EXTENDING, FORMULATING, AND REFINING CONJECTURES WITH GEOGEBRA: THE PENTAGON AND ITS MEDIAL PENTAGON

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Abstract

In this paper, the author describes how a group of preservice secondary mathematics teachers used GeoGebra to extend, formulate, and refine conjectures about medial polygons.

Keywords: Conjecturing; Discovery; GeoGebra; Geometry; Medial Pentagon; Ratios of Areas

1 INTRODUCTION

Dynamic Geometry Software (DGS) allows us to explore relationships among geometric objects as we manipulate objects and look for invariant properties. Using GeoGebra, I guided a group of preservice secondary mathematics teachers to extend, formulate, and refine a result involving areas and perimeters from triangles to quadrilaterals and then from quadrilaterals to pentagons. In this paper, I focus on exploring the relationship between the area of a pentagon and the area of its medial pentagon.

2 BACKGROUND

The students were enrolled in a college geometry class for preservice secondary mathematics teachers. In a previous lesson, the class had used GeoGebra to explore the properties of the medial triangle $\triangle DEF$ of a triangle $\triangle ABC$ (Figure 1). In particular, the class conjectured and proved the following three properties:

1. $\triangle DEF \sim \triangle ABC$ with ratio of similitude $\frac{1}{2}$.
2. $\triangle DEF$ is congruent to each of the other three triangles with area $\frac{1}{4}$ the area of $\triangle ABC$.
3. The perimeter of $\triangle DEF$ is one half the perimeter of $\triangle ABC$.

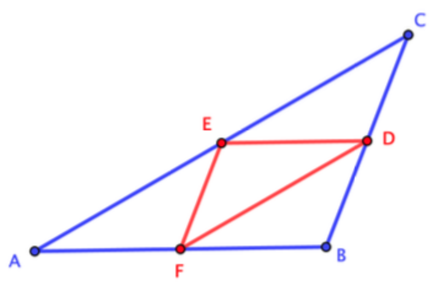

Figure 1. Triangle $\triangle DEF$ is the medial triangle of $\triangle ABC$.

We then explored the extent to which these properties extended to quadrilaterals (Figure 2). In particular, we conjectured and proved the following two properties:

1. $EFGH$ is a parallelogram.
2. The area of $EFGH$ is half the area of $ABCD$.

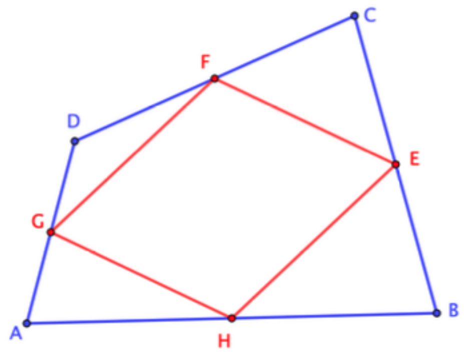

Figure 2. Quadrilateral $EFGH$ is the medial quadrilateral of $ABCD$.

We summarize the results of the medial triangle and the medial quadrilateral in Table 1.

Polygon	Medial Polygon	Ratio of areas	Ratio of perimeters
$\triangle ABC$	$\triangle DEF \sim \triangle ABC$ with scale factor = $\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{2}$
$ABCD$	$EFGH$ is a parallelogram	$\frac{1}{2}$	No apparent relationship

Table 1. Medial triangle and quadrilateral property summary.

3 The Medial Pentagon of a Pentagon

After an initial exploration of medial quadrilaterals, we shifted our attention to medial pentagons, such as the one illustrated in Figure 3.

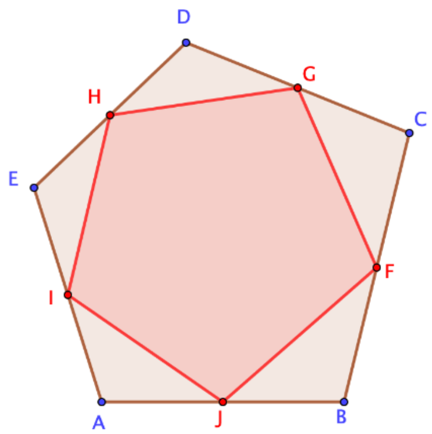

Figure 3. Pentagon $FGHIJ$ is the medial pentagon of pentagon $ABCDE$.

Specifically, I posed the following task to students.

> Let F, G, H, I and J be the midpoints of the consecutive sides of a pentagon $ABCDE$. What special properties does pentagon $FGHIJ$ have? Justify your responses.

