ORIGAMI PAPER CUP: FROM PAPER FOLDING TO DATA ANALYSIS AND ALGEBRAIC EXPLORATIONS

Lingguo Bu[1] and Harvey Henson[2]

[1,2]School of Education, Southern Illinois University Carbondale

Abstract

We present the case of an origami paper cup and its extensions to data collection, analysis, and dimensional reasoning in K–12 teacher education, featuring the integration of GeoGebra. The paper cup case has been implemented numerous times in our K–12 mathematics methods classes, targeting a variety of issues of mathematics teaching and learning—multiple representations, mathematical discourse, intuitive perceptions, rational reasoning, data analysis, as well as hands-on engagement and ownership. To implement the instructional tasks, we need an adequate supply of commercial or self-made origami paper, copy paper of various sizes, and a few pounds of pinto beans or similar beans that are safe for classroom use as well as access to GeoGebra.

Keywords: Origami, Data Analysis, Algebraic Reasoning

1 Introduction

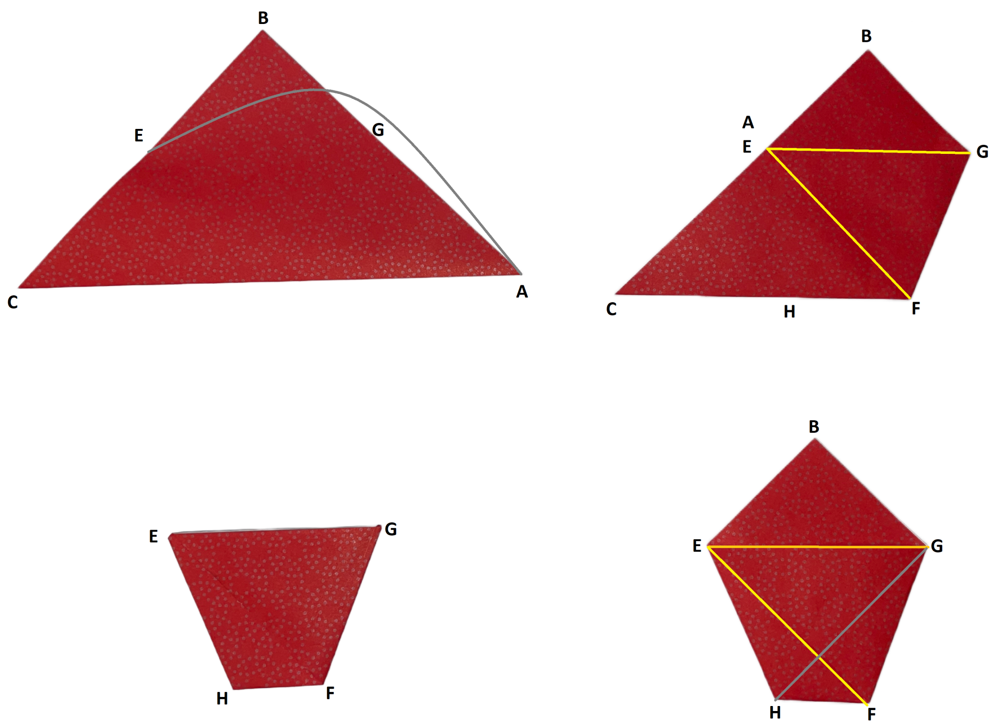
Mathematics is a coherent system of ideas and processes with ubiquitous connections to the real world and other branches of mathematics. Not only is our understanding of mathematics grounded in making connections, but the teaching of mathematics also involves the design and use of rich contexts and activities to support classroom sense-making and communication that explores the interplay among diverse aspects of mathematical connection and integration (National Council of Teachers of Mathematics[NCTM], 2000). Accordingly, teacher candidates must experience mathematics as a "connected and integrated whole" (NCTM, 2000, p. 65) through meaningful activities that embody the holistic nature of mathematics. This article showcases an origami-inspired learning sequence in our mathematics methods course that provides teacher candidates with rich opportunities to see the evolving mathematical landscape around a simple origami paper cup. The sequence has an easy entry point—unfolding along physical, visual, algebraic, and statistical dimensions in a manner that appeals to the interests of teacher candidates.

