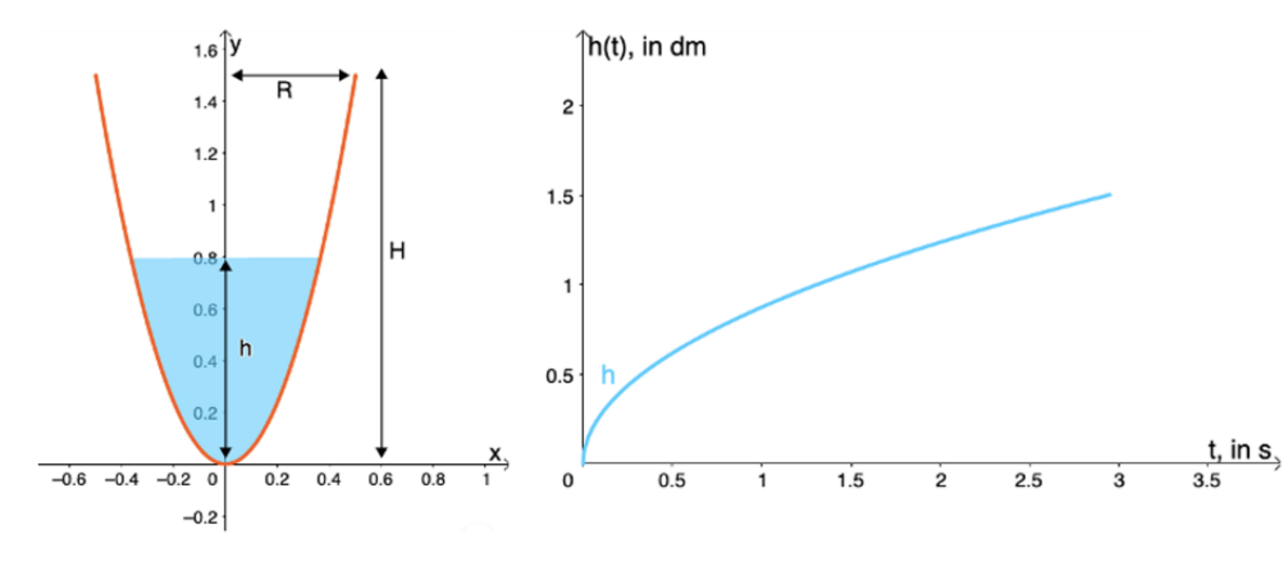
$$h(t) = \sqrt{\frac{2cta}{\pi}}.$$

If the vessel's height H, as well as the radius R of its opening, is given, the parameter a that defines the paraboloid of revolution can be expressed by solving the equation $H = aR^2$ for $a = H/R^2$. If, for example, a paraboloid-shaped vessel with a height of 1.5 decimeters (dm) and a (maximum) diameter of 1 dm is filled at a constant inflow rate of 0.2 liters per second, the parabola rotating around the y-axis would be defined by $p : y = 6x^2$, thus, the liquid level as a function of the time is expressed by

$$h(t) = \sqrt{\frac{2.4t}{\pi}}.$$

To find the definition set, the vessel's maximum capacity must be calculated: $V(1.5) = \pi\frac{1.5^2}{12} = 0.1875\pi \approx 0.59$ liters. The vessel is filled at an inflow rate of 0.2 liters per second. Therefore, after about $0.9375\pi \approx 2.95$ seconds, the vessel is filled up, and the function h is well defined for $t \in [0, 0.9375\pi]$. The parabola and corresponding filling curve are presented in Figure 3.

Figure 3. Filling a paraboloid.

