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LOCUS OF CRITICAL POINTS FOR SOME POLYNOMIALS

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Abstract

Given a polynomial with complex coefficients, the celebrated Gauss-Lucas's theorem and Marden's theorem offer us insights into the geometry of the locus of its critical points. In the following paper, the authors explore the geometry under certain restrictions of the roots of the polynomials. In particular, the authors identify some regions where all critical points do not occur.

Keywords: complex polynomial, critical points

INTRODUCTION

The locus of critical points of polynomials is an exciting field Frayer et al. (2014); Marden (1949) with numerous connections and physical interpretations. For instance, the zeros of a polynomial's derivative are the equilibrium points because of the mass on the roots in the Newtonian field Marden (1949).

Marden's theorem is one of the well-known results Kalman (2008b) relevant to the zero of the derivative of a polynomial. Inspired by Linfield (1920), the theorem reveals a connection between geometry and analysis—namely, that the critical points of the polynomial are the foci of the ellipse Kalman (2008a). A fundamental result, the so-called Gauss-Lucas theorem Lucas (1879), shows that the critical points lie in the convex hull of given polynomial roots. When the polynomials have more restricted forms, such as with real coefficients or their roots are remarkable, more precise results are available. In particular, the region of locus of the polynomial roots in the real polynomials case is smaller than the result of Gauss-Lucas theorem (Jensen, 1913; Marden, 1949; Sendov, 2014). It has been proved that every non-real zero of the derivative of a real polynomial $f(z)$ lies in or on at least one of the Jensen circles[1] of $f(z)$. Jensen's theorem provides a method to restrict the real polynomial roots to get a smaller region of the roots' locations. In this paper, we investigate the loci of polynomials with roots that occur on individual curves.

Motivated by Dan Kalman's new proof Kalman (2008a) of Marden's theorem, works such as Aghekyan and Sahakyan (2013) and Frayer et al. (2014) focus on cubic polynomials with roots on the unit circle or certain lines. Frayer and others Frayer et al. (2014) have studied the movement of the critical points of a complex cubic polynomial when its roots move. In this article, Frayer et al. made use of the result that there is a circumcircle through any three non-collinear points r_1, r_2, r_3. In particular,

[1]The Jensen circle is formed by using the line segment that joins the pair of imaginary conjugate roots of a real polynomial as its diameter.

the authors explored the unit circle with $r_3 = 1$ and $|r_1| = |r_2| = 1$ and found that critical points cannot occur in an open disk $\left\{z \in \mathbb{C} : \left|z - \frac{2}{3}\right| < \frac{1}{3}\right\}$. Relevant results are provided in Aghekyan and Sahakyan (2013).

This paper investigates the locus of critical points of polynomials with roots that occur on certain curves. We extend some results known for cubic polynomials (see Frayer et al. (2014)). We use GeoGebra, a dynamic graphing tool, to help visualize critical points based on the motion of the polynomial roots. In section one, we give explicit examples for quartic polynomials; in section two, we identify regions that critical points do not occur for polynomials of any degree.

1 Quartic Polynomials

In this section, we work exclusively with the quartic polynomials with complex coefficients. In this case, a critical point is defined to be a zero of the first derivative of the polynomial, which is a root of a cubic polynomial Larson (2012). We assume the polynomials are of the form as in Notation 1.1. In this case, the roots form a parallelogram on the complex plane, and r_1 and r_2 move on a circle with radius t, centered at $(0, 0)$.

Notation 1.1. *Let Λ denote the family of complex quartic polynomials $q(z)$ such that*

$$q(z) = (z-1)(z+1)(z-r)(z+r)$$

for $r \neq \pm i$, $|r| = t$, $t > 0$ and $r \in \mathbb{C}$.

Remark. For $r = \pm i$, $q(z) = z^4 - 1$, and $q'(z) = 4z^3$ cases, all of critical points are zero.

