VOLUME INVARIANT CUBE TWISTING: GEOGEBRA MODELING AND ALGEBRAIC EXPLORATIONS

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Abstract

If we take a cube and twist it 90 degrees, mentally or physically, without changing its volume, we obtain a twisted solid that is not only visually appealing but also algebraically entertaining. Starting with GeoGebra simulations, we discuss the algebraic nature of the twisted cube faces, the spiral curves, and further extend GeoGebra-based explorations beyond the cube.

Keywords: Cube, Twisting, Algebraic Analysis, Vector Functions, Ruled Surfaces

1 INTRODUCTION

The cube, common as it is in everyday life, provides extraordinary opportunities for students to experience mathematics in multiple domains (e.g., Senechal, 1990). It has served as the foundation for numerous mathematical toys and puzzles. It is also the basic reference framework of navigation in 3D learning and design environments. There are numerous cube-based learning activities that are both aesthetically engaging and mathematically rich for K-16 students (Bu, 2017, 2019, 2021). There is, however, little discussion in the literature about cube twisting – a playful three-dimensional transformation that opens the door to a variety of mathematical ideas, both geometrically and algebraically. It is particularly interesting to twist a cube with the assistance of GeoGebra 3D Graphics or similar modeling technologies. There are two major types of twisting we can perform on a cube. In the first case, a cube is twisted typically 90 degrees while allowing its faces to stretch as if they were made of elastic faces, which can be called an elastic or flexible twist. In the second case, a cube is twisted for a certain angle while retaining its volume.

In a thought experiment, we could cut the cube horizontally into a large number of thin square slices, which are then gradually rotated for a certain angle, such as 90 degrees, from the bottom to the top. It is evident that the twisted cube has the same volume as the original. Figure 1 shows two 3D-printed physical models with a clockwise and a counterclockwise twist of 90 degrees, respectively. The second type of twist can be called a volume-invariant twist and is the focus of the present article. We first look into the process of GeoGebra Simulation and then, using GeoGebra models as a scaffold, we explore and verify the algebraic structures around the twisted cube, including the twisted faces, the spiral curves, and the area of a twisted cube face. The ideas we unveil on the journey range from elementary to postsecondary mathematics. Readers should feel free to make stops wherever appropriate or enjoy the full picture of a twisted cube and extend the methods of inquiry to other problem situations.

Figure 1. Two 3D designed and printed models for the twisted cube with the same volume (created with Autodesk Fusion 360®).

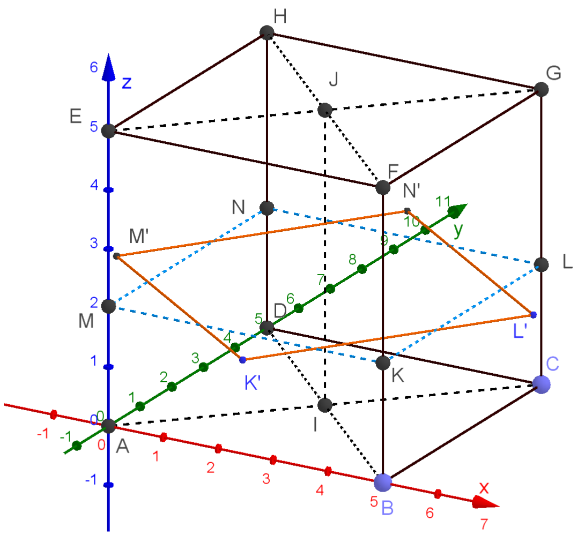

Figure 2. Setup for twisting a cube while retaining the volume (the plane containing $L'K'M'N'$ is not shown; created with GeoGebra®).

2 GeoGebra Simulation

GeoGebra 3D Graphics comes with the tools we need to twist a cube while retaining its volume. As a thought experiment, we could imagine the bottom square of the cube moving up steadily while rotating around an axis connecting the centers of its top and bottom faces, fulfilling a given amount of rotation (e.g., $\frac{\pi}{2}$ clockwise) at the end of the journey. Since its vertical movement is coordinated with its rotation, we need only one control variable – either its vertical position or the amount of rotation. Let's define a slider $t \in [0, \frac{\pi}{2}]$ to control the amount rotation in the case of twisting a cube clockwise for $\frac{\pi}{2}$. For a cube of edge length 5 units, the corresponding vertical location of the intermediate square after a rotation of t is therefore $\frac{10t}{\pi}$. Alternatively, we could use the vertical position of the square as a control parameter and calculate the corresponding angle of rotation.

