THE ARITHMETIC AVERAGE OF ALTITUDES LENGTHS FROM VERTICES OF PARALLELOGRAM TO A STRAIGHT-LINE: A SKETCH AND PROOF

Moshe Stupel[1] and David Fraivert[2]

[1]Shaanan Academic College of Education & Gordon Academic College of Education—Haifa, Israel
[2]Shaanan Academic College of Education—Haifa, Israel

Abstract

The authors illustrate a method for motivating mathematical generalization with students using GeoGebra. Building on student interest in amusement parks, the authors connect an initial question about Ferris wheel riders to general quadrilaterals in the plane. In a culminating activity, students construct a rigorous algebraic proof confirming their initial conjectures.

Keywords: Argumentation, generalization, proof, learning contexts

1 An Initial Conjecture

During a recent class, one of my students shared photos from a family vacation in Superland, an amusement park in Rishon LeZion, Israel. Since we were studying regular polygons and circles, the photo in Figure 1 was of particular relevance.

Figure 1. Ferris wheel at Superland in Rishon LeZion, Israel.

As we considered the photo together, I posed the following class challenge.

> Suppose that four students are spaced equally around the Ferris wheel. As the ride moves, the riders' distances above the ground change. Sketch a graph describing the riders' mean distance from the ground during the ride. Be ready to discuss your graph with others and justify your conclusions.

Initially, a number of my students were puzzled by the question. *Where are the numbers? You haven't given us enough information!* However, upon closer analysis, many recognized that the "distance of the riders seems to balance each other out." The following dialogue is typical.

TEACHER : What do you mean 'balance each other out'?
STUDENT 1 : I'm looking at the points around the circle. A, B, C, and D. (*Editor's Note: See Figure 2*)
TEACHER : Okay. I'm with you.
STUDENT 1 : The purple (A) and blue (C) balance each other out. A is above the center . . . the same distance as C is below.
STUDENT 2 : Same with the red and green ones (B and D). They balance each other out, too.
TEACHER : Balance each out . . . *where*?
STUDENT 2 : (*Tracing her finger along line* m) Here! Above and below this line.
TEACHER : Oh . . . yes! I see. I see!

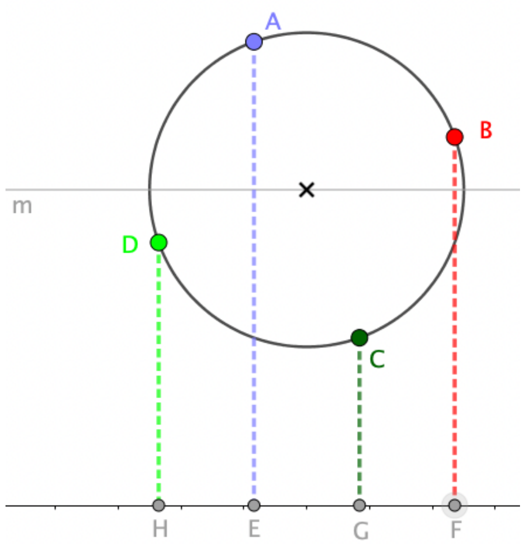


Figure 2. Diagram suggesting that mean distance of the four riders from the ground is constant (represented graphically by the distance m above the x-axis).

Some students may struggle to see m as a balance. Using a dynamic sketch of the scenario, they can drag the center of the Ferris wheel to the ground to better visualize the fixed mean distance. Such an approach is suggested in Figure 3.

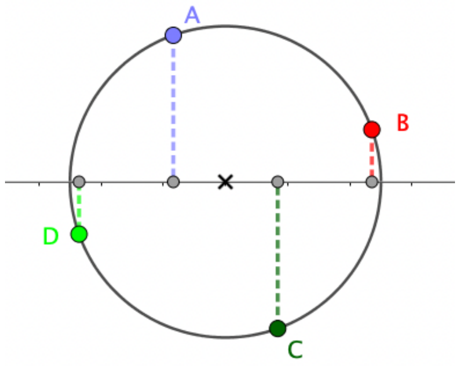


Figure 3. Diagram further suggesting that the mean distance of the four riders from the ground is constant.

After some conversation about mean as a balance, I like to move students' analysis in a more abstract direction. For instance, I ask them to consider the scenario geometrically by pointing out that $ABCD$ is a square, as illustrated in Figure 4.

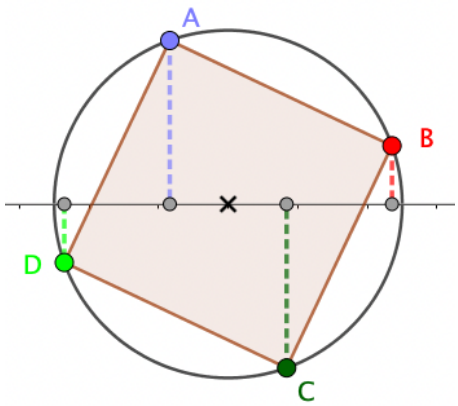


Figure 4. The riders on the Ferris wheel may be thought of as vertices of a square, $ABCD$.