Theorem 1.2. *For the polynomial defined as above, the loci of its nonzero critical points satisfy the equation $t > 0$,*

$$\left(a^2 + b^2\right)^2 = \left(a^2 - b^2\right) + \frac{t^4}{4} - \frac{1}{4}.$$

Proof. Suppose $p(z) \in \Lambda$, with roots ± 1 and $\pm r$, and let c be the critical points of p. Then

$$p(z) = (z-1)(z+1)(z-r)(z+r)$$
$$p'(z) = 4z^3 - 2\left(r^2 + 1\right)z,$$

and

$$0 = 4c^3 - 2\left(r^2 + 1\right)c.$$

For $c \neq 0$, we have

$$c^2 = \frac{r^2 + 1}{2}.$$

Let $c = a + bi$, $a, b \in \mathbb{R}$, $r = t\cos\theta + it\sin\theta$, $\theta \in (0, \pi)$, then

$$a^2 + 2abi - b^2 = \frac{t^2\cos 2\theta + 1}{2} + \frac{t^2 \sin 2\theta}{2}i.$$

Thus, we get

$$a^2 - b^2 = \frac{t^2 \cos 2\theta + 1}{2}$$
$$2ab = \frac{t^2 \sin 2\theta}{2}.$$

Compute $\left(a^2+b^2\right)^2$ by these two equations and simplify it,

$$\left(a^2+b^2\right)^2 = \left(a^2-b^2\right)^2 + 4a^2b^2 = \frac{t^2 \cos 2\theta}{2} + \frac{1}{4} + \frac{t^4}{4} = \left(a^2-b^2\right) + \frac{t^4}{4} - \frac{1}{4}.$$

We obtain the desired equation $\left(a^2+b^2\right)^2 = \left(a^2-b^2\right) + \frac{t^4}{4} - \frac{1}{4}$. $\square$

Remark. A family of Cassini ovals is represented by the equation,

$$\left(x^2+y^2\right)^2 = 2\alpha^2\left(x^2-y^2\right) + \beta^4 - \alpha^4$$

with $\alpha = \frac{\sqrt{2}}{2}, \beta = \frac{t\sqrt{2}}{2}$ as t varies. A Cassini oval is described by a point such that the product of its distances from two fixed points a distance of 2α apart is a constant β^2 $(\alpha > 0, \beta > 0)$ (Weisstein, 2004). The shape of the curve depends on the value of β and α. If $\alpha < \beta$, the curve is a single loop with an oval or dog bone shape. If $\alpha > \beta$, then the curve consists of two loops. The case $\alpha = \beta$ produces a lemniscate. We illustrate with an example.

Example 1.3. *Suppose $p(z) \in \Lambda$, with roots ± 1, $\pm r$ and $t = 1$. The locus of nonzero critical points $a + bi$ and $-a - bi$ of $p(z)$ is*

$$\left(a^2+b^2\right)^2 = a^2 - b^2$$

which is a Lemniscate with $\alpha = \frac{\sqrt{2}}{2} = \beta$.

We used GeoGebra to generate the trace. Readers are encouraged to engage in our construction at `https://www.geogebra.org/m/zhahcbs2`. Note that right-clicking on a critical point and then selecting the *Show trace* option generates clear traces of the point (See Figure 1).

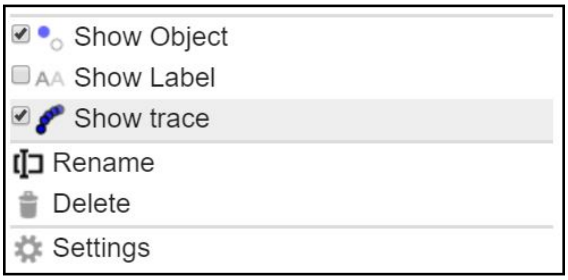


Figure 1. *Show trace* option from pop-up menu.

In Figure 2, circle u is the unit circle that the roots move on, and curve L represents the loci of the critical points of $p(z)$.

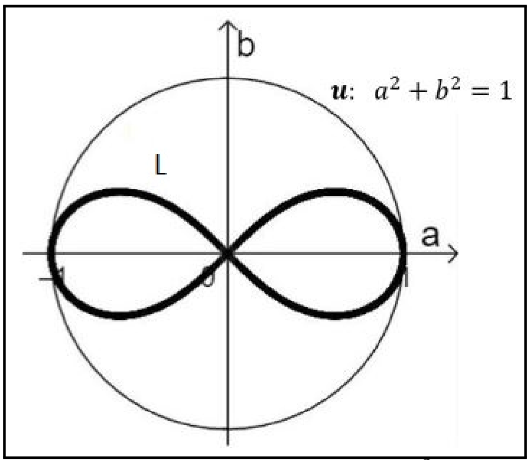


Figure 2. Lemniscate with $\left(a^2+b^2\right)^2 = a^2 - b^2$.