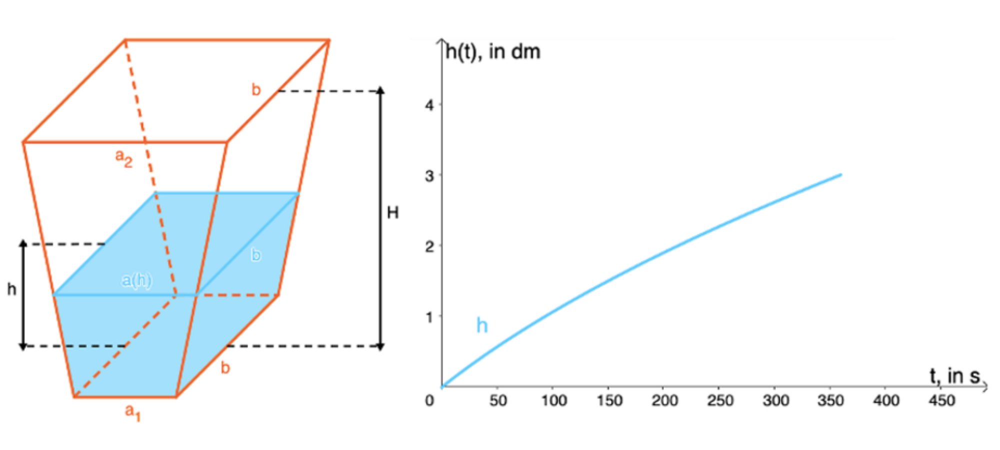


The slope of the graph of h is infinite at the point of origin. This is because of the point-shaped bottom of this vessel form. The liquid level increases instantly when starting the filling process.

2.4 Filling non-rotational bodies

This subsection will deal with filling up vessels, not bodies of revolution. To do so, a container with rectangular cross-sectional areas will be examined. The container's side length increases linearly with increasing height. The width b of the cross-sectional areas remains constant. The side length of the base area is denoted by a_1, that of the top area by a_2. The height of the vessel is expressed by the variable H. A sketch of such a container is shown in Figure 4.

Figure 4. Filling non-rotational bodies.



First, a function must be set up, which gives the side length $a(h)$ of the cross-sectional area at level h. This linear function is defined by $a : [0, H] \to \mathbb{R}$, with

$$a(h) = \frac{a_2 - a_1}{H} \cdot h + a_1.$$

Thus, the cross-sectional area Q as a function of the level h can be expressed by

$$Q : [0, H] \to \mathbb{R}, Q(h) = \left(\frac{a_2 - a_1}{H} \cdot h + a_1\right) b.$$

To find the volume of the liquid in the vessel, filled up to the liquid level h, the definite integral of Q with the limits 0 and h must be determined, expressing the volume as a function of the liquid level:

$$V : [0, H] \to \mathbb{R}, V(h) = \int_0^h \left(\frac{a_2 - a_1}{H} \cdot x + a_1\right) b\, dx = b\left(\frac{a_2 - a_1}{2H} \cdot h^2 + a_1 h\right).$$

$V(0) = 0$ expresses the minimum, $V(H) = b\left(\frac{a_2-a_1}{2} \cdot H + a_1 H\right) = bH \cdot (a_1 + a_2)/2$ the maximum volume of the vessel. By setting up the inverse function and replacing V with ct (again, assuming a constant flow rate $c \in \mathbb{R}^+$), the liquid level as a function of time is defined by the root function

$$h : [0, t_0] \to \mathbb{R}, h(t) = \frac{-a_1 + \sqrt{a_1^2 + 2 \cdot \frac{a_2-a_1}{H} \cdot \frac{ct}{b}}}{\frac{a_2-a_1}{H}}.$$

If the container is assumed to be a large flower planter with the measurements $a_1 = 2$ dm, $a_2 = 4$ dm, $b = 8$ dm, and $H = 3$ dm, the maximum capacity is 72 liters. At a flow rate of 0.2 liters per second, the flower box is filled up after $t_0 = 6$ minutes (or 360 seconds). Thus, the function h is well defined for $t \in [0, 360]$ and the water level as a function of time is given by $h : [0; 360] \to \mathbb{R}$ with

$$h(t) = -3 + \frac{3}{2}\sqrt{4 + \frac{1}{30}t},$$

as also demonstrated in Figure 4.