Further, we define a plane $z = \frac{10t}{\pi}$, which intersects the cube and yields a square to be rotated ($LKMN$ in Figure 2) for an angle of t. The resulting square $L'K'M'N'$ is what we expect at the

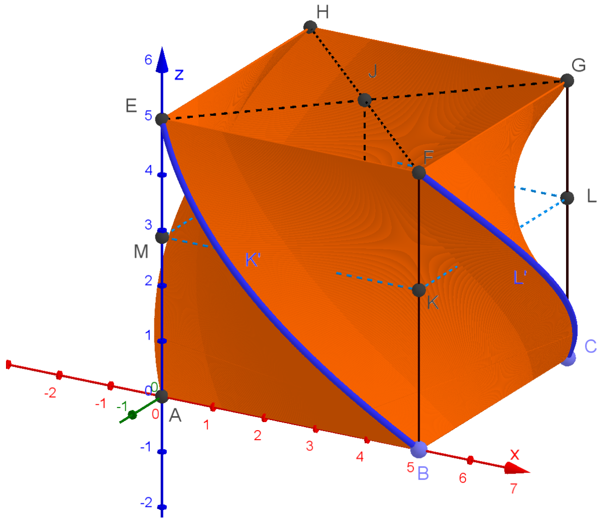

Figure 3. A twisted cube with the same volume (created with GeoGebra®).

vertical position. The `Trace` feature of GeoGebra will produce the twisted cube, as the slider t is dragged across its interval (Figure 3). Appendix A provides a summary of the steps to be taken in GeoGebra. Of course, there are always other ways to accomplish the intended simulation.

3 Parametric Equations for the Twisted Face

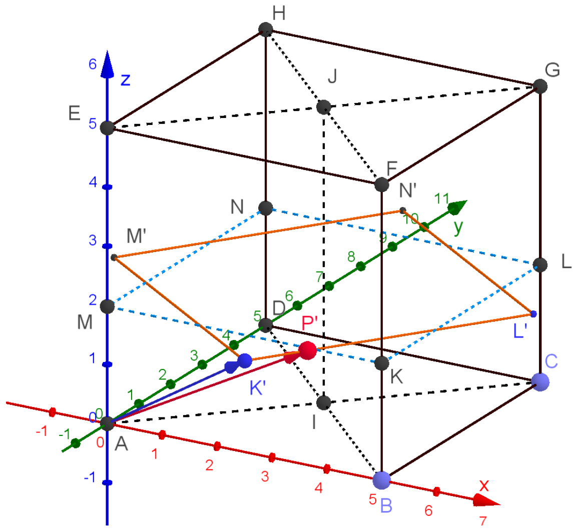

Figure 4. Toward a parametric surface for the twisted cube face (created with GeoGebra®).

The twisted surfaces in Figure 3 are algebraically inviting as well as visually appealing. To seek a vector function, and thus the corresponding parametric equations, for a twisted cube face, we recognize that a twisted cube face is swept or ruled by a spiraling cube edge. As an example, let's start with a cube with an edge length of 5 units, which is positioned in the first octant and is twisted 90 degrees clockwise (Figure 4). We further consider the twisted surface containing BC and EF, where $K'L'$

