# MODELING THE PATHS OF THE SUN USING GEOGEBRA

Santiago Cerisola & Azahara López

*Pradolongo School & Fuenlabrada Astronomy School*

**Abstract**

*The authors describe the necessary constructions for modeling the apparent position of the Sun in the sky in GeoGebra. In particular, they discuss how to build a planet that moves and rotates about its axis and a way to position the observer at a given latitude. These computed coordinates are represented in a celestial sphere where the horizon plane for the observer is fixed, giving the apparent track of the Sun at different days of the year.*

**Keywords**: GeoGebra, science, planetary motion, paths of the Sun, horizontal coordinates

## 1 Introduction

In positional astronomy, we denote the track that our main star follows through the sky over a day as the "paths of the Sun." This apparent motion is a consequence of the Earth's rotation about its axis and depends on the observer's latitude. It is well known for a school student that the Sun rises somewhere in the East, transits at the meridian, and sets somewhere in the West. Thus, if the observer is sited in the Northern hemisphere and is looking in the South direction, the Sun rises from the left side and sets on the right side. Less well known to students is that this is reversed in the Southern hemisphere, where the Sun rises from the right side and sets on the left side.

Most students also know that the altitude of the Sun is higher in the Summer season than it is in the Winter season. However, in some cases, students do not connect these ideas with the orbital motion of the Earth around the Sun or with the rotation axis's inclination from the orbital plane (the angle between the equatorial and ecliptic plane (Karttunen et al., 2016)). These ideas, along with duration of day-time and night-time and their dependence on the day of the year and position of the observer, form the building blocks of our dynamic construction in GeoGebra (Hohenwarter, 2004) and set the way for students to more fully comprehend these phenomena.

The dynamic construction that we present in this paper models the motion of the Earth around the Sun and the apparent motion of the Sun in the sky. The sketch enables the students to better grasp the geometry behind these movements and increases their motivation to study the subject further. Use of the sketch is not limited to STEM subjects and can be a launching point for interdisciplinary study. For instance, we have used the GeoGebra applet in various secondary school subjects as part of an integrated project: Latin and Greek students use the applet as they study of ancient mythology; Geography students use it to explore the Sun's relationship with the Earth; Arts and Crafts students use the applet as they create planispheres and solar clocks; and Physics students use it to better understand the motion of the moon. Of course, mathematics students use the applet in their study of geometric modeling.

We model the apparent motion of the Sun in GeoGebra, creating a planet that orbits around the Sun and setting an observer on the surface of the planet. More specifically, we use a GeoGebra slider to simulate the planet's motion around the Sun and the rotation about its axis with proper inclination. From the observer's position, students obtain the angle that the position of the Sun subtends with the horizon plane and the angle that the orthogonal projection of the Sun over the horizon plane subtends with the meridian. In this way, the applet models the horizontal coordinates of the Sun in its track through the sky. Later, we use these coordinates to simulate the motion of the Sun in a semi-sphere in which the horizon of the observer is fixed. The dynamic construction enables the user to modify the position of the observer on the surface of the planet, thereby showing the students the apparent movement of the Sun from different latitudes of the planet.

In the construction that we present in this paper, we assume these simplifications:

- The orbit of the planet around the Sun is given by a perfect circle.
- The motion of the planet around the Sun is a uniform circular motion.
- The shape of the planet is a perfect sphere.
- The rotation of the planet around its axis has constant angular velocity.

In addition, we have not synchronized the orbital and rotational motions of the planet. This way, we simulate planets with annual solar cycles of a reduced number of days. In future developments, we will include elliptic orbits given by Kepler's laws (Hasek, 2012), and adjusted values for the Earth's angular velocity. With these further developments, we will have the possibility of obtaining the equation of time and many other calculations from them as the solar analema. Furthermore, modifying the inclination of the rotation axis (also included in our model) allows modeling the paths of the Sun in different planets and other associated phenomena derived from the synchrony of the movements such as Mercury's rising Sun.