Origami paper folding is traditionally a form of art widely known for its appealing realism, cultural relevance, and values in recreational mathematics (Gardner, 2008; Lang, 2021). In recent years, origami has found its way into industrial design and applications (e.g., National Aeronautics and Space Administration, n.d.) and has also been reported in numerous articles in K–12 mathematics education, serving a host of pedagogical purposes such as student engagement, language development, geometric communication, habits of mind, and algebraic thinking (Cipoletti & Wilson, 2004;

Georgeson, 2011; Higginson & Higginson, 2001; Robichaux & Rodrigue, 2003; Wiles, 2013). In particular, Georgeson (2011) reported a case where origami and beans were used for students to explore the volume of a modular origami cube, its dimensional connections, and related algebraic analysis. The present case starts with a simple paper cup with a distinctive 3D shape and evolves into data analysis as well as geometric and algebraic explorations. As this case has been implemented with our teacher candidates with highly replicable results, we describe the case generically, supporting our descriptions with actual data and images across several implementations of the case, while highlighting turning points for pedagogical and mathematical explorations as well as details of our integration of GeoGebra.

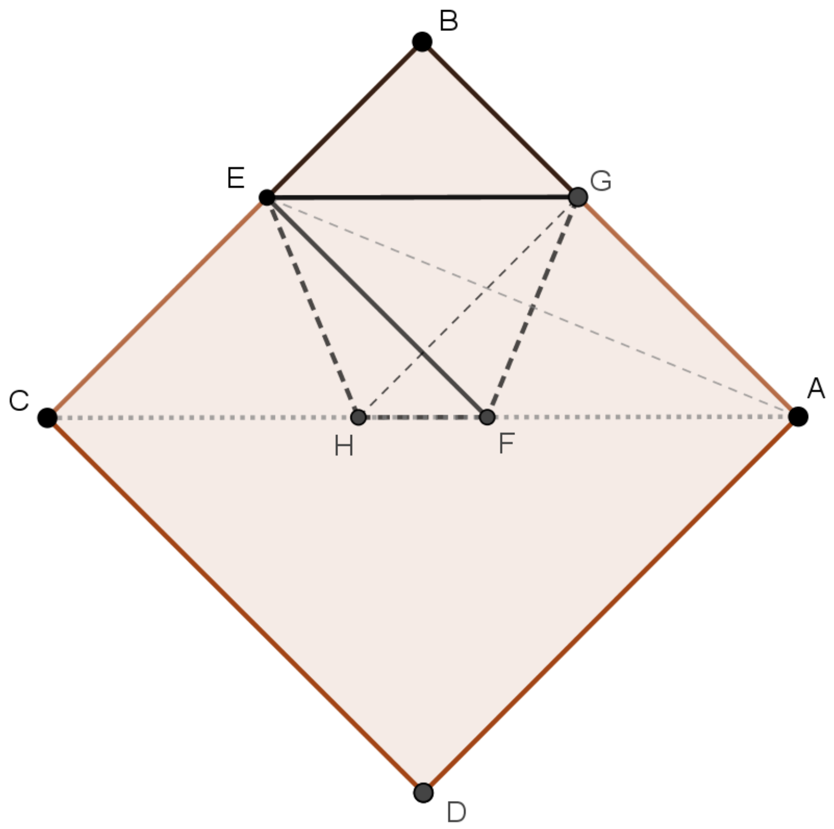
2 Making a Paper Cup

Figure 1. A square is folded into a small cup in a few steps.



We use origami projects routinely with our teacher candidates to meet a variety of pedagogical and content goals. Teacher candidates and children usually enjoy the art and emerging mathematics behind origami folding. We first came across the origami paper cup in Soong (1948, pp. 6-7). It takes a few minutes in a classroom setting, requiring little prior origami experience yet affording rich discussions.

Figure 2. Dynamic GeoGebra simulation of the processes of an origami paper cup.



As illustrated in Figures 1 and 2, we start from an origami square, typically six inches by six inches. First, we fold the square $ABCD$ along its diagonal AC for a triangle $\triangle ABC$. Second, we fold point A to point E, making sure EG (or AG) is parallel to CF, which is part of AC. After this step, we have quadrilateral $BCFG$. Third, we repeat step two on point C, folding C backward to G on the reverse side, after which we arrive at a pentagon $BEHFG$. Fourth, we fold the front triangle $\triangle BEG$ forwards along EG and tuck it into the pocket EFG at the front. Fifth, we repeat the previous step, fold the overlapping triangle $\triangle BEG$ backward, and tuck it into the back pocket EHG. After this step, we get an isosceles trapezoid $EHFG$. Finally, we open the trapezoid $EHFG$ at the top for a paper cup. All teacher candidates can get their paper cups made in about fifteen minutes in a group setting. Some may need assistance with the second step when folding parallel lines.