Initial Student Findings

Based on the results for the ratios $\frac{\text{Area(medial polygon)}}{\text{Area(polygon)}}$ for triangles and quadrilaterals, some students predicted that $\frac{\text{Area(FGHIJ)}}{\text{Area(ABCDE)}} = \frac{3}{4}$, but no conjecture was formulated for the ratio $\frac{\text{Perimeter(FGHIJ)}}{\text{Perimeter(ABCDE)}}$. Students used GeoGebra as an aid to test the initial conjecture about the ratio of the areas of the two pentagons and to try to discover a potential ratio between the perimeters of the two pentagons. To this end, they constructed configurations similar to the ones depicted in Figure 4 and then computed the perimeters, areas, and their corresponding ratios.

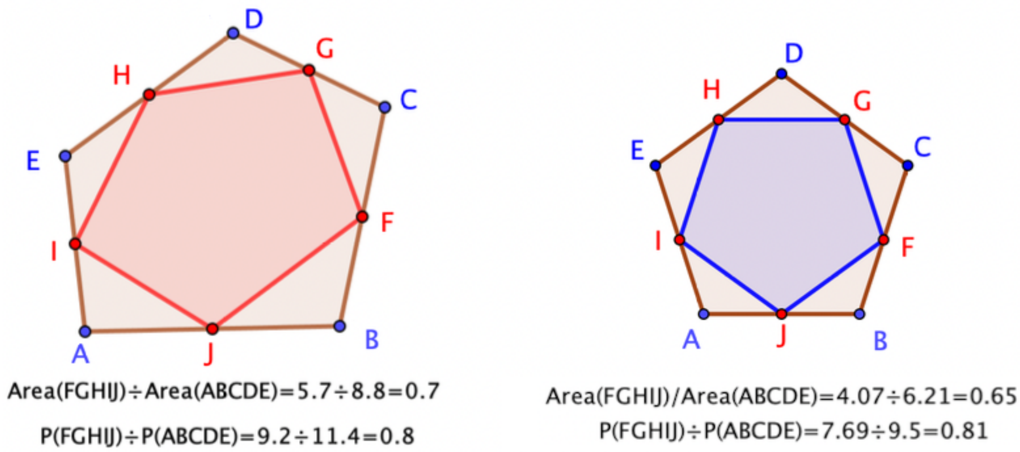

Figure 4. Medial pentagons with measurement ratios.

Notice that the pentagon $ABCDE$ displayed in Figure 4 (left) is a general pentagon while the pentagon illustrated in Figure 4 (right) is a regular pentagon. When one of the students who constructed the general pentagon and its medial quadrilateral shared their findings about the ratios being 0.7 and 0.8 with the class, the student who constructed the regular pentagon and its medial pentagon said that the ratios were, more precisely, 0.65 and 0.81. The first student used their calculator to compute $5.7 \div 8.8 (\approx 0.6477 \approx 0.65)$ and $9.2 \div 11.4 (\approx 0.8070 \approx 0.81)$ and so the first student confirmed the second student's conjecture.

Dragging

A central feature of DGS is the ability to move points within a geometric configuration (i.e., dragging). Dragging allows us to quickly generate several examples of a geometric configuration and examine its effects on geometric relationships. In this case, we can drag some of the vertices of the general pentagon $ABCDE$ and vertices A and B of the regular pentagon. Figure 5 displays some images of a general pentagon, its medial pentagon, and the corresponding ratios between their areas and their perimeters.

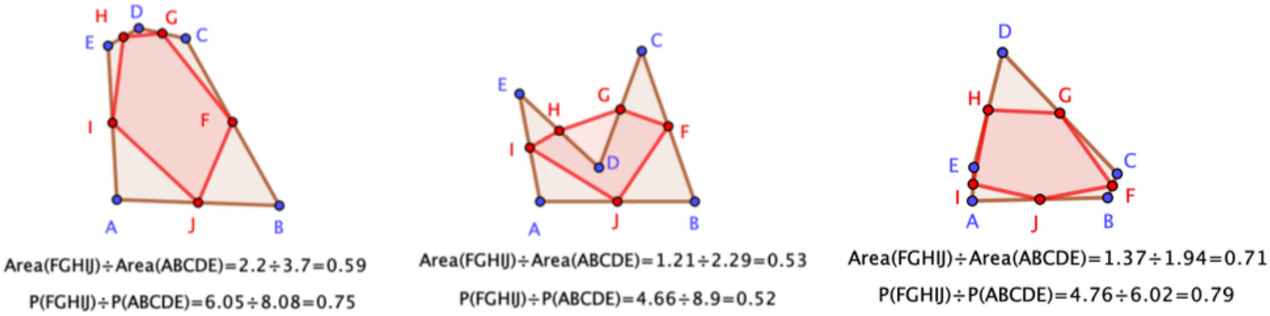

Figure 5. Three medial pentagons generated from a general pentagon in GeoGebra.