This document focuses on the description of the modeling of the objects in GeoGebra for the simulation. We model a planet with four meridians and orbital and rotational motion. We also model the inclination of its axis and a horizon in the observer's position, using North, East, South, and West directions. These basic objects enable to obtain the horizontal coordinates of the Sun. In the last section, we model the fixed horizon and the semisphere where the paths of the Sun are simulated.

## 2 Planet Modeling

We assume the orbit of the planet to be a circle in the $XY$ plane with radius $R$. We can describe this trajectory as $x^2 + y^2 = R^2$. We denote the position of the center of the planet $C$ and simulate $C(t) = (R\cos(t), R\sin(t), 0)$, where $t$ is a parameter, visualized in GeoGebra as a slider, with domain $0 \leq t \leq 360°$.

We consider the shape of the planet to be a sphere of radius $r$. We model its surface with the command `sphere(C,r)`. To appreciate the rotation motion, we model four meridians as circles through the North pole $P_N$, South pole $P_S$, and specific points in the equator. These specific points $P_1$, $P_2$, $P_3$, and $P_4$ are those corresponding to the longitudes $0°$, $45°$, $90°$ and $135°$. Thus, the starting point for the development would consider

$$\begin{aligned} P_N &= r\,(0,0,1) \\ P_S &= r\,(0,0,-1) \\ P_1 &= r\,(\cos(0°),\sin(0°),0) \\ P_2 &= r\,(\cos(45°),\sin(45°),0) \\ P_3 &= r\,(\cos(90°),\sin(90°),0) \\ P_4 &= r\,(\cos(135°),\sin(135°),0) \end{aligned}$$

As our objective is to model the rotation of the planet around its axis, we must define those points on the equator as a function of the rotation angle $\theta$, a parameter with values $0 \leq \theta \leq 360°$. The former definition is thus modified

$$\begin{aligned} P_N &= r\,(0,0,1) \\ P_S &= r\,(0,0,-1) \\ P_1 &= r\,(\cos(0°+\theta),\sin(0°+\theta),0) \\ P_2 &= r\,(\cos(45°+\theta),\sin(45°+\theta),0) \\ P_3 &= r\,(\cos(90°+\theta),\sin(90°+\theta),0) \\ P_4 &= r\,(\cos(135°+\theta),\sin(135°+\theta),0) \end{aligned}$$

The rotation axis's deviation is obtained from modifying the described points from the $OZ$ axis. Our model assumes that the rotation axis lies in the $YZ$ plane and that the inclination is measured from the $OZ$ axis. We denote $\epsilon$ as the deviation angle with $0 \leq \epsilon \leq 90°$. We consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that describes this rotation of $\epsilon$. The fundamental matrix for the transformation is given by the images of the vector in the standard basis $B = \{\vec{e_1}, \vec{e_2}, \vec{e_2}\}$.

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\\cos(\epsilon)\\-\sin(\epsilon)\end{pmatrix} \qquad T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\\sin(\epsilon)\\\cos(\epsilon)\end{pmatrix}$$

Thus, the matrix $A$ for the linear transformation $T$ that describes the inclination is

$$A = \begin{pmatrix}1 & 0 & 0\\ 0 & \cos(\epsilon) & \sin(\epsilon)\\ 0 & -\sin(\epsilon) & \cos(\epsilon)\end{pmatrix}.$$

The coordinates of the points that we need to model the sphere of the planet are the images of $P_N$, $P_S$, $P_1$, $P_2$, $P_3$ and $P_4$ under the previous transformation. We maintain the notation for those points obtaining

$$\begin{aligned} P_N &= r\,(0,\sin(\epsilon),\cos(\epsilon)) \\ P_S &= r\,(0,-\sin(\epsilon),-\cos(\epsilon)) \\ P_1 &= r\,(\cos(0°+\theta),\sin(0°+\theta)\cos(\epsilon),-\sin(0°+\theta)\sin(\epsilon)) \\ P_2 &= r\,(\cos(45°+\theta),\sin(45°+\theta)\cos(\epsilon),-\sin(45°+\theta)\sin(\epsilon)) \\ P_3 &= r\,(\cos(90°+\theta),\sin(90°+\theta)\cos(\epsilon),-\sin(90°+\theta)\sin(\epsilon)) \\ P_4 &= r\,(\cos(135°+\theta),\sin(135°+\theta)\cos(\epsilon),-\sin(135°+\theta)\sin(\epsilon)) \end{aligned}$$