While language is an important part of communication, the paper cup feels easier when approached as a hands-on task. In folding the paper cup, teacher candidates have opportunities to reflect on their use of words, pictures, and especially physical actions as they struggle through the simple processes and discuss the emerging shapes using meaningful mathematical terms. For instance, parallel lines and angle bisectors (segment AE would be a bisector of angle $\angle BAC$ in Figure 2) can be used to address the consequences of paper folding. Is $AGEF$ (or $CHGE$) a rhombus? We ask our teacher candidates to unfold the paper cup to look for patterns and to generate questions for further exploration.

3 Estimating Volume of the Paper Cup

Once everyone in the class has made a paper cup from a six-inch square, we ask teacher candidates to estimate the volume of the paper cup, which certainly has yet to have a ready-made formula. The paper cup can indeed hold water. Thus, it is appropriate to fill it up with water (or coffee for easy reading) and then pour the water into a measuring cup for an estimate in ml or fl oz. While a little messy, it should be demonstrated to students upon request. In our work with teacher candidates and young children, we have used pinto beans (or similar beans) to approach the volume of the paper cup, as did Georgeson (2011).

First, students are given a handful of pinto beans to get a feel for them. Then, they are asked to take a close look at their paper cup and estimate the number of pinto beans that can go into the cup, taking note of and reporting the number to the instructor. The instructor collects the class data in a table, using a GeoGebra spreadsheet, and leads a brief discussion of the perceptions of the whole class. We typically create a boxplot in GeoGebra (as shown in Figure 6) to illustrate the distribution of the class data. The mean can be used as a hypothesis to be tested through subsequent data collection.

Figure 3. A cupful of pinto beans to be counted.

Once the intuitive data are recorded and briefly analyzed via a Geogebra boxplot, teacher candidates will each take a cupful of pinto beans and count the actual number of beans their cup can hold (Figure 3). Counting strategies are discussed before or after students' actions according to the instructor's pedagogical intentions. While some students will count one by one, others will naturally resort to grouping (Figure 4) or even some artistic display (Figure 5).

After all the students have finished counting, they report their numbers to the instructor, who records the data in the same GeoGebra spreadsheet to compare the intuitive and the experimental data. Once the class data are recorded, students should be encouraged to make observations and propose ways of data analysis. Another boxplot can be made and placed alongside the previous one for comparison and statistical inferences.

If desired, a t-test can be conducted for students to decide their hypothesis. Figure 6 shows the boxplots of both intuitive and actual data together with the results of a t-test during our implementation in Fall 2018. The mean of the class's initial estimates is 63 beans, while the mean of the actual data is 160, which may vary with the batch of beans.

As is common in our teaching experiments, teacher candidates and children tend to underestimate the volume of the paper cup. Frequently, even the boldest estimate is less than the minimum of the actual data. This is a meaningful moment for teacher candidates to discuss the interplay between intuition and rational reasoning and the integration of data collection and analysis in teaching and learning mathematics.

Figure 4. Counting with strategy the beans from a paper cup based on an eight-inch square.

Figure 5. An artistic arrangement of a cupful of beans by a teacher candidate (eight-inch square).