As some students observed their images and shared their findings, they concluded that the initial conjectures were not only false, but that there were not apparent relationships between the areas and perimeters of the medial and outer pentagons. Thus, the dragging approach did not support the initial conjectures about relationships between the areas and perimeters of a pentagon and its medial pentagon being about 0.65 and 0.81, respectively.

Increasing the precision

The two students who initially conjectured that the ratio between the areas of the two pentagons was a constant approximately equal to 0.65 and that the ratio between the perimeters of the two pentagons was a constant approximately equal to 0.81, increased the precision of the measurements and computations (Figure 6). As they compared their images, measurements, and computations, they noticed that the ratios differed significantly and so the corresponding ratios were not a constant valid for all polygons.

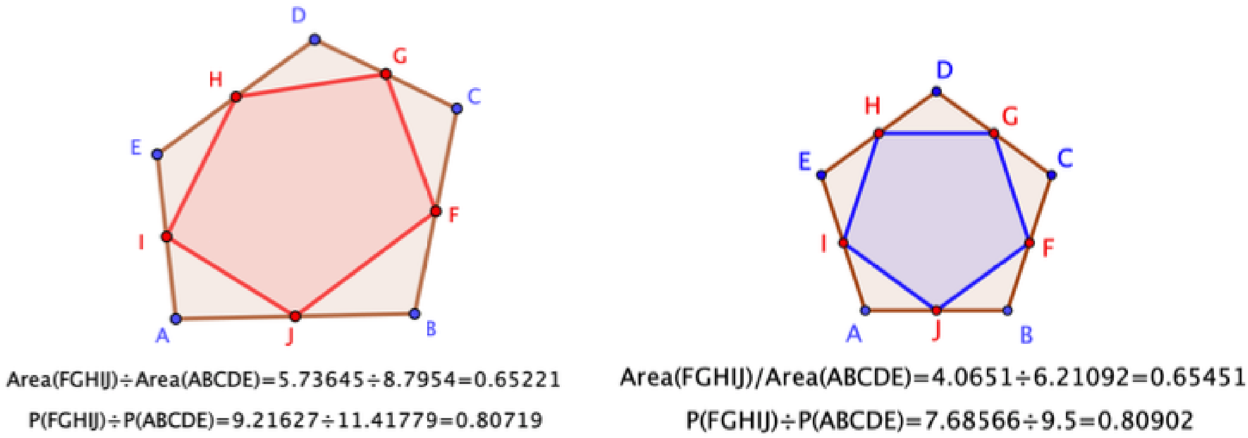

Figure 6. Medial pentagon ratios displayed with 5 decimal places of accuracy.

Other students dragged vertices of the pentagon $ABCDE$ and increased the precision of the measurements and computations (Figure 7). Once again, the images, measurements, and computations did not support the initial conjectures proposed by the two students.

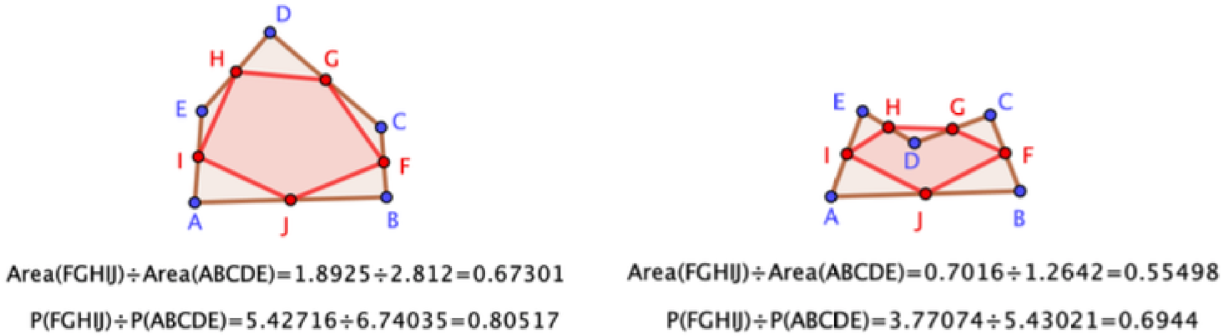

Figure 7. Medial pentagon ratios displayed with 5 decimal places of accuracy.