Finally, we model the revolution around the Sun translating the previous construction to be centered at $C$. Point $C$ tracks the position of the center of the planet. Thus, we have the coordinates that are used in our modeling.

$$\begin{aligned}
P_N &= C + r\,(0, \sin(\epsilon), \cos(\epsilon)) \\
P_S &= C + r\,(0, -\sin(\epsilon), -\cos(\epsilon)) \\
P_1 &= C + r\,(\cos(0^\circ + \theta), \sin(0^\circ + \theta)\cos(\epsilon), -\sin(0^\circ + \theta)\sin(\epsilon)) \\
P_2 &= C + r\,(\cos(45^\circ + \theta), \sin(45^\circ + \theta)\cos(\epsilon), -\sin(45^\circ + \theta)\sin(\epsilon)) \\
P_3 &= C + r\,(\cos(90^\circ + \theta), \sin(90^\circ + \theta)\cos(\epsilon), -\sin(90^\circ + \theta)\sin(\epsilon)) \\
P_4 &= C + r\,(\cos(135^\circ + \theta), \sin(135^\circ + \theta)\cos(\epsilon), -\sin(135^\circ + \theta)\sin(\epsilon))
\end{aligned}$$

As Figure 1 suggests, the `circle` command through three given points represents the meridians through the poles and the points of the equator. In the construction, we denote those circles as meridians of the planet ($mp$).

$$\begin{aligned}
mp_1 &= circle(P_S, P_1, P_N) \\
mp_2 &= circle(P_S, P_2, P_N) \\
mp_3 &= circle(P_S, P_3, P_N) \\
mp_4 &= circle(P_S, P_4, P_N)
\end{aligned}$$

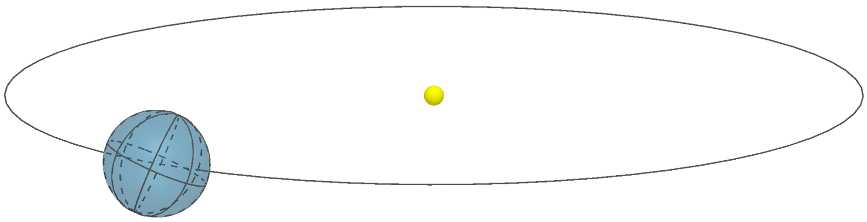


**Figure 1.** Sphere-planet orbit around the Sun

## 3 Horizon Modeling

We model the observer's position by its latitude $\phi$, and control this with a slider. Unlike the inclination $\epsilon$ of the planet, measured from the $OZ$ axis, the latitude $\phi$ is measured from the $XY$ plane. We set the domain of this parameter as $-90^\circ \leq \phi \leq 90^\circ$. We assume that at the starting time for the simulation ($t = 0$, $\theta = 0$), the point $P$ that determines the observer's position lies in the $XZ$ plane. The initial coordinates of this point are $P_0 = r\,(\cos(\phi), 0, \sin(\phi))$.

The coordinates $P$ for the position of the observer may be described as a function of the rotation angle $\theta$. Given that the fundamental matrix $B$ that describes this rotation as a linear transformation that depends on $\theta$ is

$$B = \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We find the coordinates for the observer's position as the image of the rotation $B \cdot P$. Given that the inclination of the rotation axis is also considered in our model, the coordinates of the observer's position are $A \cdot B \cdot P_0$, where $A$ is the fundamental matrix for the inclination previously described. Furthermore, after translating the construction to be centered at the center of the planet when it orbits around the Sun, we obtain the final coordinates that use in our model for the position of the observer: $P = C + A \cdot B \cdot P_0$. We note this coordinates $P$, and resume the values in the next figure.

