

MODELING AN OPTIMAL TRAJECTORY FOR A CERTAIN MINIGOLF STATION

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Abstract

Usually a minigolf course comprises a station at which the ball is supposed to enter a net, which is installed about one meter above the ground. The minigolf player uses a ramp for giving the ball the right direction. One may ask, at which velocity the ball has to be driven off in order to enter the net. In this article, we present several ways to use GeoGebra for solving this problem. We will see that the features of GeoGebra can help to visualize the problem, to implement a mathematical model, to work in this model and to validate the results by comparing to empirical data. The article gives insight in how teachers can use the problem as a modeling task in school and how different approaches lead to similar results. As part of a classroom project, filming a real tee shot on a minigolf course and analyzing it with the aid of a free cell-phone app can provide the students with a validation of their models and calculations.

Keywords: : minigolf, GeoGebra, modeling problem, empirical data

1 INTRODUCTION

Mathematical modeling is one of the central competencies, which students shall acquire (7; 1; 11). In many cases, however, realistic modeling problems are considerably complex. It is quite difficult to find problems in the world around that are suitable for modeling activities in school.

Mathematics education research has made a lot of effort to understand how students' modeling processes take place and what kind of steps are useful when building up a model (6; 4; 3). There are several theoretical concepts that describe these modeling processes, yet all of them contain the following steps: simplifying and structuring the real-world problem, translating the problem into mathematical language, operating in this mathematical model, interpreting and validating the results (2; 3).

In this paper we discuss a modeling problem, which is accessible easily, appropriate to be described by a simple mathematical model and motivating because of the possibility of empirical verification afterwards. We consider the following station from a typical minigolf course (see Figure 1). The ball is supposed to be driven off in order to roll up a ramp while aiming at a net, which is installed at a certain height. It is a quite obvious question to ask for the tee-off velocity at which the ball lands in the net (or, more easily, hits the center of the ring that holds the net).



Figure 1. Minigolf station with ramp and net.

We start by drawing a simplified sketch of this minigolf station by using the graphics window in GeoGebra. We place the origin at the upper end of the ramp, which facilitates later calculations in the model (see Figure 2). As this station is a typical one in standard minigolf courses, one can find its dimensions in the set of rules of different minigolf associations (OEBGV).

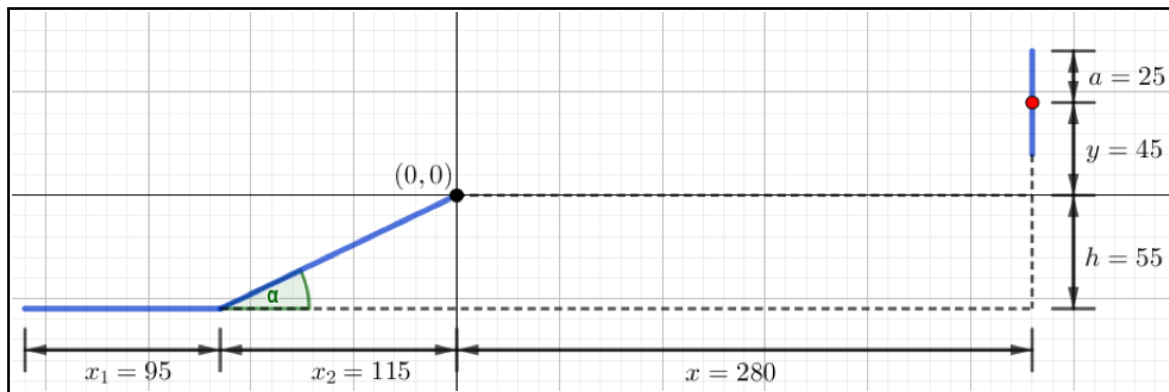


Figure 2. Sketch of the minigolf station with ramp, net (blue), center of the ring (red) and dimensions (in cm)

2 PHYSICAL BACKGROUND AND CALCULATION USING THE CAS WINDOW

In a first step, we neglect the balls' journey across the ramp and only consider its way through the air. Setting the origin at the upper end of the ramp makes the mathematical model quite easy. From a physical point of view there are (at least) two different solution approaches: One may separate the horizontal and the vertical trajectories of the ball (superposition principle) or one may use the equation of the time-independent parabolic trajectory (10; 5).