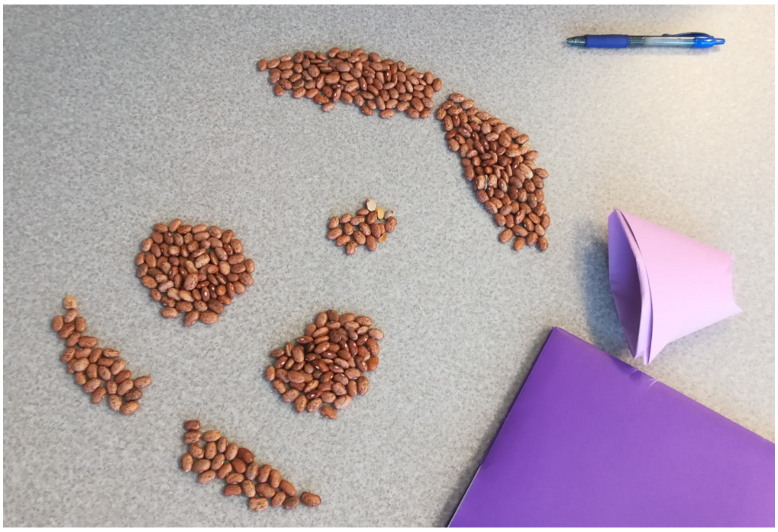
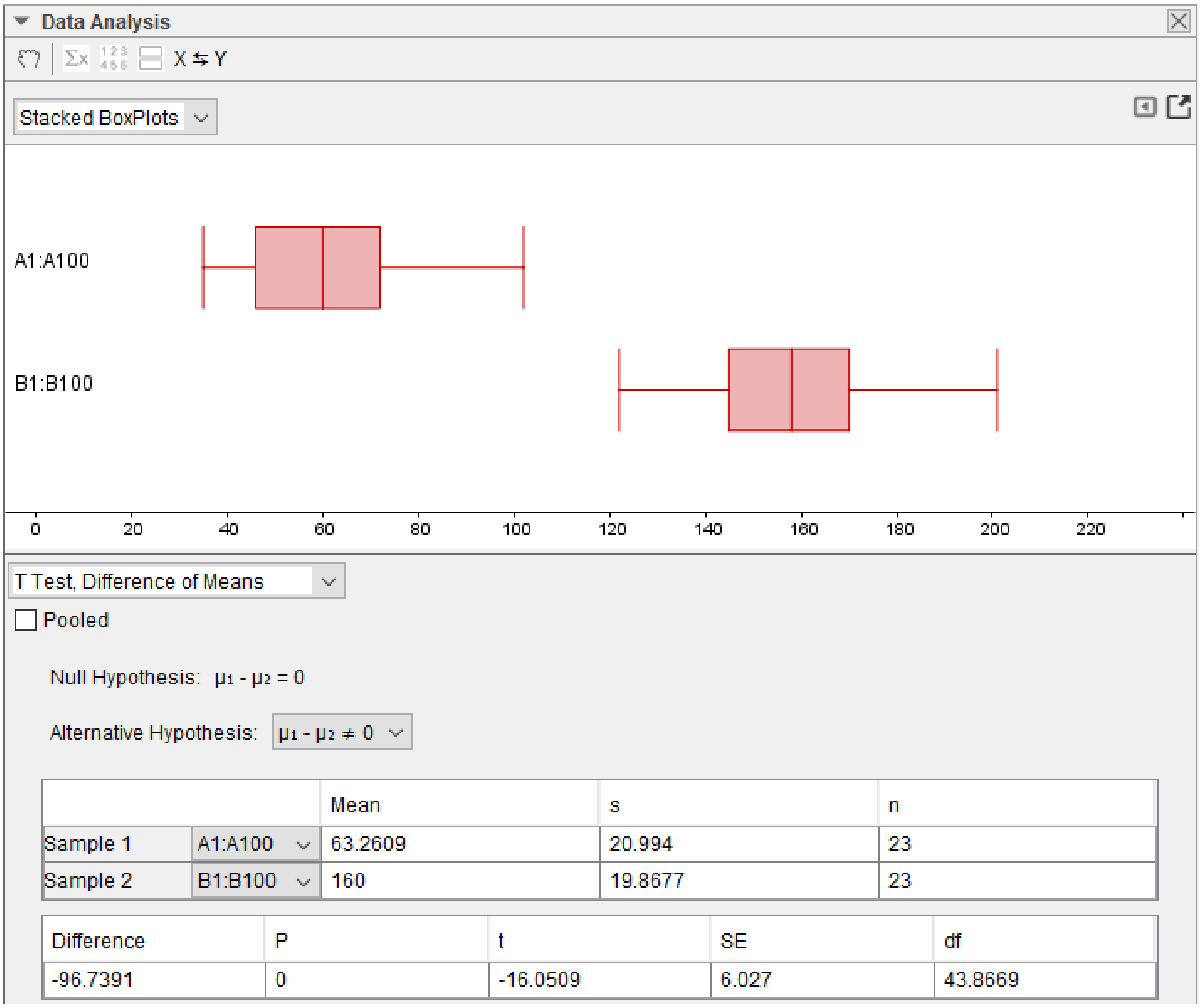


Figure 6. Boxplot comparison and t-test between the perceived volume and actual volume in number of beans (six-inch square).