2.5 Filling the frustum of a cone

The volume of a vase, shaped like a frustum of a cone, can be calculated by rotating a linear slope around the x-axis (but also with geometric considerations, if necessary). If r_1 denotes the radius of the base area (the smaller base of the cone), r_2 the radius of the top area, and H the height of the vase, the volume V as a function of the liquid level h is given by

$$\begin{aligned} V(h) &= \pi \int_0^h \left(\frac{r_2 - r_1}{H} \cdot x + r_1\right)^2 dx \\ &= \frac{H\pi}{3(r_2 - r_1)} \left[\left(\frac{r_2 - r_1}{H} \cdot h + r_1\right)^3 - r_1^3\right] \end{aligned}$$

with the inverse function

$$h : h(V) = \frac{H}{r_2 - r_1}\left(\sqrt[3]{\frac{3(r_2 - r_1)V}{H\pi} + r_1^3} - r_1\right).$$

Note, for $h = H$ we get

$$\begin{aligned} V(H) &= \frac{H\pi}{3(r_2 - r_1)} \left[\left(\frac{r_2 - r_1}{H} \cdot H + r_1\right)^3 - r_1^3\right] \\ &= \frac{H\pi}{3(r_2 - r_1)}(r_2^3 - r_1^3) \\ &= \frac{H\pi}{3}\left(r_1^2 + r_1 r_2 + r_2^2\right). \end{aligned}$$

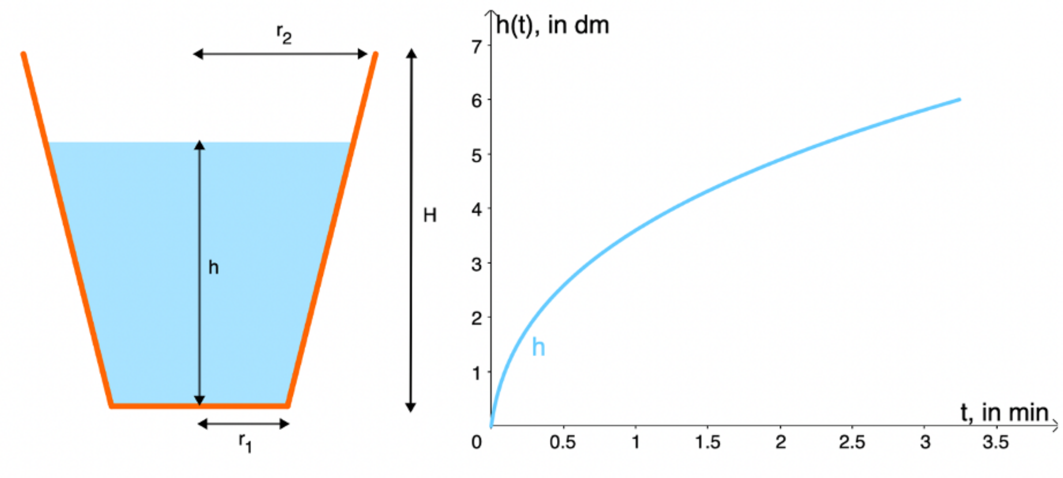
This is the well-known term for the volume of a frustum of a cone. In the case of $r_1 = 0$, the vessel is a cone with the function $h(V) = \frac{H}{r_2}\sqrt[3]{\frac{3r_2V}{H\pi}}$, showing an infinite slope at the point of origin.

Again, V can be replaced by $ct(c \in \mathbb{R}^+)$ to express the liquid level as a function of time. If the vessel is a large floor vase with the measurements $r_1 = 1$ dm, $r_2 = 5$ dm, and $H = 6$ dm, the vase holds about 194.78 liters in total, resulting in a filling time of about 3.25 minutes, if a constant inflow rate of 1 liter/second is assumed. Replacing V with $60t$ (t in minutes), the equation

$$h(t) = \frac{6}{4}\left(\sqrt[3]{\frac{720t}{6\pi} + 1^3} - 1\right)$$
$$= \frac{3}{2}\left(\sqrt[3]{\frac{120t + \pi}{\pi}} - 1\right)$$