2.1 Separation of the horizontal and the vertical motion

We start with the first one. For convenience only, we neglect air resistance in our model (in fact, it does not play an important role as you can see later in the validation of our results), we assume the ball to be point-shaped and we search for the velocity, which the ball should have at (0;0) in order

to hit the center of the ring exactly. As the horizontal movement is uniform due to the neglect of air resistance, the flight time t^* equals $\frac{X}{v_x}$ (where v_x is the constant x -component of the velocity). The vertical component of the velocity is time-dependent and changes due to gravity:

$$v_y(t) = v_y(0) - g t$$

(where g is earth's gravitational acceleration). The vertical position of the ball is therefore given by

$$y(t) = v_y(0) t - \frac{1}{2} g t^2$$

As you can see in Figure 2, the velocity vector at $(0;0)$ must be inclined by the angle θ towards the horizontal, hence $\tan \theta = \frac{v_y(0)}{v_x}$ or $v_y(0) = v_x \tan \theta$ and thus

$$y(t) = v_x (\tan \theta) t - \frac{1}{2} g t^2$$

Further on, we let $y(t^*)$ equal y and replace t^* in this equation by $\frac{X}{v_x}$. We solve the resulting equation for v_x by hand or with the aid of GeoGebra's CAS and obtain:

$$v_x = \frac{X}{2y} \sqrt{\frac{g}{\tan^2 \theta}}$$

Finally, the absolute value of the velocity vector at $(0;0)$ yields

$$v_0 = \frac{X}{2y} \sqrt{g + (v_x \tan \theta)^2}$$

2.2 Time-independent parabolic trajectory

Another approach is to use the time-independent equation of the trajectory parabola, which students who are working on this modeling problem usually find on the internet or in their physics textbooks (where its derivation can be found as well):

$$y(x) = \frac{x^2 g}{2v_0^2 (\cos \theta)^2} + x \tan \theta$$

Letting $y(x)$ equal y , one can solve the equation for v_0 (see Figure 3, using the CAS of GeoGebra). In both cases, of course, we obtain $v_0 = 7.29$ m/s. This phenomenon is not coincidental. The two models lead to the same result for v_0 in general, independent from concrete numbers. This can be seen by solving the time-independent equation for v_0 and then using the identity $1 + \tan^2 \theta = \frac{1}{\cos^2 \theta}$. Hence, the two models are in fact the same. By the way, the velocity is independent from the mass of the ball. It is therefore irrelevant for the derivations above.

2.3 Consideration of the ramp

Still the loss of velocity caused by the ball's rolling up the ramp is missing. The ball gains the potential energy mgh , while rolling uphill. Its kinetic energy decreases to the same extent. This leads to the equation

$$\frac{m(v_0 + v)^2}{2} = \frac{mv_0^2}{2} + mgh$$

where v is the loss of velocity while rolling up the ramp. We obtain $v_0 + v = 8.00$ m/s for the tee-off velocity, again using the CAS of GeoGebra.

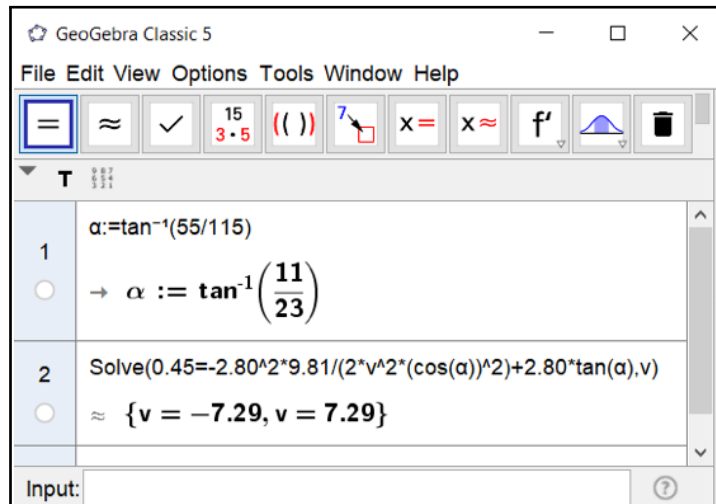


Figure 3. Calculation of v_0 , using GeoGebra’s CAS.

3 APPROXIMATION USING THE GRAPHICS WINDOW

While the above solution process focuses on the exact calculation of the velocity in the mathematical model (without air resistance), we now follow a more qualitative approach using GeoGebra’s graphics window. We plot the trajectory of the ball, using a slider for varying the value of v_0 (see Figure 4).