2 General Case

In this section, we set $p(z) = (z - z_1)(z - z_2)\cdots(z - z_n)$ and assume all $z_1, z_2, ..., z_n$ in $\mathbb{C}$ and $|z_1| = |z_2| = ... = |z_n| = 1$. We are interested in the question: Where will the critical points occur as the complex polynomial roots move on the circle? Frayer and others in Frayer et al. (2014) prove that when $n = 3$, the cubic complex polynomial has a disk $\left\{z \in \mathbb{C} : \left|z - \frac{2}{3}\right| < \frac{1}{3}\right\}$ that the critical points cannot occur in it. To extend their result to polynomials of higher degrees, we modify the open disk that are used to identify the "deserts" in Frayer et al. (2014).

Notation 2.1. *Given $\alpha > 0$, $z_t = e^{i\theta_t}, t = 1, 2, ..., n$. For each t, we denote by $D_\alpha^{z_t}$, the circle of diameter α that passes through z_t and $z_t - \alpha z_t$. Namely,*

$$D_\alpha^{z_t} = \left\{z \in \mathbb{C} : \left|z - \left(e^{i\theta_t} - \frac{\alpha}{2}e^{i\theta_t}\right)\right| = \frac{\alpha}{2}\right\}.$$

We first prove a lemma about the circle $D_\alpha^{z_t}$ (Notation 2.1) that will be used in the theorems below.

Remark. In Frayer et al. (2014), given $\alpha > 0$, they denote by T_α the circle of diameter α that passes through 1 and $1 - \alpha$ in the complex plane. That is,

$$T_\alpha = \left\{z \in \mathbb{C} : \left|z - \left(1 - \frac{\alpha}{2}\right)\right| = \frac{\alpha}{2}\right\}.$$

Lemma 2.2. *Let $z \in \mathbb{C}$ with $|z| < 1$. If $z \in D_\alpha^{z_t}$, then*

$$\mathrm{Re}\left(\frac{e^{i\theta_t}}{z_t - z}\right) = \frac{1}{\alpha}.$$

Proof. Let $A = (1, 0), O = (0, 0), |z_t| = 1, |z| < 1, r = |z_t - z|$, and set $\angle AOz_t = \theta_t$, $\angle Oz_tz = \omega$ in Figure 3. It follows that

$$\frac{z_t - z}{e^{i\theta_t}} = re^{-i\omega}.$$

That is

$$\frac{e^{i\theta_t}}{z_t - z} = \frac{1}{r}e^{i\omega}.$$

Hence,

$$\operatorname{Re}\left(\frac{e^{i\theta_t}}{z_t - z}\right) = \operatorname{Re}\left(\frac{1}{r}e^{i\omega}\right) = \frac{1}{r}\operatorname{Re}\left(e^{i\omega}\right) = \frac{\cos\omega}{r}.$$

The result holds since $\cos\omega = \frac{r}{\alpha}$. □

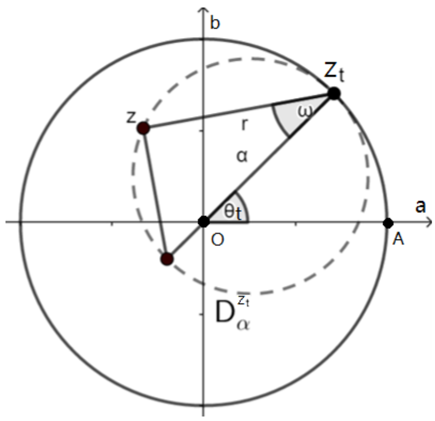


Figure 3. Relationship between the circle $D_\alpha^{z_t}$ and the point z.