4 Investigating a Special Case

Since one student had started with a regular polygon, I decided that the class investigate this particular case. Students used the Regular Polygon tool to construct a regular polygon. Further dragging supported the following two conjectures involving a regular pentagon and its medial pentagon.

Conjecture 1: The ratio of the area of a medial pentagon and the area of the outer pentagon is a constant whose approximate value rounded to two decimal places is 0.65.

Conjecture 2: The ratio of the perimeter of a medial pentagon and the perimeter of the outer pentagon is a constant whose approximate value rounded to two decimal places is 0.81.

Proof (Part 1): The Medial Pentagon is also Regular

To justify the two conjectures mathematically, we provided a proof along the following lines: Let $ABCDE$ be a regular pentagon and let $FGHIJ$ be its medial pentagon (Figure 8). We will show first that pentagon $FGHIJ$ is also a regular pentagon.

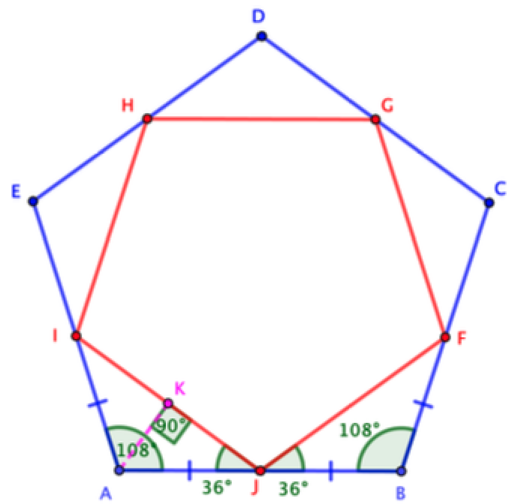

Figure 8. Regular pentagon $ABCDE$ and medial pentagon $FGHIJ$.

1. Since $ABCDE$ is a regular pentagon, we have $m(\angle A) = m(\angle B) = \frac{3\cdot 180°}{5} = 108°$.

2. Now, given that points I and J are midpoints of congruent sides, we know that $AI = AJ = BJ = BF$. Thus, $\triangle AIJ \cong \triangle BJF$ by the SAS congruence criterion and so $\overline{IJ} \cong \overline{JF}$. Similarly, we can show that $\overline{JF} \cong \overline{FG} \cong \overline{GH} \cong \overline{HI}$. Thus, pentagon $FGHIJ$ is equilateral.

3. Because $\triangle AIJ$ is an isosceles triangle, we know that $m(\angle AJI) = m(\angle AIJ)$ by the isosceles triangle theorem.

4. Thus, $m(\angle AJI) = \frac{180°-108°}{2} = 36°$. Similarly, $m(\angle BJF) = 36°$. Now, $m(IJF) = 180° - 36° - 36° = 108°$.

5. Similarly, $m(\angle JFG) = m(\angle(FGH) = m(\angle GHI) = m(\angle HIJ) = 108°$.

6. Thus, pentagon $FGHIJ$ is equiangular. Since pentagon $FGHIJ$ is equilateral and equiangular, we know that pentagon $FGHIJ$ is regular.

Proof (Part 2): Determining Ratios

Next, we will determine the ratios of the perimeters of the two pentagons and the ratio of their areas.

7. Let $AB = s$ and $IJ = t$.

8. Thus, perimeter $(ABCDE) = 5s$ and perimeter $(FGHIJ) = 5t$.

9. Construct segment $\overline{AK}$ perpendicular to segment $\overline{IJ}$. Notice that $JK = \frac{IJ}{2} = \frac{t}{2}$. Then, $\cos 36° = \frac{JK}{AJ} = \frac{t/2}{s/2} = \frac{t}{s}$ and so $t = s\cos 36°$.

10. We now have,

$$\frac{\text{Perimeter}(FGHIJ)}{\text{Perimeter}(ABCDE)} = \frac{5t}{5s} = \frac{t}{s} = \frac{s\cos 36°}{s} = \cos 36° \approx 0.8090.$$

11. Since the two pentagons are similar, we have $\frac{\text{Area}(FGHIJ)}{\text{Area}(ABCDE)} = (\cos 36°)^2 \approx 0.6545.$ □

5 Exploring the General Case

After proving that $\frac{\text{Area(medial pentagon)}}{\text{Area(outer regular pentagon)}} = (\cos 36°)^2 \approx 0.6545$ and $\frac{\text{Perimeter(medial pentagon)}}{\text{Perimeter(outer regular pentagon)}} = \cos 36° \approx 0.8090$, the class explored the general case.