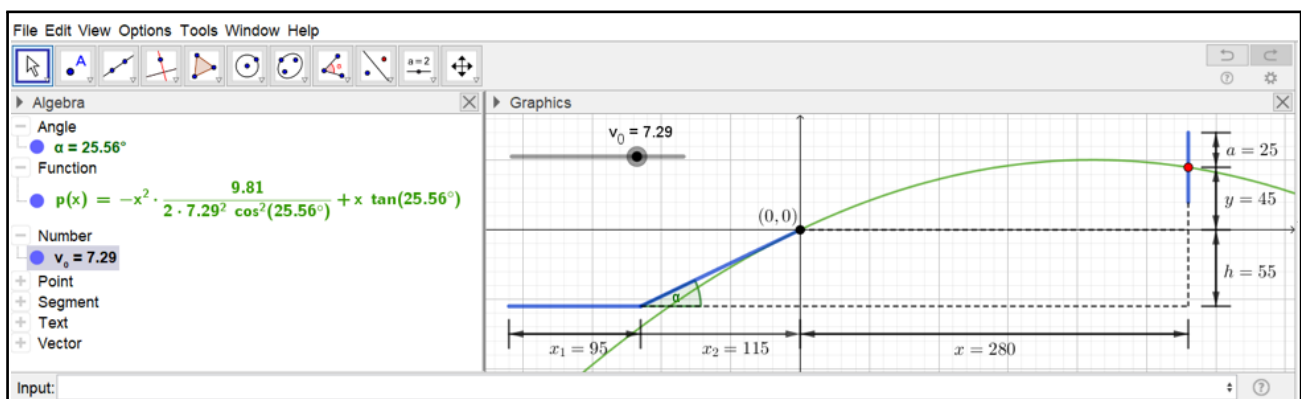


Figure 4. Trajectory of the ball, where v_0 can be varied by a slider.

As you can see, we obtain the same value $v_0 = 7.29$ m/s when aiming at the red dot in the center of the ring. This GeoGebra applet allows experimenting in the mathematical model. This way, it is easy to find out, in which interval the velocity v_0 has to be, in order to hit the net anywhere (not required in its center). Furthermore, one can easily change the input variables like the height of the ramp, its slope, the distance between the ramp and the net, and so on.

This solution process addresses visual-style learners. One can change some parameter in the model while observing the effects directly in the graphical visualization.

4 EMPIRICAL VALIDATION USING THE SPREADSHEET

One nice feature of this modeling problem is the possibility of validating the results from the mathematical model by carrying out a video analysis of a real tee shot that leads to a direct hit. You simply need a camera (e.g. the one on your smartphone) and a software, which helps you to read out the coordinates of the golf ball in each frame of the video. In the present case, we used the minigolf station that you can see in Figure 1 and the free cell phone app “VidAnalysis”, which provides a data table as output. One can then transfer this data to the spreadsheet of GeoGebra via copy and paste. By creating a list of points, these points appear in the graphics window (see Figure 5).

With the help of visual judgement, one can adjust the velocity v_0 , so that the calculated trajectory of the ball fits the data well (of course you can also use regression). In Figure 5 we obtain $v_0 = 6.9$ m/s.

The frame rate of the camera, which was used for collecting this data, is 20 frames per second, so the time-span between two pictures equals 0.05 s. We can use this information to calculate the velocity of the ball right after leaving the ramp directly from the data. Between the first two pictures, the ball covered a distance of $\sqrt{(0.37 - 0.05)^2 + (0.16 - 0.12)^2} = 0.35$ m, which gives us an average velocity of $v_0 = \frac{0.35}{0.05} = 7$ m/s in this time-span. As you can see, the model fits reality quite well.

