EXPLORING AND SOLVING FEYNMAN'S TRIANGLE THROUGH MULTIPLE APPROACHES

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Abstract

It has long been recognized that using multiple approaches to solve problems is essential for students to obtain understanding of mathematical concepts. In view of this, we consider an interesting plane geometry problem with a straightforward formulation, known as the one-seventh area triangle problem. We present solutions using GeoGebra constructions and manipulation, coordinate geometry, Euclidean geometry and linear algebra. This allows the application of many of the mathematical tools acquired at the secondary level and makes important and crucial connections between them. The linear algebra section can be used as an introduction to the subject. This section can also reinforce the close relationship between linear algebra and geometry which might not receive enough emphasis at the undergraduate level. GeoGebra diagrams, constructions and computer algebra are used throughout the paper. All explanations are done through questions and answers which allows instructors to easily format the sections into inquiry-based lessons.

Keywords: GeoGebra, one-seventh area triangle, geometry, algebra, multiple representations, inquiry-based teaching

1 INTRODUCTION

In this paper we consider the one-seventh area triangle problem as in (Wikipedia, 2019). In this problem, as in Figure 1 below, we construct an inner triangle $\triangle EDF$ by connecting each vertex of the triangle $\triangle BAC$ with a trisection point on the opposite side. The ratio of the area of the inner triangle to the area of the outer triangle is then always one-seventh. The one-seventh area triangle problem is a simple, yet rich, plane geometry problem which has many generalizations and admits many methods of proof (Clarke, 2007; Cook & Wood, 2004; de Villiers, 2005a,b; Man, 2009). We present, solve and explore the problem through GeoGebra constructions in a manner that is accessible to secondary and tertiary school students in a question and answer format. All GeoGebra constructions used in this paper are available at `https://www.geogebra.org/m/xwvn4fss`.

The primary audience for the paper is mathematicians and instructors of undergraduate mathematics. The approaches taken in the paper are perfect for secondary education majors who are taking an upper-level geometry course. For instructors, this paper provides a way to make important connections between the basic mathematical approaches of Euclidean geometry, coordinate geometry and linear algebra through the simple yet engaging one-seventh area inner triangle problem. The GeoGebra constructions, and the use of CAS, are completed – as the paper is intended for instructors. Two worksheets in Appendix A are included to demonstrate how the material can be adapted for use in a classroom. Further, any of the GeoGebra constructions can readily be modified to support inquiry-based lessons as demonstrated at the end of Exploration 1 in Section 2.1. Along these lines, at the end of Exploration 2 in Section 2.2, we provide student tasks in a GeoGebra worksheet which shows how this question may be presented to students.

The multiple approaches presented in this paper for the one-seventh area triangle problem support recommendations found in the 2015 CUPM Curriculum Guide to Majors in the Mathematical Sciences (Zorn et al., 2015):

- Major programs should include activities designed to promote students' progress in learning to: use and compare analytical, visual and numerical perspectives in exploring mathematics (Cognitive Recommendation 1: Students should develop effective thinking and communication skills, p. 9)
- Mathematical sciences major programs should teach students to use technology effectively, both as a tool for solving problems and as an aid to exploring mathematical ideas (Cognitive Recommendation 3: Students should learn to use technological tools, p. 10).
- Students majoring in the mathematical sciences should learn to read, understand, analyze and produce proofs at increasing depth as they progress through a major (Content Recommendation 2, p. 10).

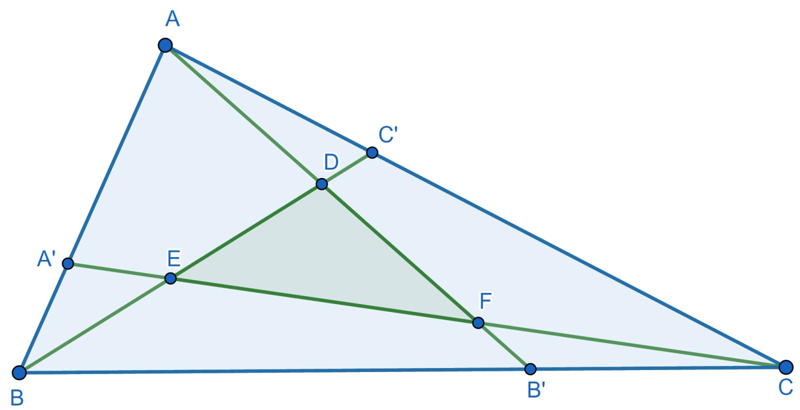

Figure 1. Construction of Inner Triangle

This paper is organized as follows. In Section 2, we present a series of explorations as GeoGebra constructions to familiarize the reader with the geometry of the problem and to present "drag tests" and "picture proofs" as suggested by de Villiers (1999) and Lehrer & Chazan (2012).