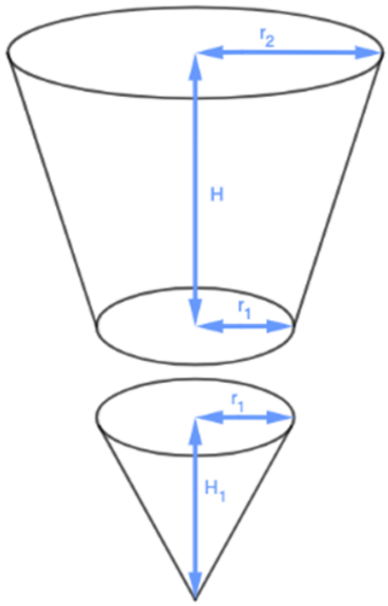
gives the liquid level in the vase (in decimeters) as a function of time (in minutes). In this case, the inequality $0 < h'(0) < \infty$ holds. This filling curve, plotted in Figure 5, is used for the applet described in the next section.

Figure 5. Filling the frustum of a cone.



It is also possible to derive the formula for V without using integral calculus. Therefore, a discussion can be provided at secondary level 1 (e.g., 9th grade). Figure 6 demonstrates the fundamental idea.

Figure 6. The volume of a frustum of a cone.



The frustum of a cone can be considered the remaining part of a small cone is cut off from a bigger one. Consequently, we calculate $\frac{r_2^2\pi(H_1+H)}{3} - \frac{r_1^2\pi H_1}{3}$. Further, the proposition $\frac{H+H_1}{r_2} = \frac{H_1}{r_1} =: \lambda$ can be derived. The volumes of the big and small cone are thus $V_1 = \frac{r_2^2\pi(H+H_1)}{3} = r_2^3 \cdot \frac{\pi}{3} \cdot \lambda$ and $V_2 = \frac{r_1^2\pi H_1}{3} = r_1^3 \cdot \frac{\pi}{3} \cdot \lambda$. The difference can then be expressed by $V_1 - V_2 = \frac{\pi}{3}\lambda\left(r_2^3 - r_1^3\right) = \frac{\pi}{3}\lambda(r_2 - r_1)\left(r_2^2 + r_2 r_1 + r_1^2\right)$. Because of $\lambda(r_2 - r_1) = \lambda r_2 - \lambda r_1 = (H + H_1) - H_1 = H$, the frustum of a cone has the volume $V = \frac{\pi H}{3}\left(r_2^2 + r_2 r_1 + r_1^2\right)$.

The next step is to describe the linear increase of the radius r of the frustum from the bottom to the top. Therefore, r depends on the height h of the frustum: $r(h) = \frac{r_2 - r_1}{H} \cdot h + r_1$. Replacing H by h and r_2 by $r(h)$, the volume of the frustum from the bottom to height h is $V(h) = \frac{\pi h}{3}\left(r_1^2 + r_1\left(\frac{r_2-r_1}{H} \cdot h + r_1\right) + \left(\frac{r_2-r_1}{H} \cdot h + r_1\right)^2\right)$, for $0 \leq h \leq H$.

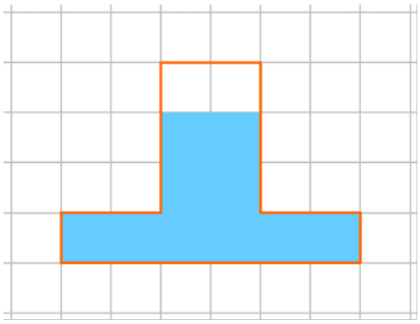
We can use GeoGebra CAS to find the inverse function $h(V)$. Suppose we compare the two terms $V(h) = \frac{H\pi}{3(r_2-r_1)}\left[\left(\frac{r_2-r_1}{H} \cdot h + r_1\right)^3 - r_1^3\right]$, given by the formula of the volume of rotational bodies, and $V(h) = \frac{\pi h}{3}\left(r_1^2 + r_1\left(\frac{r_2-r_1}{H} \cdot h + r_1\right) + \left(\frac{r_2-r_1}{H} \cdot h + r_1\right)^2\right)$, given by elementary geometry, we will not see their identity at first glance. The second one is much more difficult to invert than the first one. The computer yields $h(V) = \frac{\sqrt[3]{-H^3 r_1^3\pi^3 + 3H^2 V r_1 \pi^2 - 3H^2 V r_2 \pi^2 + H r_1 \pi}}{r_1\pi - r_2\pi}$. The denominator is negative. For that, the numerator must also be negative. Further manual transformations lead to $h(V) = \frac{H}{r_1-r_2}\left(\sqrt[3]{\frac{3V(r_1-r_2)}{\pi H} - r_1^3} + r_1\right) = \frac{H}{r_2-r_1}\left(\sqrt[3]{\frac{3V(r_2-r_1)}{\pi H} + r_1^3} - r_1\right)$. This result is identical to the inverse function of the rotational volume.