Note that the mean y-coordinate of A, B, C, and D—in other words, the mean distance of the riders from the ground—equals the y-coordinate of the Ferris wheel's center. This observation leads us to our first conjecture.

Conjecture 1. *Given square $ABCD$ with center O, the mean of the y-coordinates of the vertices $\frac{A_y+B_y+C_y+D_y}{4}$ equals O_y, the y-coordinate of O.*

2 Questions Begat Questions

Students typically accept our first conjecture without too much pushback. However what would happen if the riders were not equally spaced around the circle? Moreover, what if there were not a circle at all? Consider the following.

> Let $ABCD$ represent a quadrilateral above the x-axis with center point O, the intersection of diagonals AC and BD. $ABCD$ has the interesting property that $\frac{A_y+B_y+C_y+D_y}{4} = O_y$. What kind of quadrilateral must $ABCD$ be? How do you know?

Students explored this follow-up prompt in small groups. Some drew pictures with pencil and paper, others wrote down formulas and equations, yet others constructed ideas in GeoGebra. Figure 5 illustrates some tentative conjectures about rectangles and trapezoids.

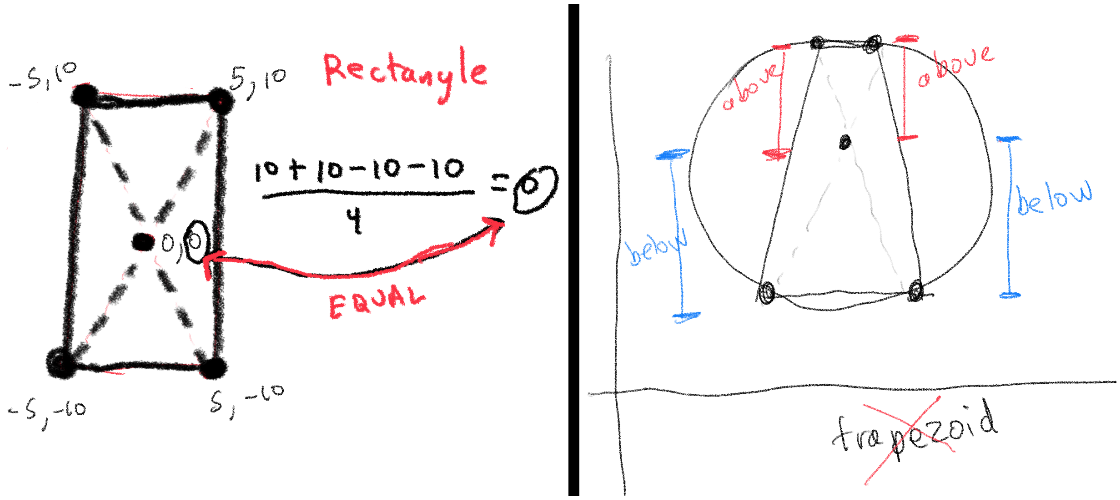


Figure 5. (Left) Drawing suggesting that rectangles satisfy the desired property; (Right) Work suggesting that some trapezoids don't share the property.

Figure 6 highlights a GeoGebra sketch that we used to search for additional quadrilaterals. As the figure suggests, students used inductive methods to arrive at a second conjecture.