$$P = C + r\begin{pmatrix} \cos(\theta)\cos(\phi) \\ \sin(\theta)\cos(\epsilon)\cos(\phi) + \sin(\epsilon)\sin(\phi) \\ -\sin(\theta)\sin(\epsilon)\cos(\phi) + \cos(\epsilon)\sin(\phi) \end{pmatrix}$$

We incorporate the cardinal points in our model for the horizon, creating vectors from the observer's position indicate North, South, East and West directions. The procedure to obtain the coordinates is similar to the constructions mentioned above that created the main points of the sphere-planet and the observer's position. We begin noticing that if we had an observer at latitude 0, ($\phi = 0°$), at the initial moment of the simulation, it would be at the position $P = (r, 0, 0)$. At that position, the North, South, East, and West directions are

$$\begin{aligned} v_{North} &= (0,0,1) \\ v_{South} &= (0,0,-1) \\ v_{East} &= (0,1,0) \\ v_{West} &= (0,-1,0) \end{aligned}$$

Thus, the initial position of the observer $P_0$ may be understood as the image of $(r, 0, 0)$ after the application of a rotation around the $OY$ axis. The fundamental matrix for this rotation is

$$D = \begin{pmatrix} \cos(\phi) & 0 & -\sin(\phi) \\ 0 & 1 & 0 \\ \sin(\phi) & 0 & \cos(\phi) \end{pmatrix}$$

This interpretation provides the means to obtain the North, South, East, and West directions at any moment of the simulation. The vectors are obtained using the application. The matrix $D$ gives the rotation around the $OY$ axis, $B$ the rotation around the $OZ$ axis, $A$ the rotation around the $OX$ axis. Thus, the cardinal directions' coordinates are the result of the multiplication of matrices $D$, $B$, and $A$ by the corresponding vectors. In particular, the North direction is computed as

$$v_{North} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\epsilon) & \sin(\epsilon) \\ 0 & -\sin(\epsilon) & \cos(\epsilon) \end{pmatrix} \begin{pmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos(\phi) & 0 & -\sin(\phi) \\ 0 & 1 & 0 \\ \sin(\phi) & 0 & \cos(\phi) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

The former calculations for the North direction and the remaining directions result in,

$$v_{North} = \begin{pmatrix} -\cos(\theta)\sin(\phi) \\ -\sin(\theta)\cos(\epsilon)\sin(\phi) + \sin(\epsilon)\cos(\phi) \\ \sin(\theta)\sin(\epsilon)\sin(\phi) + \cos(\epsilon)\cos(\phi) \end{pmatrix} \qquad v_{South} = -v_{North}$$

$$v_{East} = \begin{pmatrix} -\sin(\theta) \\ \cos(\theta)\cos(\epsilon) \\ -\cos(\theta)\sin(\epsilon) \end{pmatrix} \qquad v_{West} = -v_{East}$$

We also obtain the position of the zenith after the application of the above transformation to the vector $(1, 0, 0)$.

$$v_{Zenith} = \begin{pmatrix} \cos(\theta)\cos(\phi) \\ \sin(\theta)\cos(\epsilon)\cos(\phi) + \sin(\epsilon)\sin(\phi) \\ -\sin(\theta)\sin(\epsilon)\cos(\phi) + \cos(\epsilon)\sin(\phi) \end{pmatrix}$$

On the horizon of the observer, such directions identify the cardinal points. These points are created by adding to the position of the observer the already computed vectors. Once the cardinal points are built, we draw the horizon line at the observer's position using the command `circle` through three points of GeoGebra. In our current simulation, we draw three horizon lines as shown in Figure 2.

