EXAMINING POSSIBLE LU DECOMPOSITIONS

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Abstract

LU decomposition is a fundamental in linear algebra. Numerous tools exists that provide this important factorization. The authors present the conditions for a matrix to have none, one, or infinitely many LU factorizations. In the case where no factorization exists, the authors illustrate how to approximate an LU decomposition by considering LU factorization of nearby matrices.

Keywords: GeoGebra, linear algebra, LU decomposition

1 INTRODUCTION

Many websites (IO Tools, 2021; Codesansar, 2021) provide tools to determine an LU decomposition of a matrix; however, none of them can decompose *all* matrices. Using code, we developed a GeoGebra applet that can decompose square matrices for those that can be factored and decompose "nearby" matrices without an LU decomposition using perturbation. In this paper, we first present conditions for a matrix to have a unique LU decomposition. Several examples are included to illustrate this theorem. Next, we state and prove special cases for matrices with infinitely many and no LU factorizations.

In this paper we employ the following notation. We denote the set of $n \times n$ matrices over the real numbers by $M_n = \mathbb{R}^{n \times n}$. If $A = (a_{ij}) \in M_n$, then the determinant of A is denoted by $\det(A)$. A leading principal minor A_k of matrix $A \in M_n$ is the determinant of a principal submatrix obtained by deleting the last $(n-k)$ rows and columns of A. A matrix $A \in M_n$ is called *lower triangular* if $a_{ij} = 0$ for $i < j$. A *unit* lower triangular matrix is a lower triangular matrix with $a_{ii} = 1$ for $1 \leq i \leq n$. The transpose of $A \in M_n$ is denoted by $A^T \in M_n$. The matrix $A \in M_n$ is *upper triangular* if A^T is lower triangular. For a complete background in linear algebra, see Strang (1993).

2 LU DECOMPOSITION

For $A \in M_n$, the factorization $A = LU$, where L is unit lower triangular and U is upper triangular, is called the *LU decomposition*, or *LU factorization*. We can use such a factorization, when it exists, to solve the system $A\mathbf{x} = \mathbf{b}$ by first solving for the vector $\mathbf{y}$ in $L\mathbf{y} = \mathbf{b}$ and then solving $U\mathbf{x} = \mathbf{y}$. However, not every $n \times n$ matrix A has an LU decomposition. The following theorem provides conditions for the existence and uniqueness of an LU decomposition of a $n \times n$ matrix. A proof can be found in Johnson and Horn (1985, p. 160).

Theorem 1 (Existence). *Suppose that $A \in M_n$ is rank k. If $\det(A_j) \neq 0$ for all $j = 1, \ldots k$, (all leading principal minors are non-zero), then A has a LU factorization. Moreover, if $k = n$, then this factorization is unique.*

The following examples illustrate Theorem 1.

Example 1. The 3×3 matrix $A = \begin{bmatrix} 1 & 5 & 1 \\ 1 & 4 & 2 \\ 4 & 10 & 2 \end{bmatrix}$ has all non-zero principle minors, A_1, A_2 and A_3. Therefore, there is a unique LU factorization with both L and U nonsingular given by

$$\begin{bmatrix} 1 & 5 & 1 \\ 1 & 4 & 2 \\ 4 & 10 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 10 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 12 \end{bmatrix}.$$

Example 2. Consider the rank 2 matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Since A_1 and A_2 are nonzero, Theorem 1 says an LU decomposition exists, for example

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}.$$

Example 3. The following rank 1 matrix has an LU decomposition since the first principle minor is nonzero, however, it is not unique.

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & x & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

What happens if a matrix does not meet the hypothesis in Theorem 1? The next two theorems classify those matrices.

Theorem 2 (Matrices with Infinitely Many LU Factorizations). *For $A \in M_n$, if two or more of any first $(n-1)$ columns are linearly dependent or any of the first $(n-1)$ columns are 0, then A has infinitely many LU factorizations.*

Proof. We will prove only for the the case when $A \in M_3$.

Case 1: Suppose column one is equal to 0. In particular, $A = \begin{bmatrix} 0 & d & g \\ 0 & e & h \\ 0 & f & i \end{bmatrix}$. We have, $A_1 = A_2 = 0$. Suppose the factorization, $A = LU$, exists, then

$$\begin{bmatrix} 0 & d & g \\ 0 & e & h \\ 0 & f & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ m & 1 & 0 \\ n & p & 1 \end{bmatrix} \begin{bmatrix} 0 & d & g \\ 0 & r & s \\ 0 & 0 & t \end{bmatrix}.$$

From this we get the following equalities:

$$A = \begin{bmatrix} 0 & d & g \\ 0 & e & h \\ 0 & f & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ m & 1 & 0 \\ n & p & 1 \end{bmatrix} \begin{bmatrix} 0 & d & g \\ 0 & r & s \\ 0 & 0 & t \end{bmatrix} = \begin{bmatrix} 0 & d & g \\ 0 \times m & dm + r & gm + s \\ 0 \times n & dn + rp & gn + sp + t \end{bmatrix}.$$

Equate each cell from A and the product LU, we have $(0)m = 0$, thus m can be any number. Similarly, $(0)n = 0$, thus n can be any number. From this we have,

$$dm + r = e \Rightarrow r = e - dm \tag{1}$$

$$dn + rp = f \Rightarrow p = \frac{f - dn}{r} \tag{2}$$

$$gm + s = h \Rightarrow s = h - gm \tag{3}$$

$$gn + sp + t = i \Rightarrow t = i - sp - gn \tag{4}$$

While solving for r, p, s, and t, (Equations (1) - (4)), we see they are dependent of variables m or n. Thus, A has infinitely many LU factorizations. A similar argument verifies the case where the second column is 0.