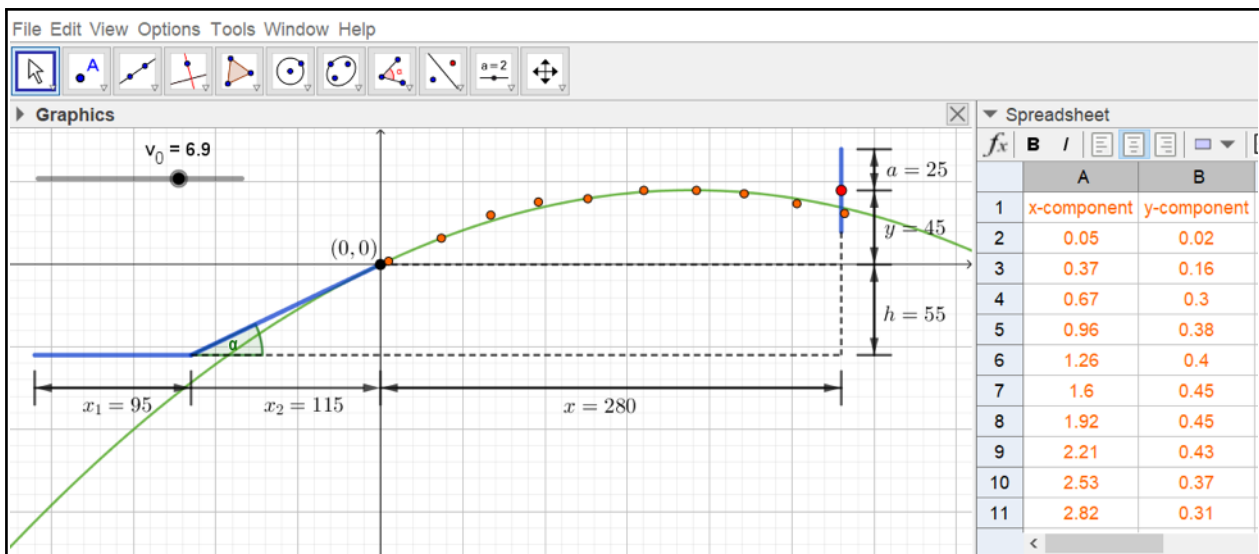


Figure 5. Collected data (orange, dimensions in cm) and fitted trajectory (green).

5 DISCUSSION

Realistic modeling problems are often too complex to use them in mathematics lessons. Conversely, situations that are easy to model are often not realistic (8). In this article, we describe a problem, which is easily accessible for learners and enhances their motivation. Anyway, various solutions can be found using quite simple school mathematics. We want to show that GeoGebra can serve as a tool in many ways for visualizing, structuring and solving the present modeling problem.

With the help of the graphics window the situation can be sketched easily. GeoGebra's CAS helps with the calculations, especially when working in a general model without concrete numbers. The ball's trajectory can be drawn in the graphics window as well, which provides a visual representation of the whole situation. Finally, the slider allows experimenting with the parameter v_0 . With its help, it is very natural to find the interval for the velocities that lead to a trajectory entering the net.

The present modeling problem leads to the activation of all four central phases of the modeling process mentioned above:

simplifying and structuring the real-world problem (neglecting air resistance and friction, reducing the problem into a 2D-problem, neglecting spin, etc.),

translating the problem into mathematical language (sketching the situation, determining suitable variables, recognizing relationships between certain concepts, etc.),

operating in this mathematical model (calculations in the model, using different approaches, experimenting with parameters, etc.),

interpreting and validating the results (collecting real data, comparing them to the results of the models).

Teachers can give this problem as a project to students in a quite simple manner. My own teaching experience indicates that posing the question "At which velocity do you have to tee off the ball in order to hit the net?" and providing a picture as shown in figure 1 is sufficient for the beginning. Students start talking about the problem in small groups, sketch the situation and search for physical relationships on their own (possibly by using physics textbooks or the internet). The teacher should hold back in this phase of the lesson and only should be available for questions. Students should be free to choose their own approach, solution process and mathematical techniques. The teacher's role is to tutor these learning processes with possible expedient solutions in mind. Therefore, the above-mentioned approaches are not binding for students but relevant for teachers in order to compare with students' results. Of course, an outdoor trip to a minigolf course is advisable at some stage of the solution process, either for starters or as an opportunity for validating the results afterwards. Basic skills in GeoGebra and competencies in using a video analysis app (e.g. VidAnalysis) are prerequisites for working on this modeling problem.

One can vary the problem according to the students' capabilities. For example, the ball does not have to be seen as "point-shaped", air resistance can be included into the solution process, as well as the friction between the ball and the ramp. Quite similar approaches can be used to solve problems like the perfect free-throw in basketball or the perfect serve in tennis. These problems are a bit more sophisticated because of the additional degree of freedom one gets by varying the throwing angle.

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