By Gauss-Lucas theorem, the critical points of a polynomial lie in the convex hull of its roots. Since we assume all roots are on the unit circle, the critical points are also in the interior of the unit disk.

Notation 2.3. *Let Ω denote the family of complex polynomials w,*

$$w(z) = (z - z_1)(z - z_2)\cdots(z - z_n)$$

where $z_i = e^{i\theta_i}$, $1 \le i \le n$, and $z_i \ne z_j$ for any $1 \le i, j \le n$.

Lemma 2.4. *Let $p(z) \in \Omega$ and $c_1, c_2, \ldots, c_{n-1}$ denote critical points of $p(z)$ and $c_k \in D_{\alpha_k}^{z_t}$, $1 \le k \le n-1$. Then*

$$\sum_{k=1}^{n-1} \frac{1}{\alpha_k} = n - 1.$$

Proof. We may write $p'(z) = C(z - c_1)(z - c_2)\ldots(z - c_{n-1})$ for some $C \in \mathbb{C}$. Since $(\ln(p'(z)))' = \frac{p''(z)}{p'(z)}$, and $\ln(p'(z)) = \ln C + \ln(z - c_1) + \cdots + \ln(z - c_{n-1})$, we have

$$\frac{p''(z)}{p'(z)} = \sum_{k=1}^{n-1} \frac{1}{z - c_k}.$$

Put $z = z_t$,

$$\frac{p''(z_t)}{p'(z_t)} = \sum_{k=1}^{n-1} \frac{1}{z_t - c_k}.$$

Note by our assumption and Notation 2.3, we have $p'(z_t) \ne 0$, $z_t \ne z_k$ for any k.
Hence, we have

$$\operatorname{Re}\left(\frac{p''(z_t)}{p'(z_t)}e^{i\theta_t}\right) = \operatorname{Re}\left(\sum_{k=1}^{n-1} \frac{e^{i\theta_t}}{z_t - c_k}\right) = \sum_{k=1}^{n-1} \frac{1}{\alpha_k}$$

by Lemma 2.2.

On the other hand, write $p(z) = (z - z_t)g(z)$, we get $p'(z) = (z - z_t)g'(z) + g(z)$ and $p''(z) = (z - z_t)g''(z) + 2g'(z)$. So,

$$\frac{p''(z_t)}{p'(z_t)} = \frac{2g'(z_t)}{g(z_t)} = 2\,(\ln(g(z)))'|_{z=z_t} = 2\sum_{m\neq t}^{n-1} \frac{1}{z_t - z_m}$$

Again, by Lemma 2.2, we get

$$\mathrm{Re}\left(\frac{p''(z_t)}{p'(z_t)}e^{i\theta_t}\right) = 2\sum_{m=1}^{n-1} \mathrm{Re}\left(\frac{e^{i\theta_t}}{z_t - z_m}\right) = 2\sum_{k=1}^{n-1}\frac{1}{2} = n - 1$$

Since both z_t and z_m are on the unit circle whose diameter is 2, we use Lemma 2.2 to obtain the second equality.

Together, we get

$$\sum_{k=1}^{n-1}\frac{1}{\alpha_k} = n - 1.$$

□

Finally, we obtain:

Theorem 2.5. *If $z_1, z_2, ..., z_k$, $k \leq n$, are the roots of a complex polynomial $p(z) \in \Omega$, there are k disks $D_{\alpha_k}^{z_k}$, in which critical points cannot occur no matter how remaining roots varied on the unit circle.*

Proof. Let $c_1, c_2, \ldots, c_{n-1}$ denote the critical points of $p(z)$, and $c_k \in D_{\alpha_k}^{z_k}$ for each k. By Gauss-Lucas theorem, we know that for any $1 \leq k \leq n-1$, $|c_k| < 1$, which means that $\frac{1}{\alpha_k} \geq \frac{1}{2}$ for each k. By Lemma 2.4, $\sum_{k=1}^{n-1} \frac{1}{\alpha_k} = n - 1$. Then we find that

$$n - 1 = \sum_{k=1}^{n-1}\frac{1}{\alpha_k} = \frac{1}{\alpha_g} + \sum_{k\neq g}^{n-2}\frac{1}{\alpha_k} \geq \frac{1}{\alpha_g} + (n-2)\frac{1}{2}.$$