Case 2: Assume column one and two are linearly dependent, then there exists a real number k such that $A = \begin{bmatrix} a & ka & g \\ b & kb & h \\ c & kc & i \end{bmatrix}$ (same as below)

where $A_2 = kba – kab = 0$.

Now if $A = LU$ then,

$$A = \begin{bmatrix} a & ka & g \\ b & kb & h \\ c & kc & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ m & 1 & 0 \\ n & p & 1 \end{bmatrix} \begin{bmatrix} a & ka & g \\ 0 & r & s \\ 0 & 0 & t \end{bmatrix} = \begin{bmatrix} a & ka & g \\ am & kam + r & gm + s \\ an & kan + rp & gn + sp + t \end{bmatrix}.$$

From this we get the following equations and implications.

$$b = am \Rightarrow m = b/a. \tag{5}$$

$$c = an \Rightarrow n = c/a. \tag{6}$$

$$(ka)m + r = kb \Rightarrow ka(b/a) + r = kb \Rightarrow kb + r = kb \Rightarrow r = 0. \tag{7}$$

$$(ka)n + rp = kc \Rightarrow ka(c/a) + rp = kc \Rightarrow kc + rp = kc \Rightarrow (0)p = 0 \Rightarrow \forall p \in \mathbb{R}. \tag{8}$$

$$gm + s = h \Rightarrow s = h - gm. \tag{9}$$

$$gn + sp + t = i \Rightarrow t = i - sp - gn. \tag{10}$$

While solving for t, we can observe from Equation (8) that t relies on p which is free to be any number, resulting in the matrix having infinitely many LU decompositions.

$\square$

Example 4. *The first two columns of the matrix* $\begin{bmatrix} 20 & 5 & 9 \\ 16 & 4 & 7 \\ 4 & 1 & 3 \end{bmatrix}$ *are dependent and therefore has infinitely many factorization for some* t*. In particluar, we have*

$$\begin{bmatrix} 20 & 5 & 9 \\ 16 & 4 & 7 \\ 4 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{4}{5} & 1 & 0 \\ \frac{1}{5} & t & 1 \end{bmatrix} \begin{bmatrix} 20 & 5 & 9 \\ 0 & 0 & \frac{-1}{5} \\ 0 & 0 & \frac{1}{5}t + \frac{6}{5} \end{bmatrix}.$$

Example 5. *The matrix* $\begin{bmatrix} 0 & 2 & 3 \\ 0 & 4 & 6 \\ 0 & 6 & 9 \end{bmatrix}$ *has infinitely many factorizations since the first column is 0.*

Notice that if any first $(n-1)$ column's entries are all 0, or if any 2 or more of any first $n-1$ are linearly dependent, then there is a principal minor $A_k = 0$ where $k < n$.

The following theorem discusses the case of the matrices that have no LU decompositions for both invertible and singular matrices.

Theorem 3. *[Matrices with No LU Decompositions] Let $A \in M_n$, if the first $(n-1)$ columns are non-zero and linearly independent and at least one leading principal minor is zero, then A has no LU decomposition.*

Proof. We consider only matrices $A \in M_3$ and proceed by cases.

Case 1: $A_1 = 0$. Suppose A is given by,

$$A = \begin{bmatrix} 0 & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ m & 1 & 0 \\ n & p & 1 \end{bmatrix} \begin{bmatrix} 0 & d & g \\ 0 & r & s \\ 0 & 0 & t \end{bmatrix} = \begin{bmatrix} 0 & d & g \\ 0 & dm + r & gm + s \\ 0 & dn + rp & gn + sp + t \end{bmatrix}.$$

Since $0 \times m = b$ and $b \neq 0$, there is no solution for m. Thus, A has no LU factorization.

Case 2: $A_2 = 0, a \neq 0$.

$$A = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ m & 1 & 0 \\ n & p & 1 \end{bmatrix} \begin{bmatrix} a & d & g \\ 0 & r & s \\ 0 & 0 & t \end{bmatrix} = \begin{bmatrix} a & d & g \\ am & dm + r & gm + s \\ an & dn + rp & gn + sp + t \end{bmatrix}$$

where $A_2 = ae - bd = 0$

$$am = b \Rightarrow m = b/a$$
$$an = c \Rightarrow n = c/a$$
$$dm + r = e \Rightarrow r = e–dm = e–(bd/a) = (ae - bd)/a. \text{ Since } ae - bd = 0, r = 0$$
$$dn + rp = f \Rightarrow p = (f–dn)/r.$$

Since $r = 0$, we cannot solve for the variable p. Therefore, there is no LU decomposition for this matrix. □

Below, we share several examples of matrices with no LU factorization.