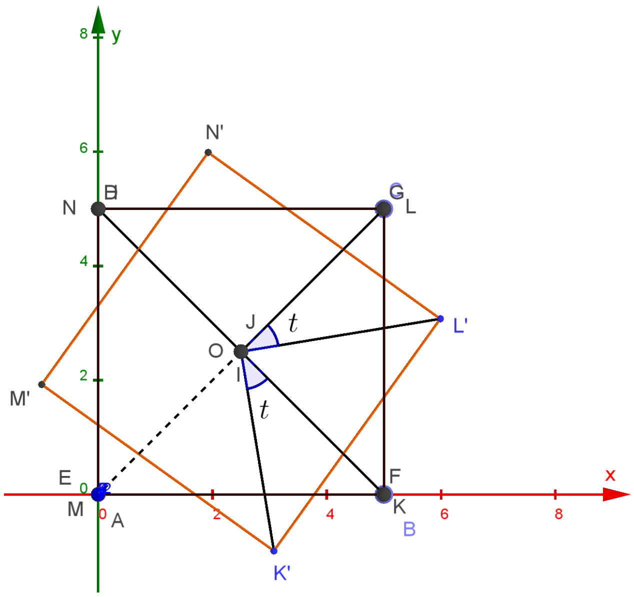

Figure 5. Top view of an intermediate twisted square, where $KL = 5$ units (created with GeoGebra®).

is an intermediate instance of BC as it spirals up toward EF. $K'L'$ has rotated for a certain angle t with respect to its reference edge KL in the plane $z = z_0 = \frac{10t}{\pi}$. We can thus use the amount of rotation t and the z-coordinate of K' and L' as intermediate parameters and make adjustments later on.

First, let's find the coordinates of K' and L', using the top view of a twisting cube. In Figure 5, $L'K'M'N'$ has rotated clockwise for an angle t with respect to its initial position at $LKMN$. Thus,

$$\begin{aligned} K' &= \left(\frac{5}{2} + \frac{5\sqrt{2}}{2}\sin(\frac{\pi}{4} - t), \frac{5}{2} - \frac{5\sqrt{2}}{2}\cos(\frac{\pi}{4} - t), z_0\right) \\ &= \left(\frac{5}{2} + \frac{5}{2}(\cos(t) - \sin(t)), \frac{5}{2} - \frac{5}{2}(\cos(t) + \sin(t)), z_0\right) \end{aligned} \tag{1}$$

and

$$\begin{aligned} L' &= \left(\frac{5}{2} + \frac{5\sqrt{2}}{2}\cos(\frac{\pi}{4} - t), \frac{5}{2} + \frac{5\sqrt{2}}{2}\sin(\frac{\pi}{4} - t), z_0\right) \\ &= \left(\frac{5}{2} + \frac{5}{2}(\cos(t) + \sin(t)), \frac{5}{2} + \frac{5}{2}(\cos(t) - \sin(t)), z_0\right) \end{aligned} \tag{2}$$

where z_0 is the height of the plane $z = z_0$ that contains $LKMN$ and $L'K'M'N'$.

Therefore,

$$\overrightarrow{K'L'} = \langle 5\sin(t), 5\cos(t), 0\rangle\,. \tag{3}$$

Let P' be a point on $K'L'$ (Figure 4). Then, $\overrightarrow{K'P'} = s(\overrightarrow{K'L'})$ for $0 \le s \le 1$. Further, since $\overrightarrow{AK'} = \left\langle \frac{5}{2} + \frac{5}{2}(\cos(t) - \sin(t)), \frac{5}{2} - \frac{5}{2}(\cos(t) + \sin(t)), z_0 \right\rangle$ and $z_0 = \frac{10t}{\pi}$, the vector function for the twisted face under consideration is:

$$
\begin{aligned}
\vec{r}(s,t) &= \overrightarrow{A'P'} \\
&= \overrightarrow{AK'} + s(\overrightarrow{K'L'}) \\
&= \left\langle \frac{5}{2} + \frac{5}{2}(\cos(t) - \sin(t)), \frac{5}{2} - \frac{5}{2}(\cos(t) + \sin(t)), \frac{10t}{\pi} \right\rangle + s\,\langle 5\sin(t), 5\cos(t), 0\rangle \\
&= \left\langle \frac{5}{2} + \frac{5}{2}(\cos(t) - \sin(t)) + 5s\sin(t), \frac{5}{2} - \frac{5}{2}(\cos(t) + \sin(t)) + 5s\cos(5), \frac{10t}{\pi} \right\rangle, \qquad (4)
\end{aligned}
$$

where $0 \le t \le \frac{\pi}{2}$ is the amount of rotation and $0 \le s \le 1$ determines the position of P' along $\overrightarrow{K'L'}$. The vector function in (4) can be readily graphed in GeoGebra using the `Surface` command and the three corresponding parametric equations. The result is shown in Figure 6.