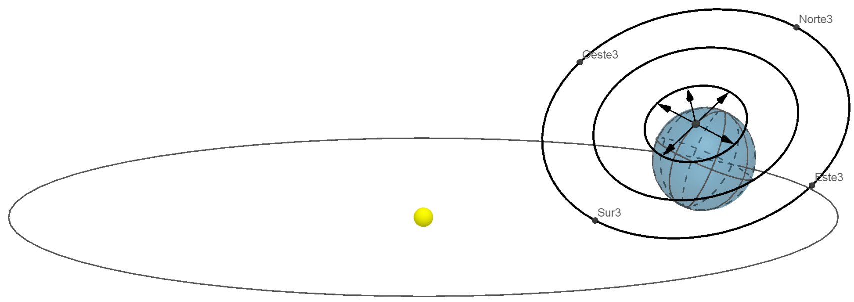



**Figure 2.** Sphere-planet horizon at the position of the observer

## 4 HORIZONTAL COORDINATES

The constructions described in the previous sections allow the computation of the horizontal coordinates of the Sun at any moment of the simulation. It is necessary to rescale the sphere-planet's size according to the size of the orbit it follows. For the simulations we present in this article, we modify the planet's radius $r$ to value zero. The parameter $r$ is visualized in GeoGebra as a slider with a null lower bound. The reader and students will likely appreciate that our modeling permits this modification because it collapses the sphere planet's main points and the meridians circles to the center $C$ of the planet. Furthermore, the North, South, East, and West directions are still computable, and so are the cardinal points, which permits the computation of the horizon plane for the observer.

We build this horizon plane for the observer with the command `plane` through three points. The necessary points are three out of the four cardinal points built in Section 2. We make the orthogonal projection of the Sun over this plane and denote it $P_{Sun}$. The orthogonal projection is given as the intersection of the horizon plane with the Sun's line with direction $v_{Zenith}$. Note that this direction is orthogonal to the horizon plane. Thus

$$P_{Sun} = \begin{cases} \pi_{horizon} & \equiv \quad plane(North, East, South) \\ r_{orthogonal} & \equiv \quad Sun + [v_{Zenith}] \end{cases}$$

A horizontal framework for the stars' position takes the South direction as a reference for calculations. Care must be taken because other fixed directions may be used in other frameworks (Karttunen et al., 2016; Asín, 1990). The azimuth $A$ is the angular distance of a star's orthogonal projection from the South direction. This angle is measured counter-clockwise and can be calculated with the command `angle` in GeoGebra.

$$Azimuth_{Sun} = A = angle(South, P, P_{Sun})$$

We note that GeoGebra returns the angle $0 \leq A \leq 180°$. In our construction, we measure that angle counter-clockwise (i.e., positive when the measure goes from South to East). We further note, it is convenient to identify the azimuth sign because it provides information about the semi plane. The Sun's vertical is positive for the East semi plane and negative for the West semi plane.

A numerical method to obtain this information is to cross-product vector $v_{South}$ with vector $\overrightarrow{PP}_{Sun}$ (vector from position $P$ to the vertical of the Sun $P_{Sun}$). The result is a normal vector to the horizon's plane and thus parallel to the vector $v_{Zenith}$. In case this cross-product has the same direction of

$v_{Zenith}$ the sign of the $azimuth$ is positive, and in case it has the opposite direction, the sign of the $azimuth$ is negative. The comparison of directions can be made comparing the signs of the first components of both vectors.

$$sign(A) = \begin{cases} 1 & \text{if } direction\left(v_{South} \otimes \overrightarrow{PP}_{Sun}\right) = direction\left(v_{Zenith}\right) \\ -1 & \text{if } direction\left(v_{South} \otimes \overrightarrow{PP}_{Sun}\right) \neq direction\left(v_{Zenith}\right) \end{cases}$$

In this manner we obtain the first horizontal coordinate for the position of the Sun.

$$Azimuth_{Sun} = sign(A)A$$

The second horizontal coordinate is the altitude or elevation, $a$, of the Sun. This altitude is the angular distance of the position of the Sun from the plane of the horizon. As this angle is measured from the horizon plane, its value lies in the range $[-90°, 90°]$. In our work, we compute this angle as $a = 90° - \beta$, where $\beta$ is the zenith distance, the angular distance of the position of the Sun with the vertical direction $v_{Zenith}$. This calculation directly provides the correct sign for the Sun's altitude, being positive during day-time and negative during night-time.

$$Altitude_{Sun} = a = 90° - angle\left(\overrightarrow{PP}_{Sun}, v_{Zenith}\right)$$

Figure 3 is taken from our construction and shows the planet reduced to a single point and the angles we calculate during the simulation: angle $A$ is where the vertical of the Sun subtends with the South direction and angular elevation $a$ where the Sun subtends with the horizon.