		Mean	s	n
Sample 1	A1:A100	63.2609	20.994	23
Sample 2	B1:B100	160	19.8677	23

Difference	P	t	SE	df
-96.7391	0	-16.0509	6.027	43.8669

4 What-if's and Dimensional Explorations

The paper cup activities can be readily extended to launch a discussion about dimensions and their implications, which we typically pursue in a three-hour class period. A series of *what-if* questions (Brown & Walter, 2005)) can be posed and investigated using data collection and algebraic reasoning. For example, *what if we make a similar paper cup out of a* 3*-inch square? What if we make a similar paper cup out of a* 12*-inch square? How about an* 8*-inch square? How about a* 5*-inch square?* Usually, it is a pleasant surprise for students to face the consequence of doubling the starting square from six inches to 12 inches — the new cup can hold eight times as many beans. This can be demonstrated by filling the bigger cup with the small cup when there are enough pinto beans.

Ultimately, the 3D nature of volume will become part of the discussion, which leads naturally to discussions about the paper cup's exterior surface area — two overlapping trapezoids. *What would happen to the exterior surface area if the original square is doubled, or tripled, in terms of its dimensions?* We have also had teacher candidates making tiny paper cups using a 3-inch square, which holds about 20 beans. As paper folding becomes easier with each replication, the mathematical ideas become more curious! Indeed, an algebraic representation is possible. We usually bring in the cube as a scaffold. In short, if a cup based on a six-inch square can hold 160 beans, a cup made from a N-inch square can be expected to hold $160\left(\frac{N}{6}\right)^3$ beans.

4.1 Classroom Examples

To show what may unfold in a real class, we share some details from an implementation of the paper cup activity with one of our classes back in the fall of 2018.

After the initial data collection with cups constructed from six-inch squares, the mean bean count stood at 160 beans, clearly violating our hypothesis. Next, teacher candidates discussed the cup made from a 12-inch square. After observing the big cup, one student, Amy,[1] conjectured that it would hold four times as many beans. She supported her argument with a demonstration, showing that four small cups fit onto a big cup—using *flattened* cups in the shape of a trapezoid. The instructor repeated Amy's idea to the class, confirming Amy's claim as a valid 2D perspective and calling attention to the 3D nature of volume.

Next, Amy discovered she could dump more than four small cupfuls into the big cup using real beans. Amy's classmate, Kara, asked how many beans could go into a smaller cup made from a three-inch square (i.e., half the side length of the original square). However, the instructor explained that the resulting volume should be one-eighth the volume of the original, which Kara did not clarify. Then, Madi went ahead, tore a six-inch square into four equal pieces, and made a smaller cup. She walked across the room and handed it to Kara. Kara put some beans in it and counted 19. It all happened naturally as the teacher candidates got curious about the problem and sought solutions as a group.

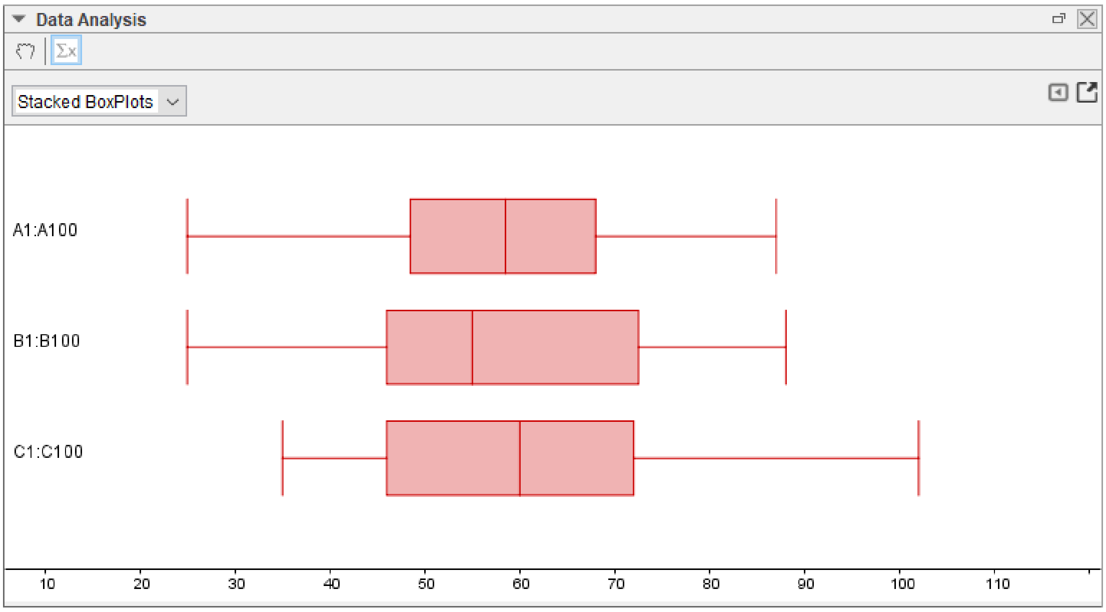
Indeed, there are multiple directions a teacher can proceed in response to the dynamics of student learning, blending a host of pedagogical and mathematical discussions. The paper cup and beans are merely tools of exploration and communication in mathematics.