I asked the class who had the lowest and highest ratios between the areas of the medial pentagon and the outer pentagon. I then projected preconstructed GeoGebra images similar to those displayed in Figure 5 (with slightly lower values than 0.59 and 0.71) on the projector. I then asked the class to find a lower ratio than 0.59. After some students found a value lower than 0.59, I invited one student to share findings with class (see Figure 9). I asked the class to examine the pentagon and formulate a conjecture about why the ratio area $\frac{\text{Area(FGHIJ)}}{\text{Area(ABCDE)}}$ was lower for this pentagon.

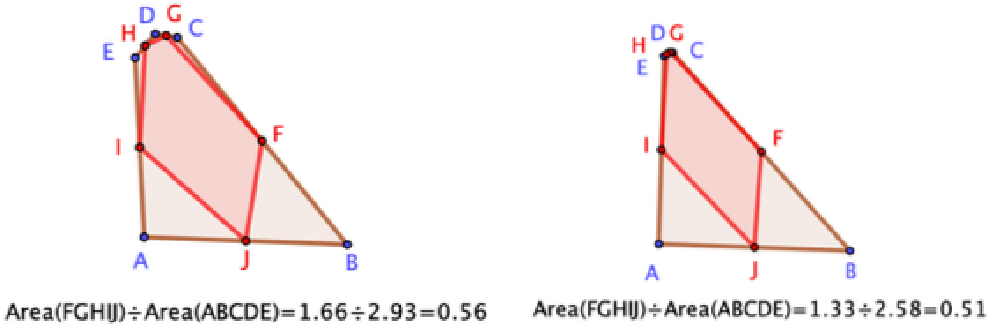

Figure 9. The ratio $\frac{\text{Area(FGHIJ)}}{\text{Area(ABCDE)}}$ approaches to 0.5 as pentagon $ABCDE$ approaches $\triangle ABD$.

Exploring the Greatest Lower Bound of Area Ratios

Some students stated that pentagon $ABCDE$ was "almost" a triangle. I then asked the class how we can make the ratio between the two areas even lower while maintaining point D fixed. After some dragging, several students noticed that the ratio $\frac{\text{area(FGHIJ)}}{\text{area(ABCDE)}}$ approaches 0.50 when points C and E approach point D, as suggested in Figure 9. Moreover, my students noticed that in the "limit" pentagon $ABCDE$ degenerates into $\triangle ABD$ while the medial pentagon $FGHIJ$ degenerates into quadrilateral $FDIJ$ whose area is half the area of $\triangle ABD$. This result is suggested in Figure 10.

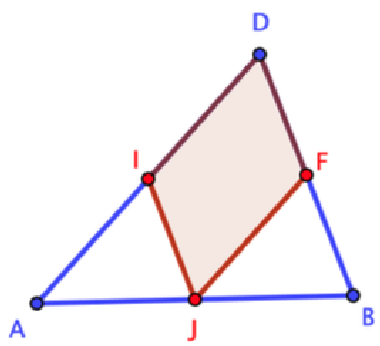

Figure 10. The area of degenerate pentagon $FGHIJ$ (i.e., $FDIJ$) is 1/2 the area of degenerate pentagon $ABCDE$ (i.e., $\triangle ABD$).

Since $\triangle ABD$ (Figure 10) is no longer a pentagon, the class conjectured that $\frac{\text{area(FGHIJ)}}{\text{area(ABCDE)}}$ does not reach its (greatest) lower bound and so $\frac{\text{area(FGHIJ)}}{\text{area(ABCDE)}} > 1/2$.

Exploring the Least Upper Bound of Area Ratios

After the class conjectured that the ratio $\frac{\text{area(FGHIJ)}}{\text{area(ABCDE)}}$ is greater than 0.50, we explored under what conditions the value of the ratio $\frac{\text{area(FGHIJ)}}{\text{area(ABCDE)}}$ approaches its (least) upper bound, if any. Using a similar process, some students noticed that by letting point E approach point A and point C approach point B, the ratio of the two areas approaches 0.75, as suggested in Figure 11.