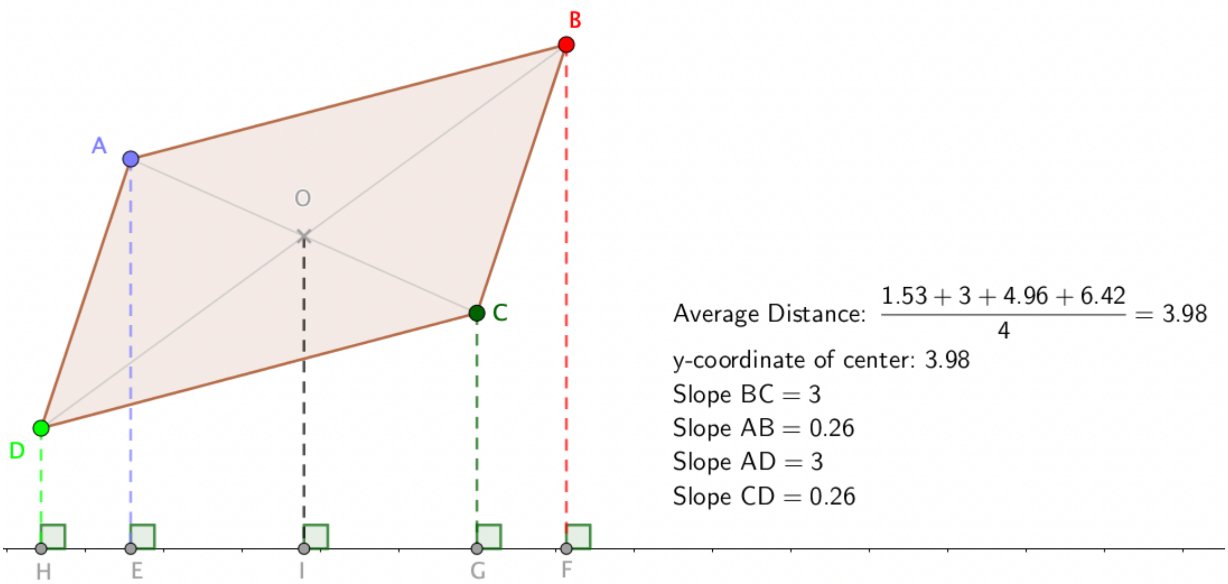


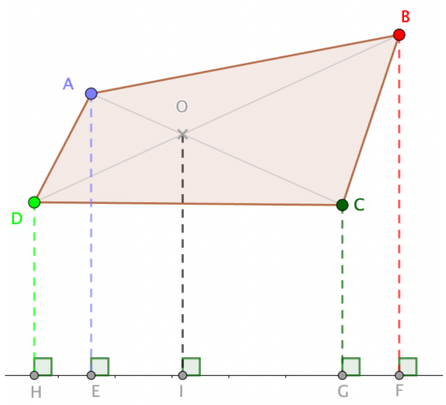
Figure 6. Dynamic sketch suggesting that parallelograms satisfy the desired property.

Conjecture 2. *Given parallelogram $ABCD$ with diagonals intersecting at center O. The mean of the y-coordinates of the vertices $\frac{A_y+B_y+C_y+D_y}{4}$ equals O_y, the y-coordinate of O.*

3 Proof of Findings

Since all squares are parallelograms, proof of conjecture 2 also addresses conjecture 1.

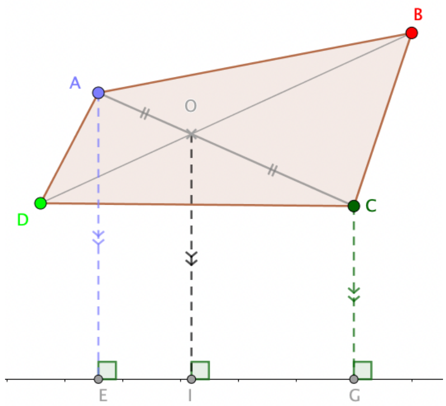
3.1 Givens



Given a parallelogram $ABCD$, whose diagonals meet at point O. l is a line passing outside the parallelogram. From points A, B, C, D, O, move down five heights to the straight line l, as shown in Figure 7.

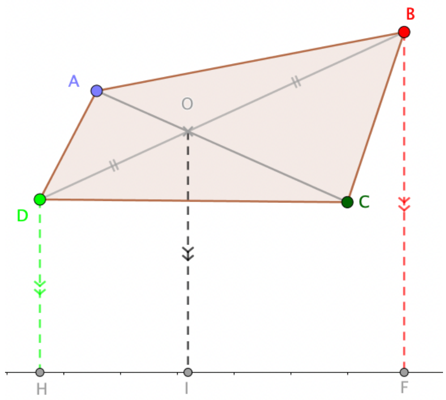
The following must be proved. The length of the height descending from the point of intersection of the diagonals O to line l is the arithmetic mean of the four heights descending from the vertices of the parallelogram to the straight line l. By mathematical formulation: $OI = \frac{AE+BF+CG+DH}{4}$.

3.2 Step 1



Consider quadrilateral $ACGE$.
$AE \parallel OI \parallel CG$ and $AO = OC \implies OI = \frac{AE+CG}{2}$.

3.3 Step 2



Consider quadrilateral $DBFH$.
$DH \parallel OI \parallel BF$ and $DO = OB \implies OI = \frac{DH+BF}{2}$.

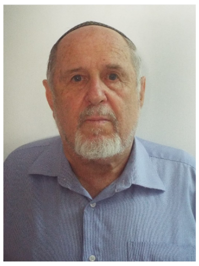
3.4 Step 3

By Steps 1 and 2, $OI = \frac{AE+CG}{2}$ and $OI = \frac{DH+BF}{2} \implies OI = \frac{AE+BF+CG+DH}{4}$. $\square$

4 Dynamic Research

We have created a series of GeoGebra applets that allow teachers and their students to explore various figures and results. The applets can be accessed via the following link: `https://www.geogebra.org/m/zgujqutp`. Note that the last sketch (i.e., parallelogram) allows one to drag the vertices of the parallelogram, thereby changing its lengths and angles. The sketch provides evidence that the conjectures hold.

When all vertices of the parallelogram fall below line l, distances are calculated as negative values, and conjectures 1 and 2 still hold. However, further consideration is needed when the line l passes through the parallelogram (i.e., some, but not all vertices, fall below line l). We leave this investigation to the reader.



Moshe Stupel, stupel@bezeqint.net, is a Professor of Mathematics Education at Gordon - Academic College of Education, and Shaanan - Academic College of Education, Haifa, Israel. Dr. Stupel published more than 200 articles in journals on technology and mathematics education, two books, and several mathematical enrichment materials. He also develops Geometric constructions and finds different solutions to the same problem by combining different mathematical tools.

David Fraivert, davidfraivert@gmail.com, is a Senior Lecturer and Researcher in the Department of Mathematics at Shaanan College in Haifa, Israel. David's research interests include the use of dynamic geometry software in the study of advanced topics.