2.6 Resume

Subsections 2.1 to 2.5 show that filling curves can be discussed at different levels. Filling a cylinder leads to simple linear functions (Subsection 2.1). However, if one wants to fill a non-rotational body with increasing cross-sectional areas (Subsection 2.4), the change in the length of specific edges with increasing height has to be described first. Suppose the areas of the cross sections depend on one variable. In that case, students can quickly evaluate the definite integral to find the volume of this solid with specific cross sections on an interval. This application is an attractive alternative to the well-known rotational bodies like a paraboloid in Subsection 2.3.

Simple compound bodies, composed of cylinders and cones (cf. Subsection 2.2), allow analyzing filling curves qualitatively. Similar to Figure 2, we can construct the filling curve of the vessel shown in Figure 7 by noting that the cross-sectional area of the top cylinder is the ninth part of the cross-sectional area of the bottom cylinder. This comes from the proportion of the basic diameters (3:1). The height of the top cylinder is three times bigger than the height of the bottom one.

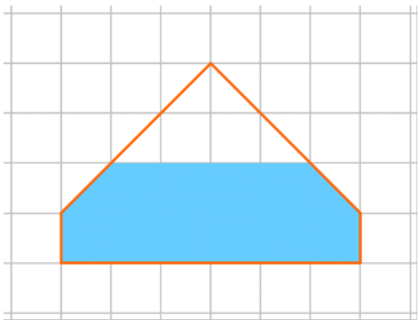
Figure 7. Two cylinders with different cross-sections.



Therefore, filling the upper part of the vessel will be three times faster than filling the cylinder that makes up the bottom of the vessel (Lambert & Hilgers, n.d.).

Similar observations can be made when analyzing a compound body, as shown in Figure 8, with a cone three times higher than the cylinder. Because of the same bottom area, the volumes of the two bodies are equal and, consequently, the filling times.

Figure 8. A cone at the top of a cylinder.