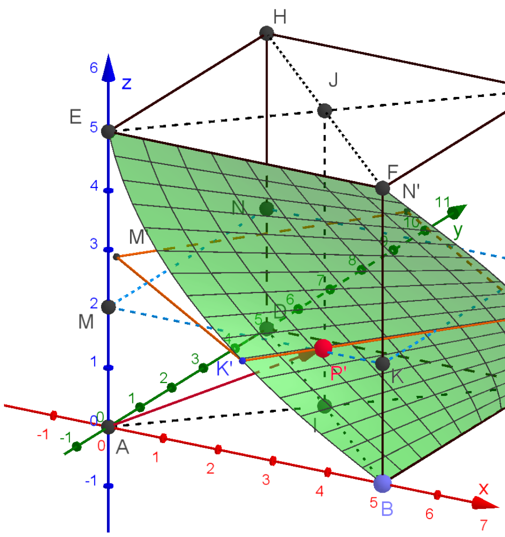

Figure 6. Graph of a twisted cube face from BC to EF after a 90-degree clockwise twist (created with GeoGebra®).

4 Parametric Equation for the Spiral Curve

As the cube is twisted, the four vertical edges around the axis of rotation become spiral curves, as shown in Figures 1 and 3. In a dynamical sense, each curve is the pathway of a square vertex as it spirals up. As an example, let's consider the vertex $B = (5,0,0)$ on the x-axis in Figure 3, which becomes K' as it moves up. After a clockwise rotation of 90 degrees, K' reaches vertex $E = (0,0,5)$ on the z-axis. Using $0 \le t \le \frac{\pi}{2}$ as a parameter, we can calculate $z(K') = \frac{10t}{\pi}$. Therefore, using the information in (1), we can establish the vector function for the spiral curve from B to E:

$$
\vec{r}(t) = \left\langle \frac{5}{2} + \frac{5}{2}(\cos(t) - \sin(t)), \frac{5}{2} - \frac{5}{2}(\cos(t) + \sin(t)), \frac{10t}{\pi} \right\rangle, \quad 0 \le t \le \frac{\pi}{2}, \qquad (5)
$$

which can be graphed using the GeoGebra `Curve` command in its 3D Graphics. An example can be found in Figure 3.

The length of one spiral curve across the cube is therefore:

$$\begin{aligned}\text{CL} &= \int_0^{\frac{\pi}{2}} \left|\vec{r'}(t)\right| \mathrm{d}t \qquad (6)\\ &= \int_0^{\frac{\pi}{2}} \left|\left\langle -\frac{5}{2}(\cos(t)+\sin(t)), \frac{5}{2}(\cos(t)-\sin(t)), \frac{10}{\pi}\right\rangle\right| \mathrm{d}t\\ &= \int_0^{\frac{\pi}{2}} \sqrt{\frac{25}{4}(\cos(t)-\sin(t))^2+\frac{25}{4}(\cos(t)+\sin(t))^2+\frac{100}{\pi^2}}\ \mathrm{d}t\\ &\approx 7.473 \text{ units.} \qquad (7)\end{aligned}$$

5 Surface Area of a Twisted Face

While the cube retains its volume after the twist described above, the surface area of each cube face may have changed during the process of twisting. To find the surface area of a twisted face, we take the partial derivative of $\vec{r}(s,t)$ in (4) with respect to s and t, respectively:

$$\vec{r}_s(s,t) = 5\sin(t)\vec{i} + 5\cos(t)\vec{j} + 0\vec{k}, \qquad (8)$$

$$\begin{aligned}\vec{r}_t(s,t) = &\left(-\frac{5}{2}\cos(t)-\frac{5}{2}\sin(t)+5s\cos(t)\right)\vec{i}\\ &+\left(-\frac{5}{2}\cos(t)+\frac{5}{2}\sin(t)-5s\sin(t)\right)\vec{j}\\ &+\frac{10}{\pi}\vec{k}. \qquad (9)\end{aligned}$$

