FROM SEMI-PROOF TO PROOF: MOTIVATING DEDUCTIVE THINKING THROUGH INDUCTIVE EXPLORATION

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Abstract

In this short article, the authors introduce and prove a geometrical property involving externally tangent circles. Using an applet to model the problem situation, the authors illustrate how GeoGebra applets can serve as an accelerator of understanding, helping aid in the progression from inductive to deductive proof.

Keywords: circles, argumentation, proof, GeoGebra

INTRODUCTION

The use of dynamic software (DGS) enhances the mathematical and technological knowledge of learners. Alqahtani and Powell (2016) claim that "learners develop their own knowledge of how to use the tool [in this case the GeoGebra software] which turns tool into an instrument that mediates an activity between learners and a task" (p. 72). Stupel and Ben-Chaim (2017) echo this claim, indicating that "the advent of dynamic geometry environment (DGE) software serves as an intermediatory tool that bridges the gap between a mathematical problem or concept and its symbolic proof by providing a clear visual representation of the equation involved" (p. 86). Stupel and Ben-Chaim (2017) refer to the inductive nature of the DGE as "semi proof" and simultaneously warn that "students must be aware that formal proof based on mathematical arguments (using deductive approaches rather than relying on virtual experiments) is still required" (p. 86). The following geometrical problem provides an illustrative example.

Two circles O_1 and O_2 are tangent externally at point M. Two straight lines passing through point M intersect the circles at points A, B, C, D as can be seen in Figure 1.

1. Prove that $AC \parallel DB$.
2. Is this property also holds when the two circles are tangent internally?

Using two separate interactive applets, one can experience this property actively. We've constructed a sketch illustrating the case of external tangent point at `https://www.geogebra.org/m/eddv2aaf` and another one highlighting an internal tangent point at `https://www.geogebra.org/m/f5df9pmf`. In both sketches, EF is a tangent line to both circles at point M.

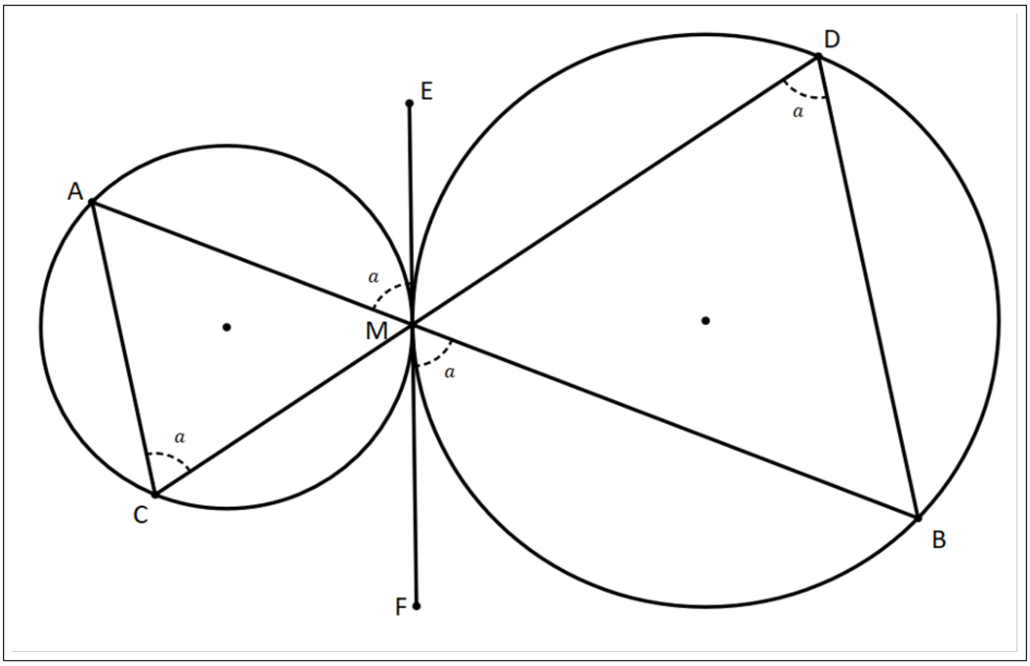

Figure 1. Problem situation with two externally tangent circles.