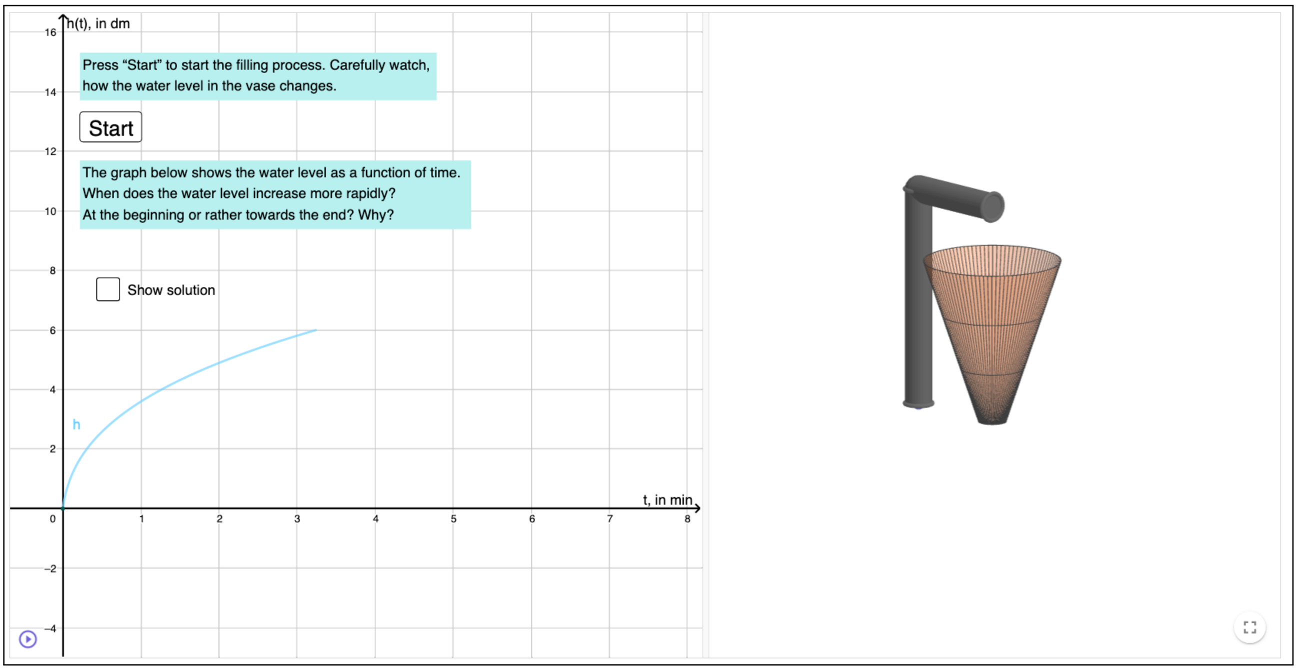


The volume of the top half of the cone is an eighth of the rest. Therefore, filling the lower half of the cone takes $\frac{7}{8}$ of the time needed to fill the entire cone. The filling curve's graph has an infinite slope at the cone's peak. The continuous decrease of the diameter of the entire body leads to a graph without buckles, dissimilar to Figure 2 (Lambert & Hilgers, n.d.). With the results of Subsection 2.1 and Subsection 2.5, it is easy to compose the filling function of the body shown in Figure 8.

3 The applet

Next, we demonstrate how GeoGebra applets can be used to help students grasp the concept of filling curves. While schoolbooks typically present graphs to be matched with the corresponding vessel, the entire dynamic process can be made visible using GeoGebra. An applet (`https://www.geogebra.org/m/ykh56bys`) demonstrating the process of filling a vase and its relationship to the filling curve has been developed in the course of a master's thesis on GeoGebra applets (Sergi, 2022, pp. 76–83). Figure 9 shows the applet as it appears in its initial state.

Figure 9. Applet in the initial state.

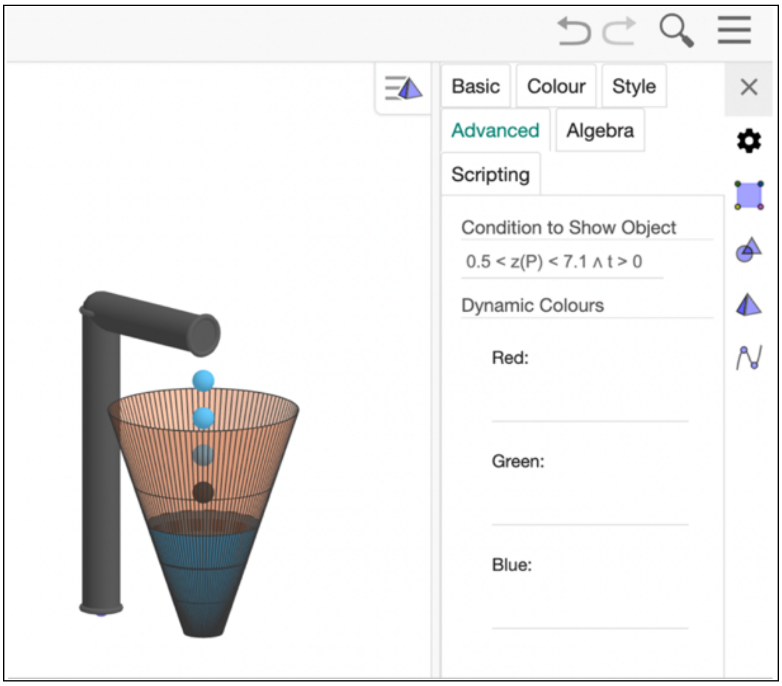


The vessel in this applet is a large floor vase, a frustum of a cone with the measurements $r_1 = 1$ dm, $r_2 = 5$ dm, and $H = 6$ dm as described in Subsection 2.5. When a user presses "Start," a virtual push button programmed with `StartAnimation()`, an invisible slider is set in motion, changing the value of a variable t. The value of this variable corresponds to the time passed in minutes. The vase is filled at a constant rate of one liter per second (60 liters per minute). The slider runs through the interval $[0, 3.25]$, the time (in minutes) needed to fill the vase (Subsection 2.5).