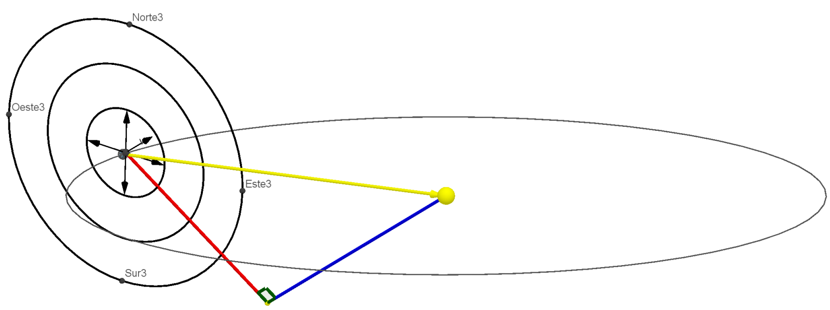



**Figure 3.** Angles for calculating horizontal coordinates

## 5 Paths of the Sun

To model the paths of the Sun, we must build a fixed horizon for the observer and model from this observer the position of the Sun using the horizontal coordinates azimuth and altitude described in the previous section. We note that those coordinates depend on the variables $t$ and $\theta$ (declared as sliders in GeoGebra), that parameterize the orbital motion and the rotation of the sphere-planet around its axis. Thus the coordinates are functions of these variables ($A(t, \theta)$ and $a(t, \theta)$). The animation of the sliders permits the simulation of the paths of the Sun.

In our work we denote this fixed horizon as the model for the simulation. For its construction we build a plane parallel to the $XY$ plane at a height ($\pi \equiv z = k$) such that in the same GeoGebra graphic view both the representation of the orbital motion of the planet and the paths of the Sun in the model can be seen at the same time. In that plane we declare point $Q = (0, 0, k)$ as the position of the observer. In the same plane we select cardinal points at a distance $r_C$ from point $Q$. These cardinal points of the model are fixed, and we build them directly as $Q_{North} = (r_C, 0, k)$, $Q_{East} = (0, r_C, k)$, $Q_{South} = (-r_C, 0, k)$ and $Q_{West} = (0, -r_C, k)$. In the same way we build a zenith point, as $Q_{Zenith} = (0, 0, k - r_C)$.

These cardinal points permit the construction of the meridian and the first vertical. We build the inter-cardinal points for representing two additional vertical planes.

$$Q_{NW} = \left( r_C \frac{\sqrt{2}}{2}, r_C \frac{\sqrt{2}}{2}, k \right),$$

$$Q_{SW} = \left( -r_C \frac{\sqrt{2}}{2}, r_C \frac{\sqrt{2}}{2}, k \right),$$

$$Q_{SE} = \left( -r_C \frac{\sqrt{2}}{2}, -r_C \frac{\sqrt{2}}{2}, k \right), \text{ and}$$

$$Q_{NE} = \left( r_C \frac{\sqrt{2}}{2}, -r_C \frac{\sqrt{2}}{2}, k \right).$$

Using the command `arc` through three points of GeoGebra we represent the meridians above the horizon. Similarly, we use the command `circle` through three points to represent the horizon line in the simulation model.

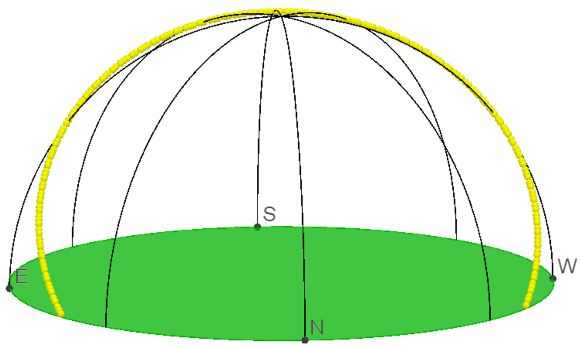
In the model that we are building the North direction coincides with the $OX$ axis and the West direction with the $OY$ direction. As the azimuth coordinate is measured with respect to the South direction, we model the apparent path of the Sun shifting the variable $180°$. Thus the path of the Sun is given by the next spherical coordinates depending on $A$ and $a$.