5 Ubiquity of Intuitive Underestimation

Human intuitions are natural and persistent (e.g., De Bock, Van Dooren, Janssen & Verschaffel, 2007), affording rich opportunities for teacher candidates and teacher educators to address the roles of mathematical reasoning and data-based inferences. In the case of teacher candidates' initial perceptions of the paper cup volume, there has been no significant change over the past few years with different groups of teacher candidates. Figure 7 shows the boxplots of intuitive data collected in three iterations of the case, where the mean remains around 60 beans for a paper cup made from a six-inch square. After the teaching experiment, teacher candidates tended to be cautious in making predictions about bigger or smaller cups. However, they still tend to underestimate the volume. In addition to physical modeling, a transition toward algebraic analysis is thus warranted and will allow them to make further connections and grow as resourceful classroom teachers.

[1] all student names in this paper are pseudonyms

Figure 7. Multisemester boxplot comparison of intuitive data about the paper cup volume based on a six-inch square($n = 24,\ 28,\ 23$, respectively).



6 Mathematical Extensions

From paper folding to GeoGebra data analysis and algebraic discussions, the origami paper cup provides a rich context for teacher candidates to experience the connections among mathematical concepts and processes. Whenever time permits, the paper cup can be further mathematized to facilitate the exploration of other mathematical ideas. We discuss two extensions in the following sections.

6.1 Angle Bisectors

In folding an origami square to two overlapping triangles and then to a pentagon (Figures 1, 2), we directed students to bring point A to a certain point E on the other side of the triangle and make sure EG is parallel to the base CF. In paper folding, we just estimate. But what is point E (or G) in a rigorous geometric sense? And how can we construct it in GeoGebra?

The location of point E is an interesting and worthwhile task to explore before we can simulate the paper cup using GeoGebra. A close examination of the processes in Figure 2 leads to a few observations:

1. $AF \cong EF$ (as the consequence of folding).
2. $AG \cong EG$ (as the the consequence of folding).
3. $EG \parallel AF$ (as is the requirement).
4. $\angle GEA \cong \angle GAE$ (because $\triangle EGA$ is isosceles).
5. $\angle GEA \cong \angle EAF$ (because $EG \parallel AF$).
6. Thus, $\angle GAE \cong \angle EAF$.