The magnitude of the cross product $\vec{r}_s \times \vec{r}_t$ is thus

$$\begin{aligned}|\vec{r}_s \times \vec{r}_t| &= \left\| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ 5\sin(t) & 5\cos(t) & 0 \\ -\frac{5}{2}\cos(t)-\frac{5}{2}\sin(t)+5s\cos(t) & -\frac{5}{2}\cos(t)+\frac{5}{2}\sin(t)-5s\sin(t) & \frac{10}{\pi}\end{matrix}\right\|\\ &= \frac{25}{2\pi}\sqrt{\pi^2(1-2s)^2+16}. \qquad (10)\end{aligned}$$

Therefore, the surface area of one twisted cube face after a 90-degree twist is

$$\begin{aligned}\text{SA} &= \iint_D |\vec{r}_s \times \vec{r}_t|\ \mathrm{d}A\\ &= \int_0^{\frac{\pi}{2}}\int_0^1 \frac{25}{2\pi}\sqrt{\pi^2(1-2s)^2+16}\ \mathrm{d}s\mathrm{d}t \qquad (11)\\ &= \int_0^{\frac{\pi}{2}} \mathcal{K}\ \mathrm{d}t\\ &= \frac{\mathcal{K}\pi}{2}\\ &\approx 27.373 \text{ square units,}\end{aligned}$$

where
$$\mathcal{K} = \frac{25\left(\pi\sqrt{16+\pi^2} + 16\sinh^{-1}(\frac{\pi}{4})\right)}{4\pi^2} \approx 17.426.$$
Interestingly, $\mathcal{K}$ is a constant, approximately 17.426 for a cube with an edge length of 5 units and after a twist of 90 degrees. It is worth noting that $\mathcal{K}$ also depends on the amount of rotation, $\frac{\pi}{2}$ in the present example. As expected, a twisted cube face gets larger in terms of its surface area. With a 90-degree twist on a 5-cube, each lateral cube face increases from 25 to about 27.373 square units. The analysis above applies to cubes and rectangular boxes of different sizes and various amounts of twisting.

6 Playful Extensions in GeoGebra

Once we have established the parametric equations or vector functions for the twisted cube faces, we can look into the big picture of the surfaces and spiral curves by posing a host of *what-if* or *what-if-not* questions (Brown and Walter, 2005). In the case discussed above, we used a cube with an edge length of five units. At the pace of twisting 90 degrees clockwise over the height of the cube, we could extend the surface and spiral curve vertically by simply extending the interval for the rotation parameter t in equations (4) and (5) to any upper bound. Figure 7 shows the twisted surface and spiral curve with $t \in [0, 2\pi]$.

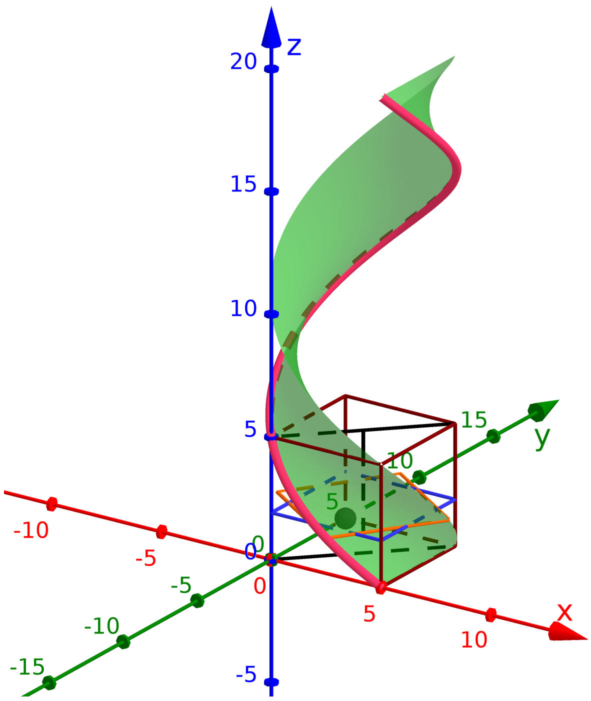

Figure 7. The twisted surface and the spiral curve are extended beyond the cube (created with GeoGebra®).

