PROOF WITHOUT WORDS: FERMAT-TORRICELLI THEOREM

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Abstract

The authors explore the Fermat-Torricelli Theorem in a proof without words using GeoGebra. They provide two alternative approaches to the theorem and discuss connections to other geometry problems accessible to secondary level teachers and their students.

Keywords: Proof and argumentation, measurement, GeoGebra, Fermat-Torricelli Theorem

1 INTRODUCTION

In *Standards for the Preparation of Secondary Mathematics Teachers*, the National Council of Teachers of Mathematics (NCTM) state that a secondary mathematics teacher needs to be knowledgeable about mathematics content and be able to demonstrate this knowledge by applying conceptual understanding, procedural fluency, and factual knowledge in each of six major mathematical domains: Numbers; Algebra and Functions; Statistics and Probability; Geometry; Trigonometry and Measurement; Calculus; and Discrete Mathematics.

In this paper, we present examples of mathematical problems that address NCTM's recommendations in the domain of Geometry, a natural area for sense making and argumentation using visual approaches. Geometry's natural connections to trigonometry and measurement make it a rich area for mathematical exploration at the secondary level.

Specifically, we explore a visual confirmation of the Fermat-Torricelli Minimum Distance Theorem using a proof without words approach. Our method provides two distinct arguments, each utilizing an interactive visual representation in GeoGebra. We've designed our sketches to deepen student understanding of the theorem and its proof.

FERMAT-TORRICELLI THEOREM

In the 17th century, Fermat became interested in determining a method for finding a point within a given triangle such that the total distance from the point to each of the triangle's vertices was minimized. Refer to Figure 1. Fermat's question can be posed as follows: Find a point D within the triangle $\triangle EFG$ such that $m(\overline{DE}) + m(\overline{DF}) + m(\overline{DG})$ is minimal. A few years later, Torricelli showed that $m(\overline{DE})+m(\overline{DF})+m(\overline{DG})$ is minimal only if $m(\angle EDF) = m(\angle FDG) = m(\angle GDE) = 120^\circ$ (Eriksson, 1997).

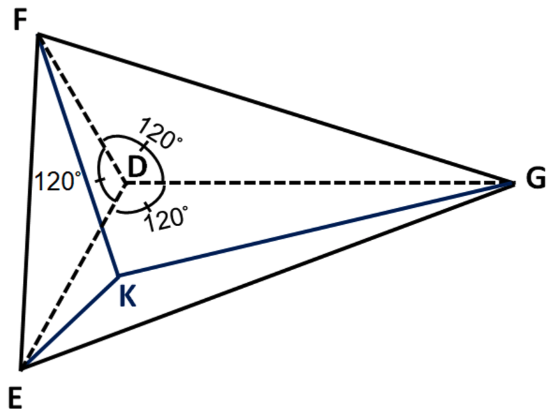

Figure 1. Fermat-Torricelli Theorem

2 Alternative Approaches

Here we use proof without words approach to present two distinct approaches to confirming the Fermat-Torricelli minimal distance theorem. As Figure 2 suggests, we use an auxiliary equilateral triangle to build our case.

Problem

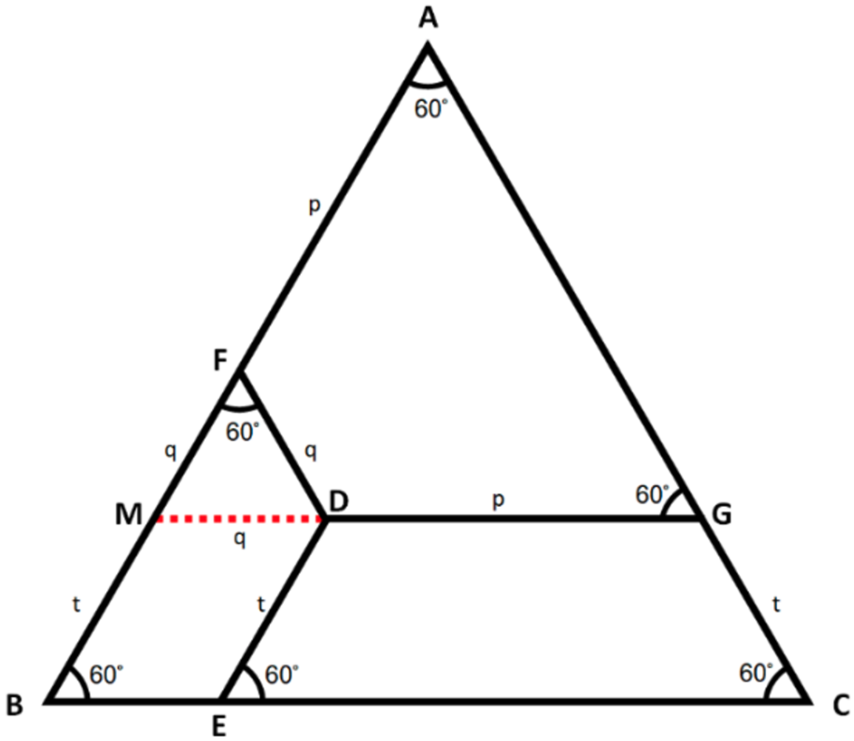

Figure 2. Proof Without Words

In Figure 2, the equilateral triangle $\triangle ABC$ and a point D inside it are given so that $\overline{DE} \parallel \overline{AB}$, $\overline{DF} \parallel \overline{AC}$, and $\overline{DG} \parallel \overline{BC}$. Prove that $m(\overline{DE}) + m(\overline{DF}) + m(\overline{DG}) = m(\overline{AB})$.

PROOF WITHOUT WORDS 1

$$m(\overline{DG}) = m(\overline{AF}) = p$$
$$m(\overline{DF}) = m(\overline{FM}) = q$$
$$m(\overline{DE}) = m(\overline{MB}) = t$$
$$\therefore m(\overline{DG}) + m(\overline{DF}) + m(\overline{DE}) = m(\overline{AF}) + m(\overline{FM}) + m(\overline{MB}) = m(\overline{AB})$$

PROOF WITHOUT WORDS 2

$$m(\overline{AM}) = m(\overline{MG}) = m(\overline{DF}) = m(\overline{DG})$$
$$m(\overline{MB}) = m(\overline{DE})$$
$$\therefore m(\overline{AB}) = m(\overline{AM}) + m(\overline{MB}) = m(\overline{DF}) + m(\overline{DG}) + m(\overline{DE}).$$

It is remarkable to note that $\angle EDF = \angle FDG = \angle GDE = 2\pi/3$ as required by the Fermat-Torricelli Minimal Distance Theorem.

3 GEOGEBRA APPLET

We've constructed a GeoGebra applet to help students gain insight into the aforementioned arguments. We invite you to engage with the applet at `https://www.geogebra.org/m/szrxkzmd`. Specifically, try dragging the point D within $\triangle ABC$. Note how the lengths of segments $\overline{DG}$, $\overline{DF}$, and $\overline{DE}$ change, while angle measurements and the sum of these segments remain fixed.

REFERENCES

Eriksson, F. (1997). The Fermat-Torricelli problem once more. *The Mathematical Gazette*, 81(490):37–44.

National Council of Teachers of Mathematics (NCTM) (2020). *Standards for Teacher Preparation Programs*.

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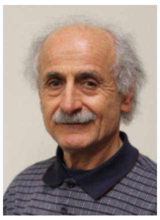

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