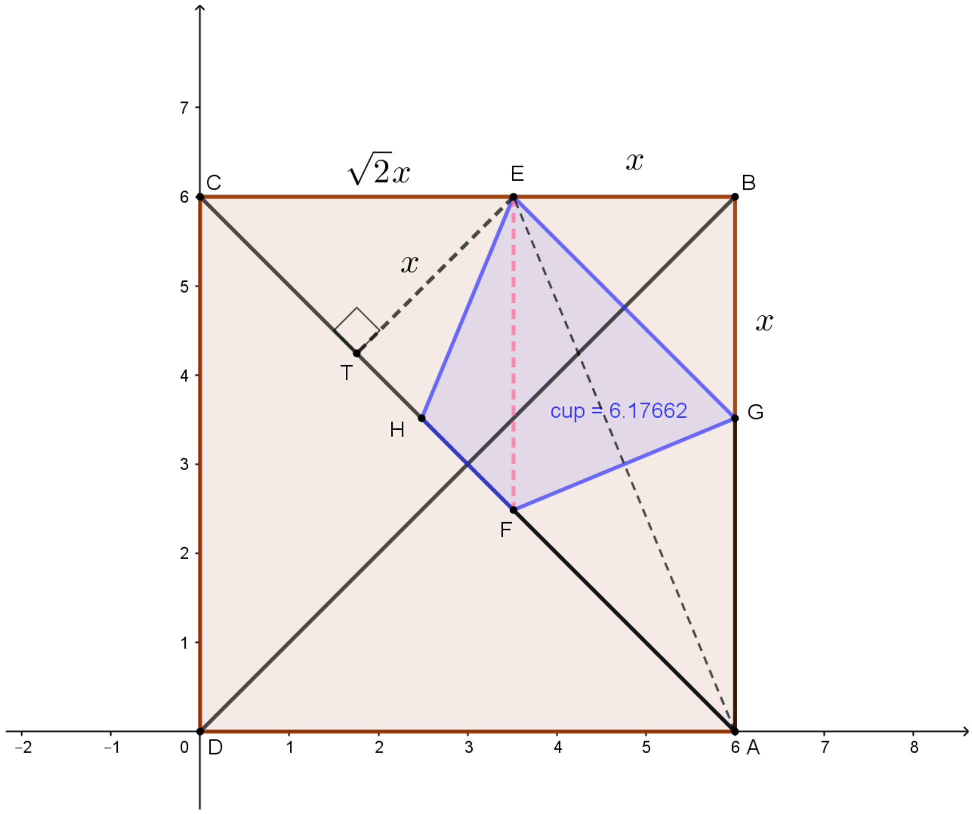
Therefore, point E is on the angle bisector of $\angle BAC$, which can be readily constructed in GeoGebra using the angle bisector tool or other geometric processes. By symmetry, point G is on the bisector of $\angle ACB$. It then follows that in Figure 2, $AGEF$ is a rhombus; so is $CHGE$.

These observations facilitate the construction a dynamic paper cup using GeoGebra: Starting from a square, we construct a diagonal AC, an angle bisector AE, and some parallel lines ($EG \parallel CA$, $EF \parallel BA$, and $GH \parallel BC$) for the paper cup $FGEH$ in Figure 2.

6.2 Surface Area of the Paper Cup

The surface area of the finished paper cup is another worthwhile inquiry. As shown in Figure 8, the paper cup is made up of two overlapping trapezoids, namely $FGEH$. Using GeoGebra, we can read the area of $FGEH$ from the algebra panel: approximately 6.17622 square units for a six-inch origami square. If we shrink the original square to three units, its area becomes 1.54416 square units, which is one-fourth of the previous one. Dynamic manipulations help highlight the two-dimensional nature of the area of $FGEH$.

Figure 8. Finding the surface area of the paper cup.



Indeed, we can delve deeper and find a general formula for the area of the trapezoid $FGEH$. Let's assume the origami square has a side length of s units. Further, we use x to represent the length of EB in Figure 8, and construct $ET \perp CA$. Since point E is on the bisector of $\angle BAC$, then $|ET| = |EB| = x$. Also, $\triangle CET$ is an isosceles right triangle, which means $|CE| = \sqrt{2}x$. Therefore, we have

$$\sqrt{2}x + x = s,$$

and

$$x = \frac{s}{\sqrt{2}+1}.$$

Further, since $CH \cong CE$ (because $CEGH$ is a rhombus), we can calculate the area of $\triangle CEH$:

$$\text{Area}_{\triangle CEH} = \frac{1}{2} \times \sqrt{2}x^2.$$

The trapezoid $FGEH$ is geometrically $\triangle ABC$ minus two copies of $\triangle CEH$ and $\triangle EBG$. It is worth noting that two copies of $\triangle CEH$ make the area of rhombus $AGEF$ or $CEGH$. Thus, we have

$$\text{Area}_{FGEH} = \frac{1}{2}s^2 - 2 \times \frac{1}{2} \times \sqrt{2}x^2 - \frac{1}{2}x^2,$$

which can be simplified to

$$\text{Area}_{FGEH} = (3 - 2\sqrt{2})s^2.$$

The formula above can be used to calculate the area of the trapezoid (one side of the paper cup) based on an arbitrary square with a side length of s. The surface area of the paper cup is accordingly twice as much.