As students engage with the applets, they surmise that chords AC and DB are parallel when the two circles are tangent externally or internally. As they actively explore this property, students may also conjecture that the measure of the angles between the tangent line EF and the chords AM and MB remain equal to the measure of the peripheral angles ACM and BDM based on the chords AM and MB.

Mathematical Proof A (see Figure 1)

Proposition 32 in Book III of Euclid's Elements (Heath et al., 1956) states that "If a straight line touches a circle, and from the point of contact there is drawn across, in the circle, a straight line cutting the circle, then the angles which it makes with the tangent equal the angles in the alternate segments of the circle." Applying this proposition to the problem at hand, the following result holds: $\angle AME = \alpha \Rightarrow \angle ACM = \alpha$ (see Figure 1). By the same reason $\angle BMF = \alpha \Rightarrow \angle BDM = \alpha$. Hence, by using the theorem of vertical angles for $\angle AME$ and $\angle BMF$ it can be concluded that $\angle ACM = \angle BDM$. Then, by the theorem that if two alternating interior angles between two lines are equal, then the two lines are parallel, we achieve the result $AC \parallel DB$.

Note if presenting this activity to middle or high school students, a teacher might require his/her students to write the proof in a two-column format. We provide a proof in this style in the following section.

Second Proof

Another way to prove that chord AC is parallel to chord DB is by connecting O_1 with points A, C, M and connecting O_2 with points B, D, M to form three isosceles triangles in each circle. This idea is illustrated in Figure 2.

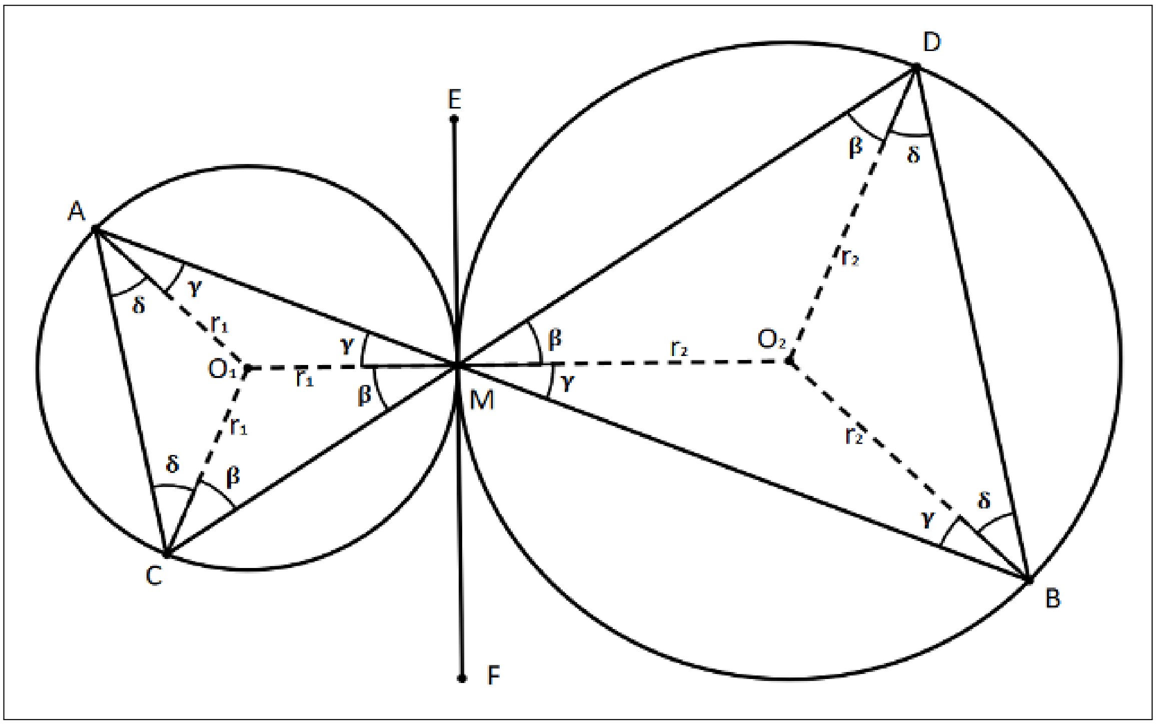

Figure 2. Forming isosceles triangles within each circle.