$$Q_{Sun} = Q + r_C \begin{pmatrix} \cos(a)\cos(A + 180°) \\ \cos(a)\sin(A + 180°) \\ \sin(a) \end{pmatrix}$$

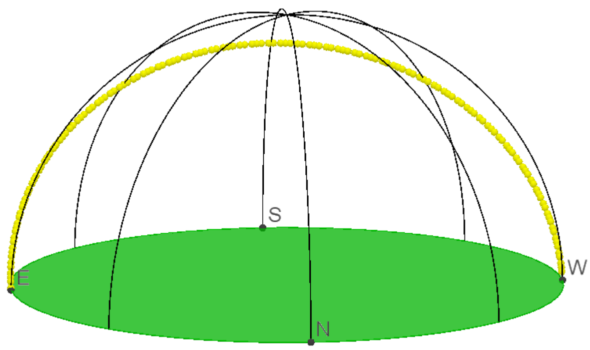
Next, we show examples of our simulation of the paths of the Sun from a position of $40°$ and from the North pole position. Figure 4a shows the paths of the Sun during the summer solstice. Figure 4b show the path of the Sun during the vernal equinox and Figure 4c during the winter solstice. In Figure 4d we can see the paths of the Sun during the summer solstice from the position of an observer situated in the North Pole. The reader may observe that the Sun does not set along the day, a phenomenon that lasts for six months. In our simulations, the elevation of the Sun at the North pole ranges between $22.5°$ and $24.5°$.

## 6 Didactic experiences and conclusions

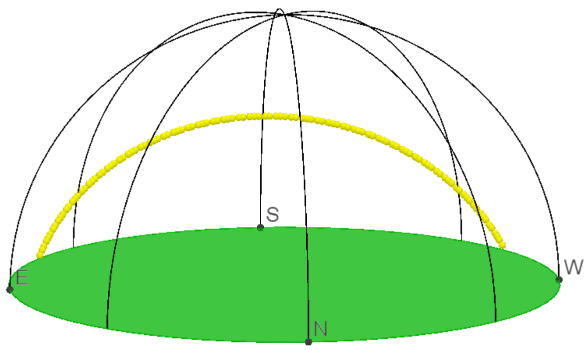
The dynamic construction we have presented in this paper was developed by the authors to improve the visual and spatial reasoning of the students in a multidisciplinary project that took place over an entire academic year. The aim of the project was to motivate the student to study STEM subjects,



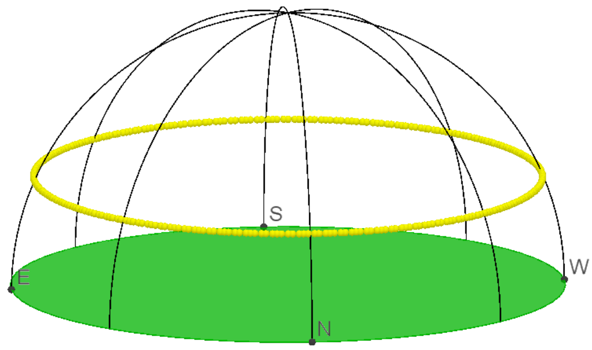

**(a)** During the summer solstice




**(b)** During the vernal equinox




**(c)** During the winter solstice




**(d)** From the North pole

**Figure 4.** Paths of the Sun

emphasizing their connection with many other disciplines, typically considered non-scientific. Astronomy was the chosen subject for this goal. Our exploration took advantage of alternative methods of thinking (Guzmán, 1994) and essential technological tools in education (Hohenwarter, 2004; Chan, 2013). In particular, the dynamic tools for the "paths of the Sun" were used in one of our sessions, where the motion of the Earth and the apparent motion of the Sun in the sky were explained. We used the sketch in conjunction with physical demonstrations using globes with students in a circle to answer questions such as:

- Where are you if the Sun rises from your right and sets on your left?
- How will the Sun behave if you are situated on the equator?
- Is the Sun higher in France or Spain during the summer?