All dynamic elements that can be found in this applet are dependent on the value of t: The two 3-dimensional truncated cones on the right were created by rotating straight lines around the z-axis, with the rotation interval of the blue line corresponding to the calculated height of the water level at time t to ensure a correct increase in the liquid level (rotation interval $[0, h(t)]$). The bullets, representing water dropping from the water tap, are dotted with z-components changing their value as the slider moves along. In total, eight dots were defined, ranging from $(0, 0, 12 - \frac{6}{3.25}t)$ to $(0, 0, 5 - \frac{6}{3.25}t)$. Thus, the dots move down the vertical axis as the value of t increases and reach their lowest point at $t = 3.25$.

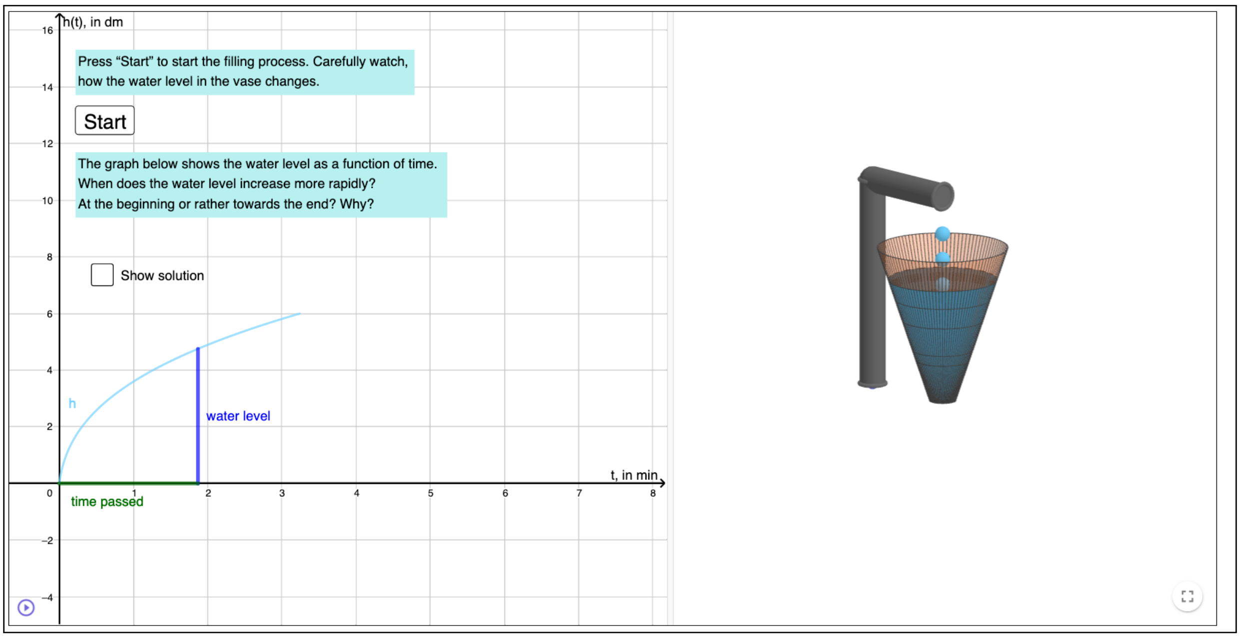
Additionally, conditions for the visibility of these dots have been set to ensure that they are only visible as they are below the tap and within the vase $(0.5 < z(P) < 7.1)$ and as soon as the slider is set in motion ($t > 0$). Figure 10 shows the conditions for the visibility of one of those dots.

Figure 10. Conditions for visibility.