Example 6. *Although the matrix* $\begin{bmatrix} 4 & 2 & 3 \\ 6 & 3 & 6 \\ 5 & 7 & 9 \end{bmatrix}$ *is nonsingular, Theorem 1 tells us that it does not have an LU Factorization since not all principle minors are nonzero, in particular* $A_2 = 0$.

Using the setup in Case 2 of Theorem 3, we can use GeoGebra to program the LU decomposition for any 3×3 matrix as seen at `https://www.geogebra.org/m/s8yajtw5`. However, for higher dimensions, it is easier to write a program in javascript to perform the calculations. We constructed such a program to decompose any $n \times n$ matrix into an LU decomposition using row operations [4]. Readers are encouraged to explore the app at `https://lyjacky.github.io/ludecomp/dist`.

3 How to Approximate LU when no LU factorization exists

For a matrix with no LU factorization, we approximate the LU of a "nearby" matrix instead, then take a limit. The steps in the process follow.

1. Locate the leading principal with linearly dependent columns/rows.
2. Add or subtract ϵ to any cell from that leading principal minor so that it becomes non-zero. Thus, there will now be a unique LU decomposition for this new "nearby" matrix.
3. Let $\epsilon \to 0$.

Example 7. *Consider the matrix from Example 6. Let,*

$$A = \begin{bmatrix} 4 & 2 & 3 \\ 6 & 3 & 6 \\ 5 & 7 & 9 \end{bmatrix}.$$

Since $A_2 = 0$, we change cell a_{11} to 3.999 to get

$$A_\epsilon = \begin{bmatrix} 3.999 & 2 & 3 \\ 6 & 3 & 6 \\ 5 & 7 & 9 \end{bmatrix}.$$

A_ϵ is a "near-by" matrix of A with LU factorization

$$A_\epsilon = \begin{bmatrix} 3.999 & 2 & 3 \\ 6 & 3 & 6 \\ 5 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{2000}{1333} & 1 & 0 \\ \frac{5000}{3999} & -\frac{17993}{3} & 1 \end{bmatrix} \begin{bmatrix} 3.999 & 2 & 3 \\ 0 & -\frac{1}{1333} & \frac{1998}{1333} \\ 0 & 0 & 8995 \end{bmatrix} = L_\epsilon U_\epsilon.$$

Note that $\det(A_\epsilon) = \det(U) = 3.999 \cdot (-1/1333) \cdot 8995 = -26.985$. If we change cell a_{11} to 3.9999, then the new $\det(U) = -26.9985$, closer to $\det(A) = -27$. We can see that as ϵ approaches 0, A_ϵ is closer to A, and so are their determinants.

4 Conclusion

Using this GeoGebra application, we can find the LU decompositions of many square matrices without pivoting. If a matrix has infinitely many LU factorizations, users can arbitrarily select a value to get an LU decomposition. If, on the other hand, a matrix has no LU factorizations, users can approximate the LU decomposition by finding the LU decomposition of a nearby matrix.

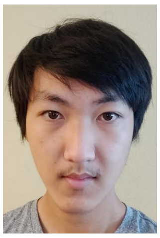
References

Codesansar (2021). Online LU Decomposition (Factorization) Calculator. *Available on-line at* `https://www.codesansar.com/numerical-methods/online-lu-decomposition-factorization-calculator.htm`.

IO Tools (2021). LU Factorization Calculator.

Johnson, C. R. and Horn, R. A. (1985). *Matrix analysis*. Cambridge University Press.

Strang, G. (1993). *Introduction to Linear Algebra*, volume 3. Wellesley-Cambridge Press.



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Appendix - Code

```
for(int i = 0; i < dim - 1; i++){
    for(int k = i+1; k  < dim; k++ ){
        for(int m =  i+1; m < dim; m++){
            if(upper[i][i] == 0 && upper[m][i] != 0 ){
                'THERE IS NO LU FACTORIZATION'
            }
        }
        if(upper[i][i] != 0 && upper[k][i] != 0){
            multiplier =upper[i][i]/upper[k][i];
            lower[k][i] = multiplier;
        }else{
            if(upper[i][i] == 0 && upper[k][i] == 0){
                INFINITELY MANY LU FACTORIZATIONS
                multiplier = INPUT FROM USER;
                lower[k][i] = multiplier;
            }
        }
    for(int j = 0; j < dim; j++){
        row[j] = upper[i][j]*multiplier;
    }
    for(int r = 0; r < dim; r++){
        upper[k][r] = upper[k][r] - row[r];
    }
}
Display Lower And Upper
(Perform Matrix Multiplication on L and U)
Display L and U
}
```