7 CONCLUSION

Through numerous iterations, we found that the origami paper cup provides rich and affordable mathematical experiences for both teacher candidates and young children. It allows all students to see the unfolding nature of mathematical ideas, ranging from hands-on art, mathematical conversations, basic geometry, algebraic reasoning, measurement, and data analysis, with moderate struggles and memorable surprises. The survey data confirm our self-reflection on teacher candidates' learning experience. Our teacher candidates were very positive about the origami lesson sequence, and they wrote, "This activity was very engaging and fun!!! . . . I loved it! I'm going to use it in my classroom!:) . . . It lets everybody have a part in the lesson. . . . I thought the origami was simple. Still, I ended up being a great resource to teach lessons like estimation, probability, volume, area, mean, median, mode, minimum, maximum, outlier, average, etc. . . . It seemed difficult. Still, it ended up being very easy to make. The concept was interesting, and I enjoyed calculating the difference in the number of pinto beans each size held. . . . It makes the lecture part of the lesson come to life. . . . It showed students that you can always have many options. . . . It was fun to listen to my peers' problem solve out loud with one another. . . . Data always helps to have a deeper understanding."

As we seek to enrich the paper cup case in future iterations, we have come to see the surface area of the flattened paper cup (two overlapping trapezoids) as an algebraic treasure hidden in the paper cup. Furthermore, as our teacher candidates become fluent with modeling tools such as GeoGebra, it is worthwhile to demonstrate or even have them try to simulate the geometric transformations behind the process of making a paper cup (`https://www.geogebra.org/m/peez9med`), which will help make meaningful connections between informal and formal mathematical explorations and facilitate the communication of mathematical ideas. GeoGebra, with its simple and efficient modeling tools, promises to bridge various origami art projects to informal and formal mathematical inquiries.

REFERENCES

Brown, S. I., & Walter, M. I. (2005). *The art of problem posing* (3rd ed.). Mahwah, NJ: Lawrence Erlbaum Associates.

Cipoletti, B., & Wilson, N. (2004). Turning origami into the language of mathematics. *Mathematics Teaching in the Middle School*, *10*(1), 26-31.

De Bock, D., Van Dooren, W., Janssen, D., & Verschaffel, L. (2007). *The illusion of linearity: From analysis to improvement*. New York, NY: Springer.

Gardner, M. (2008). *Origami, Eleusis, and the Soma Cube*. New York, NY: Cambridge University Press.

Georgeson, J. (2011). Fold in origami and unfold math. *Mathematics Teaching in the Middle School*, *16*(6), 354-361.

Higginson, W., & Higginson, L. (2001). Algebraic thinking through origami. *Mathematics Teaching in the Middle School*, 6(6), 343-349.

Lang, R. J. (2011). *Origami design secrets: Mathematical methods for an ancient art* (2nd ed.). Boca Raton, FL: CRC Press.

National Aeronautics and Space Administration (n.d.). *Space Origami: Make your own starshade*. Retrieved October 1, 2022, from https://www.jpl.nasa.gov/edu/learn/project/space-origami-make-your-own-starshade/

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Robichaux, R. R., & Rodrigue, P. R. (2003). Using origami to promote geometric communication. *Mathematics Teaching in the Middle School*, 9(4), 222-229.

Soong, M. (1948). *The art of Chinese paper folding for young and old*. New York, NY: Harcourt, Brace and Company.

Wiles, P. (2013). Folding corners of the habits of mind. *Mathematics Teaching in the Middle School*, 19(4), 208-213.

Lingguo Bu, lgbu@siu.edu, is a Professor of Mathematics Education at Southern Illinois University Carbondale in the School of Education. He is interested in multimodal STEM modeling and simulations in the context of K–16 mathematics education and teacher development, including the integration of GeoGebra and 3D design and printing.

Harvey Henson, henson@siu.edu, is an Associate Professor of Science Education in the School of Education and of Geology in the School of Earth Systems & Sustainability at Southern Illinois University Carbondale. Henson is Interim Director of the STEM Education Research Center at SIUC and his research interests include STEM education and leadership, applied geophysics, and STEM assessment.