Instead of twisting a cube clockwise for 90 degrees as discussed above, what if the cube is twisted 270 degrees or $\frac{3\pi}{2}$? Then, the vector function in (4) should be slightly adjusted, the z-component in particular, to reflect the changing rate between the rotation and the vertical movement of the imagined dynamic square:
$$\vec{r}_b(s,t) = \left\langle \frac{5}{2} + \frac{5}{2}(\cos(t) - \sin(t)) + 5s\sin(t), \frac{5}{2} - \frac{5}{2}(\cos(t) + \sin(t)) + 5s\cos(5), \frac{10t}{3\pi} \right\rangle \quad (12)$$

with $0 \leq t \leq \frac{3\pi}{2}$. Figure 8 shows the GeoGebra graph of a corresponding cube face. The surface area of one twisted face is approximately 40.4435 square units for a square with an edge length of five units. In general, if a cube with an edge length of five units is twisted for an arbitrary angle $\mathcal{A}$, then the z-component of the parametric equation should be $\frac{5t}{\mathcal{A}}$ and the plotting interval for t is $[0, \mathcal{A}]$.

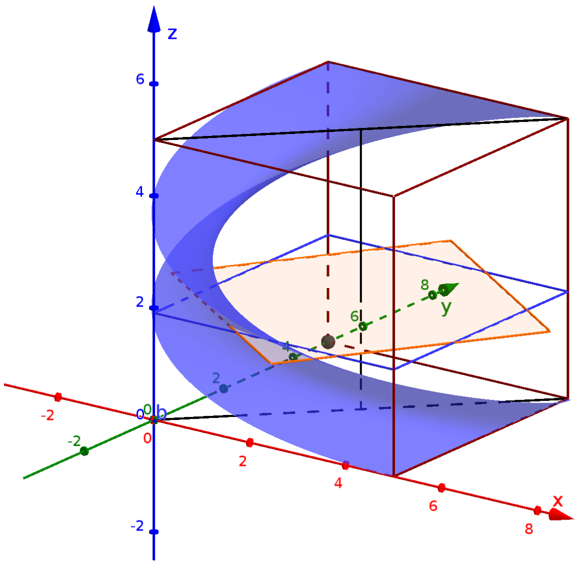

Figure 8. A cube is twisted 270 degrees clockwise (created with GeoGebra®).

Furthermore, we could extend volume-invariant twisting to other solids such as triangular or pentagonal prisms and explore the visual and algebraic nature of such twisted solids, using GeoGebra modeling and similar algebraic setups. Figure 9 shows a regular triangular prism twisted 120 degrees clockwise using GeoGebra simulation. The parametric equations for the surfaces and spiral curves can be established using the methods described earlier. These equations can be subsequently verified using GeoGebra 3D graphing commands.

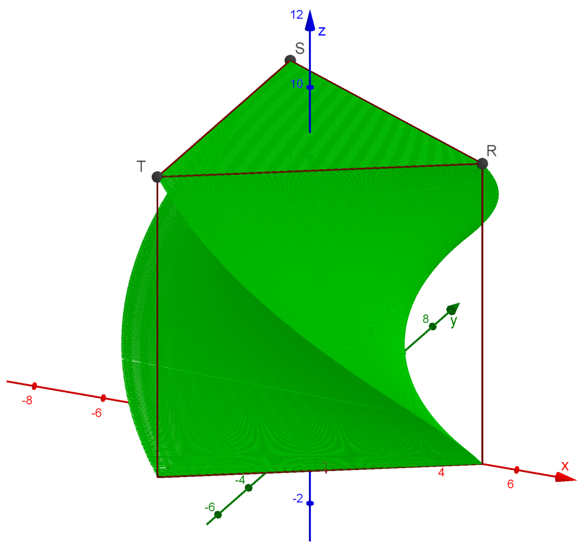

Figure 9. A regular triangular prism is twisted 120 degrees clockwise (created with GeoGebra®).