Another dot in the graphics window on the left, defined by the coordinates $(0, t)$, specifies the end-point of the horizontal as well as the vertical line, illustrating how the time passed, and the water level can be read off the graph h, as demonstrated in Figure 11.

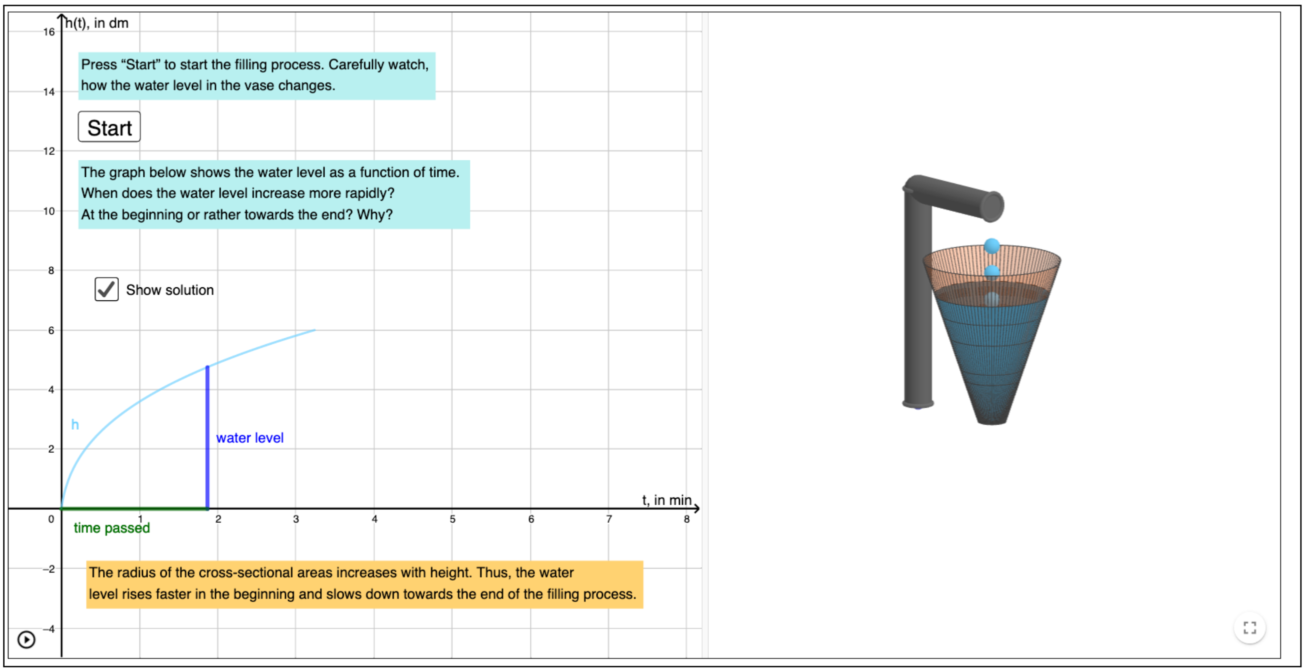
Figure 11. The filling process.



As suggested by Dorner (2014, p. 35f.), applets that are not interactive—so-called demonstration applets—should always include guiding questions to help students focus on the essential mathematical aspects. Thus, the question “When does the water level increase more rapidly? At the beginning or rather towards the end? Why?” have been included as a text element.

By ticking the checkbox “Show solution”, an explanation—a text element with visibility conditions depending on the value of the checkbox—appears, as shown in Figure 12.

Figure 12. Showing the solution.



The applet primarily supports students’ functional thinking and helps them grasp the concept of covariation. Not only does it demonstrate that a change in the argument causes a change in the function value, but the nature of this change is thematized and illustrated (especially with the help of the additional tasks). In contrast to static illustrations, as they are often found in textbooks, the dynamic elements focus on how changing one quantity affects the other. If students cannot perform and observe dynamic changes, this relationship between elements of the domain and codomain becomes less noticeable. Since the instructions are exact and the usability of this applet is straightforward, students can use the applet without the teacher’s guidance. If the teacher does the demonstration, the additional questions could be deleted and replaced by a dialog guided by the teacher.

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