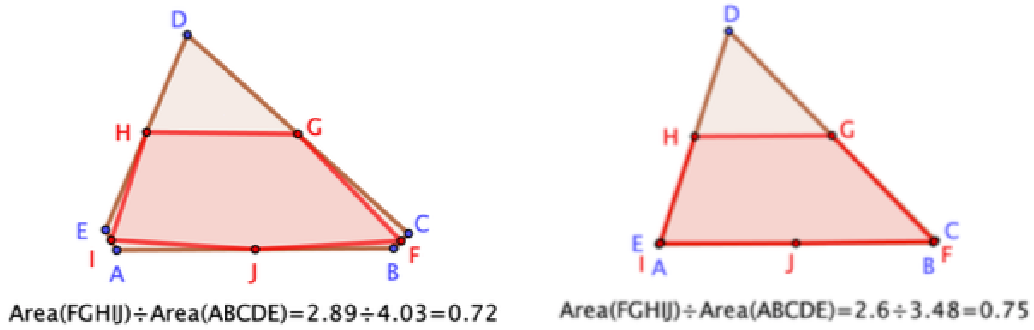

Figure 11. By transforming pentagon $FGHIJ$ into a degenerate pentagon, we see that the area ratio will not reach its least upper bound.

They observed that in the "limit" pentagon $ABCDE$ degenerates into $\triangle ABD$ while pentagon $FGHIJ$ degenerates into quadrilateral $ABGH$ whose area is $3/4$ the area of $\triangle ABD$, as shown in Figure 12.

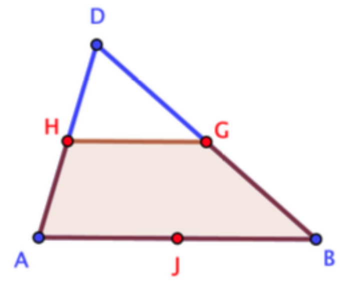

Figure 12. The area of degenerate pentagon $FGHIJ$ (i.e., $BGHA$) is $3/4$ the area of degenerate pentagon $ABCDE$ (i.e., $\triangle ABD$)

Since $\triangle ABD$ (Figure 12) is no longer a pentagon, the class conjectured that *area*$(FGHIJ)/$*area*$(ABCDE)$ does not reach its (least) upper bound and so *area*$(FGHIJ)/$*area*$(ABCDE) < 3/4$.

Limiting Values and Extreme Cases

By using GeoGebra and considering extreme cases, the class formulated the following conjecture.

> The ratio of the area of the medial pentagon to the area of the original pentagon is between $1/2$ and $3/4$.

Values that are extremely low or extremely high are called extreme values and the geometric configurations that generate the extreme values are commonly called *extreme cases* (Polya, 1962). They are called extreme cases because they produce extreme values and their construction involves a limiting process.

In this particular exploration, a triangle and a quadrilateral are extreme, or degenerate, cases of a pentagon because they are constructed or generated by a limiting process: In one limiting process, pentagon $ABCDE$ became $\triangle ABD$ while the medial pentagon $FGHIJ$ became $FDIJ$ as points C and E approached point D (Figures 9 and 10). In this limiting process, two vertices of the pentagon $ABCDE$, vertices C and E, approached a single vertex of said pentagon, vertex D.

In the second limiting process, pentagon $ABCDE$ became triangle ABD while the medial pentagon $FGHIJ$ became $ABGH$ as points C and E approached points B and A, respectively (Figures 11 and 12). In this limiting process, two vertices of the pentagon $ABCDE$, vertices C and E, approached two vertices of said pentagon, vertices B and A, respectively. The class did not notice any relationship between the perimeter of the medial pentagon and the perimeter of the outer pentagon and so we did not pursue this line of exploration. Perhaps the reader can offer some suggestions or insights into whether there is a relationship between the perimeter of the medial pentagon and the perimeter of the outer pentagon.

6 Conclusions

Teaching and learning mathematics is enhanced with the proper use of technology as it empowers students to learn mathematics actively (National Council of Teachers of Mathematics, 1989, 1991, 2000). Exploring the medial pentagon of a pentagon with GeoGebra, or other types of Dynamic Geometry Software, as well as considering extreme cases, provides students with experiences in formulating and refining conjectures and visualizing geometric relationships that are difficult to achieve with pencil and paper alone. Exploring the "medial pentagon of a pentagon" task with GeoGebra, coupled with extreme case reasoning (Polya, 1962), provides also experiences for learners to engage in the process of discovering new geometric relationships, even if the relationships are only new to them. As a result, they may "experience the tension and enjoy the triumph of discovery" (Polya, 2004).

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