7 CONCLUSION

The cube is one of the most common objects in everyday life and school mathematics, affording surprisingly rich opportunities for mathematical engagement. In twisting the cube, we have explored the three worlds (Tall, 2013) of the mathematics around and within a twisted cube – the physical, the visual, and the algebraic dimensions of the problem situation, taking advantage of the 3D modeling tools of GeoGebra. A 3D printable model, if desired, can be readily designed in OpenScad® using the `linear_extrude` command or the `sweep` command in Autodesk Fusion 360® or similar 3D design environments. In a traditional sense, we could certainly start with a stack of paper squares, drill a hole in the center, and mount them on a rod for twisting. Alternatively, we could slice a cube, using 3D design tools, to demonstrate the process of cube twisting (Figure 10).

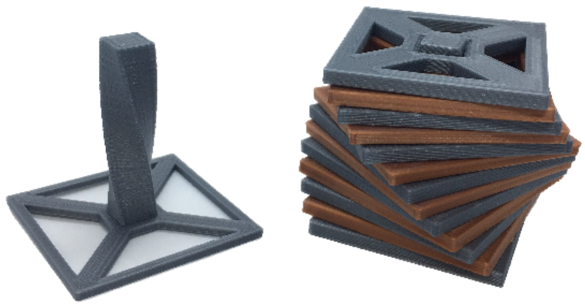

Figure 10. A cube is sliced for modeling the twisting process; the twisted center prism provides rotational guidance (created with Autodesk Fusion 360®).

In a classroom setting, we can start with physical manipulation, which serves as a means of student engagement and scaffolds mathematical analysis of the problem scenario (e.g., Madden, 2010). GeoGebra simulation subsequently shifts the mode of exploration from physical to dynamic modeling, where we learn about GeoGebra 3D tools while making sense of the proportional relationship between the rotation and vertical movement of an intermediate square. The algebraic analysis is also built on dynamic GeoGebra views, where we can look inside and around the cube to clarify and formulate the rich mathematical relationships. GeoGebra further helps verify the validity of our tentative parametric equations or vector functions, indicating possible errors or yielding satisfying 3D visuals. The same protocols can be applied to a variety of other solids that lend to 3D modeling in GeoGebra. In summary, GeoGebra 3D graphics supports a new approach to the teaching and learning of school geometry and advanced mathematics particularly in a problem-based instructional environment. When integrated with emerging 3D design and printing technologies, GeoGebra helps create a multidimensional world of mathematical sense-making and exploration.

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APPENDICES

A GUIDANCE FOR GEOGEBRA SIMULATION

The following summarizes the major steps in GeoGebra to twist a cube:

1. Define a cube of edge length five in GeoGebra 3D Graphics.
2. Define the axis of rotation by connecting the centers of the cube bottom and top faces.
3. Define a slider t over the interval $[0, Pi/2]$.
4. Define a plane $z = 10t/Pi$.
5. Intersect the plane with all four vertical edges of the cube for four points.
6. Define a polygon using the four points constructed from the previous step.
7. Rotate the polygon (square) from the previous step clockwise or counterclockwise for an angle of t.
8. Drag the slider t to simulate a spiraling square.
9. Turn on `Trace` for the rotated polygon (square) and drag the slider t to simulate a twisted cube.

B GRAPHING SURFACES AND CURVES

B.1 Graphing surfaces

To graph a parametric surface in GeoGebra 3D Graphics, use the following command, without the line breaks, and adjust the equations and parameter intervals as needed:

```
Surface(5/2+5/2(cos(t)-sin(t))+s*5sin(t),5/2-5/2(cos(t)+sin(t))+
s*5cos(t),10t/Pi,t,0,Pi/2,s,0,1)
```

B.2 Graphing curves

To graph a parametric spiral curve in GeoGebra 3D Graphics, use the following command, without the line breaks, and make adjustments to the parameter interval as needed:

```
Curve(5/2+5/2(cos(t)-sin(t)),5/2-5/2(cos(t)+sin(t)),10t/Pi,t,0,Pi/2)
```

C 3D PRINTABLE TWISTED CUBES AND SCAFFOLDS

To download 3D printable models for the twisted cube and other related pedagogical models, please visit the following URL for the STL files:
`https://www.thingiverse.com/thing:4790797`