A final evaluation of the project gave us important insight about the increased motivation that our students acquired. Of course part of the merit comes from the project itself, and part of the merit from the possibility of using this geometrical constructions that GeoGebra provides.

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**Santiago Cerisola** is Head of the Department of Mathematics at Pradolongo School and part-time professor at Carlos III University in Madrid, Spain. He is also an affiliated researcher at Comillas University in Madrid and a member of the Madrid Kepler Group. His research interests include applied optimization methods and the development of educational tools for didactics in mathematics.

**Azahara López Romero** is a mathematics teacher and Head of the Fuenlabrada Astronomy School at Comunidad de Madrid, Spain. She is also a member of the Madrid Kepler Group devoted to astronomy at the school. Her research interests include the elaboration and implementation of technology for astronomy teaching for undergraduate students.

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## APPENDIX: LIST OF SYMBOLS

| Symbol | Definition |
|---|---|
| $R$ | radius of orbital motion |
| $r$ | radius of sphere-planet |
| $r_C$ | radius of celestial sphere |
| $t$ | Slider to model orbital motion $0 \leq t \leq 360°$ |
| $\theta$ | Slider to model rotation of the sphere-planet $0 \leq \theta \leq 360°$ |
| $\epsilon$ | Slider for the inclination of the rotation axis $0 \leq \epsilon \leq 90°$ |
| $\phi$ | Slider for the latitude of the observer $-90° \leq phi \leq 90°$ |
| $C$ | Center of the planet |
| $P$ | Position of the observer |
| $P_N$ | North pole of the sphere-planet |
| $P_S$ | South pole of the sphere-planet |
| $P_1$ | Equator point of longitude $0°$ in the sphere-planet |
| $P_2$ | Equator point of longitude $45°$ in the sphere-planet |
| $P_3$ | Equator point of longitude $90°$ in the sphere-planet |
| $P_4$ | Equator point of longitude $135°$ in the sphere-planet |
| $P_N$ | North point in the horizon of the observer |
| $P_S$ | South point in the horizon of the observer |
| $P_E$ | East point in the horizon of the observer |
| $P_W$ | West point in the horizon of the observer |
| $Sun$ | Sun position : origin of coordinates $(0, 0, 0)$ |
| $P_{Sun}$ | Orthogonal projection of the Sun over the horizon |
| $Q_{Sun}$ | Apparent position of the Sun in the model |
| $Q$ | Position for the observer in the model |
| $Q_N$ | North position in the model |
| $Q_{NE}$ | North-East position in the model |
| $Q_E$ | East position in the model |
| $Q_{SE}$ | South-East position in the model |
| $Q_S$ | South position in the model |
| $Q_{SW}$ | South-West position in the model |
| $Q_W$ | West position in the model |
| $Q_{NW}$ | North-West position in the model |
| $mp_1$ | Meridian of the sphere-planet through point $P_1$ |
| $mp_2$ | Meridian of the sphere-planet through point $P_2$ |
| $mp_3$ | Meridian of the sphere-planet through point $P_3$ |
| $mp_4$ | Meridian of the sphere-planet through point $P_4$ |
| $A$ | Inclination matrix of the rotation axis |
| $B$ | Rotation matrix of the sphere-planet |
| $D$ | Rotation matrix of the latitude of the observer |

| Symbol | Definition |
|---|---|
| $v_{North}$ | North direction from position $P$ |
| $v_{South}$ | South direction from position $P$ |
| $v_{East}$ | East direction from position $P$ |
| $v_{West}$ | West direction from position $P$ |
| $v_{Zenith}$ | Zenith direction from position $P$ |
| $\pi_{horizon}$ | Plane of the horizon of the observer |
| $r_{orthogonal}$ | Orthogonal line to the horizon through the Sun |
| $A$ | Azimuth of the Sun form the South $-180° \leq A \leq 180°$ |
| $a$ | Elevation of the sun $-90° \leq a \leq 90°$ |