So that,

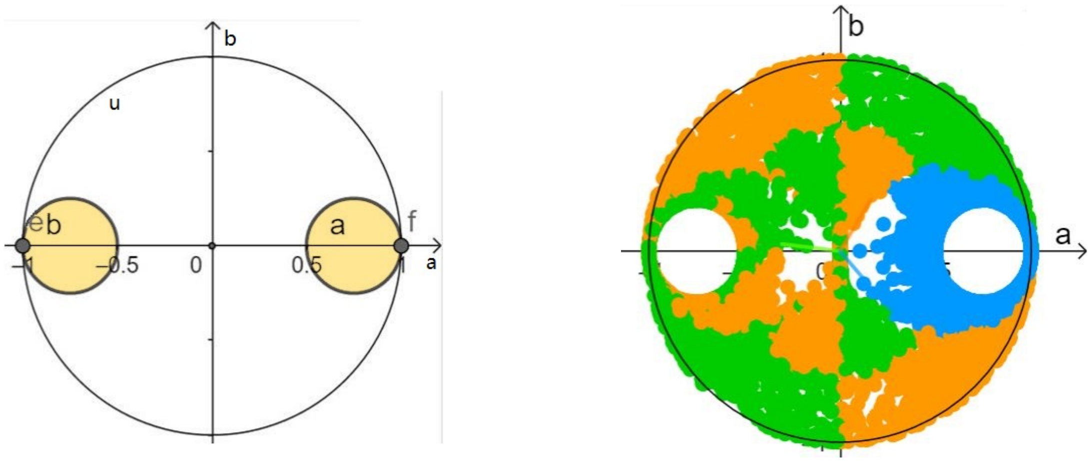
$$\alpha_g \geq \frac{2}{n}.$$

This means that the diameter of the disk $D_{\alpha_g}^{z_g}$ which critical points occur on cannot be smaller than $\frac{2}{n}$. As a result, for all of the roots r_k, there is no critical point of $p(z)$ inside the disks $\left\{z \in \mathbb{C} : |z - \left(z_k - \frac{z_k}{n}\right)| < \frac{1}{n}\right\}$.

□

Remark. The theorem works out the critical points' locations of polynomial, which roots are concyclic. For critical points' locations in other polynomials cases, such as a polynomial with roots on an ellipse, we find that Lemma 2.2 cannot be used because roots are on an ellipse instead of the circle. It means that the way we use in this paper to derive Theorem 2.5 cannot be used.

Here is an example that illustrates Theorem 2.5.



(a) Disks with no critical points.

(b) Traces of three critical points.

Figure 4. Critical points on Disk

Example 2.6. *Suppose the complex quartic polynomial*

$$p(z) = (z - z_1)(z - z_2)(z - z_3)(z - z_4) \in \Omega$$

for $z_1 = 1$, $z_2 = -1$, *and* $|z_3| = 1 = |z_4|$.

Because $z_1 = 1$, $z_2 = -1$, and $n = 4$, by Theorem 2.5, there are two disks $\left\{a \in \mathbb{C} : \left|a - \frac{3}{4}\right| < \frac{1}{4}\right\}$ and $\left\{b \in \mathbb{C} : \left|b + \frac{3}{4}\right| < \frac{1}{4}\right\}$ that the critical points of $p(z)$ will not occur wherever other two roots z_1 and z_2 are (for $p(z) \in \Omega$, $|z_1|$ and $|z_2|$ should equal to 1).

In Figure 4a, points e and f are the roots ± 1 of the complex quartic polynomial. The yellow disks a and b are $\left\{a \in \mathbb{C} : \left|a - \frac{3}{4}\right| < \frac{1}{4}\right\}$ and $\left\{b \in \mathbb{C} : \left|b + \frac{3}{4}\right| < \frac{1}{4}\right\}$ that the critical points cannot occur in them as other roots z_1, z_2 varied on the unit circle u. In GeoGebra, we can see these two disks directly by using the *Show trace* tool (Figure 4b). See `https://www.geogebra.org/classic/ere36kg4` for the construction. The orange, blue, and green parts are traces of three critical points.

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