Below, we use the idea from Figure 2 to generate a two-column proof to rigorously confirm that $AC||DB$. As previously noted, this format is popular in many secondary school classrooms.

Theorem 1. *If segment EF is a tangent at point M to the circles O_1 and O_2 (see Figure 2), then $AC \parallel DB$.*

Proof.

Statement	Reason
1. Segment EF is a tangent at point M to the circles O_1 and O_2 (see Figure 2)	1. Given.
2. $\angle O_1ME = \angle O_2ME.$	2. By the theorem: the angle measure between a tangent to a circle and radius of the circle from the tangent point is $90°$.
3. $\triangle AO_1M, \triangle CO_1M, \triangle AO_1C, \triangle BO_2M,$ $\triangle DO_2M,$ and $\triangle BO_2D$ are isosceles.	3. $AO_1 = O_1M = O_1C = r_1$ and $BO_2 = O_2M = O_2D = r_2$
4. $m(\angle AMO_1) = m(\angle BMO_2) = \gamma$ $m(\angle CMO_1) = m(\angle DMO_2) = \beta$	4. Vertical angles have equal measures.

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Statement	Reason
5. $m(\angle O_1MA) = m(\angle O_1AM) = \gamma$ $m(\angle O_2MB) = m(\angle O_2BM) = \gamma$ $m(\angle O_1MC) = m(\angle O_1CM) = \beta$ $m(\angle O_2MD) = m(\angle O_2DM) = \beta$ $m(\angle O_1AC) = m(\angle O_1CA) = \delta$	5. By step 4 and base angles of isosceles have equal measures.
6. $m(\angle AO_1M) = m(\angle BO_2M) = 180° - 2\gamma$ $m(\angle CO_1M) = m(\angle DO_2M) = 180° - 2\beta$	6. The angle sum in a triangle is $180°$.
7. $m(\angle AO_1C) = m(\angle BO_2D)$	7. The angle sum around a point is $360°$.
8. $m(\angle O_1AC) =$ $m(\angle O_1CA) = \ m(\angle O_2BD) =$ $m(\angle O_2DB) = \delta$	8. By step 7, angle sum in a triangle is $180°$ and base angles of an isosceles triangle are equal.
9. $m(\angle MAC) = m(\angle MBD) = \beta + \delta$	9. Steps 5 and 8.
10. $AC \| DB$	10. $\angle MAC$ and $\angle MBD$ are congruent alternating interior angles formed by AC and BD, hence $AC \| BD$.

CONCLUSION

In this paper, we explored two externally tangent circles. Specifically, we constructed a dynamic sketch of the situation within GeoGebra. We used the software's visualization capabilities to motivate formal proof. Dragging the circles to numerous locations within our sketch, we noted that two chords appeared parallel. While our actions did not constitute formal proof, our sketch provided us with compelling evidence that supported our conjecture (what we refer to as "semi-proof").

While an inductive tool like GeoGebra can't replace rigorous proof, engaging students in an examples-first approach provides them with a motivation to prove (e.g., "I wonder if those sides are *really* parallel in *all* cases?"). Moreover, the experience with GeoGebra lays a sound logical and mathematical foundation for crafting deductive arguments. We believe that students' immersion in proofwriting and argumentation improves their ability to cope with challenging problems while enhancing their ability to explain steps in a process and building convincing arguments.

References

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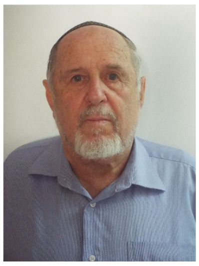

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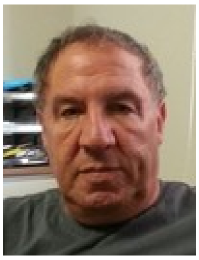

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