In Section 3, we present a proof of the problem in the case of an equilateral triangle using coordinate geometry. A parameter is introduced by taking "n-sections" of the sides in the inner triangle construction. This results in the derivation of an interesting formula for the ratio of the area of the inner to the outer triangle in terms of this parameter. The discussion and solution in this section come directly from Warkentin (1992) with additional explanations, an interactive GeoGebra construction, and computer algebra. In Section 4, a proof of the one-seventh inner triangle problem, developed by the authors, is given using Euclidean geometry. In Section 5.1, we develop a proof using the tools of linear algebra. The proof can be done directly on the construction using vector algebra (Cook & Wood, 2004), but we first solve the problem on a right triangle and then consider a linear transformation. This involves a simpler, initial vector computation in the right triangle and then introduces and brings to bear crucial properties of linear transformations. Lastly, we give some final remarks.

2 USING GEOGEBRA TO EXPLORE THE ONE-SEVENTH TRIANGLE

In this section we will use GeoGebra constructions to explore the ratio of the area of a constructed inner triangle to the area of the outer triangle. We will investigate the problem through a "drag test", where we can observe the ratio for many types of triangles, by considering component parts of the construction and through division and rotation of the construction in GeoGebra. We provide a similar construction in a parallelogram in Appendix B.

2.1 Exploration 1: Inner Triangle Construction

As in Figure 1, construct an inner triangle $\triangle EDF$ by connecting each vertex of the triangle $\triangle BAC$ with a trisection point on the opposite side. Each trisection point is one-third of the way along the line segment opposite the vertex in the clockwise direction, so that

$$BA' = (1/3)BA,\ AC' = (1/3)AC \text{ and } CB' = (1/3)CB. \tag{1}$$

To find trisection points, use the dilate tool from the transform menu.

Question 2.1 *Measure* area($\triangle EDF$) *and* area($\triangle BAC$). *Then compute the ratio*

$$\frac{\text{area}(\triangle EDF)}{\text{area}(\triangle BAC)}.$$

Drag the vertices of the triangle and observe the ratios of the areas of the triangles. Based on this "drag test" what can you say about this construction? Does the "drag test" prove your conjecture?

See `https://www.geogebra.org/m/kmkdbfkp` for the construction. Based on the drag test, and inspecting the area measures, the ratio of the area of the inner triangle (formed by segments from each vertex to an opposite trisection point) to the area of the outer triangle is always 1 to 7. The "drag test" strongly suggests this conjecture, but does not prove it as it is impossible to check all cases this way. Using different methods, we give proofs of this ratio in Sections 3, 4, and 5.

In this construction, we use the `FractionText` command to represent the ratio as a fraction rather than a rounded decimal. To modify the construction for an inquiry-based approach, the instructor can open the file in the GeoGebra application, delete the four text boxes and ask students to use the input bar, the spreadsheet, or a calculator to make the measurement for themselves. Note that the default setting will show 0.1 for the ratio. In order to see the repeating pattern for 1/7, the student would need to change the global setting to 10 decimal places.

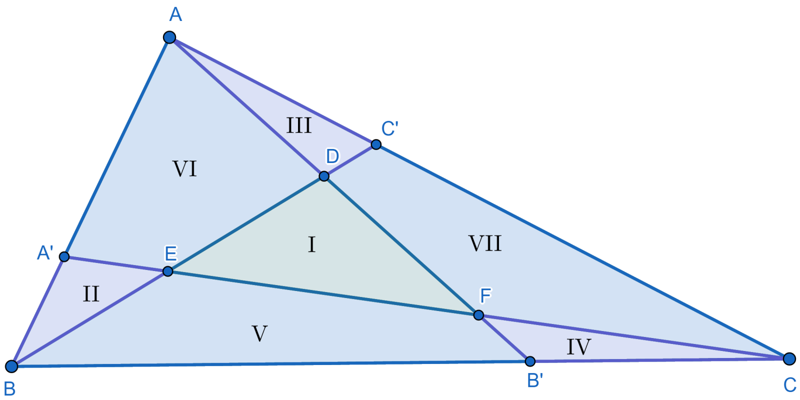

Figure 2. Other Components of Triangle

2.2 *Exploration 2: Other Components of Inner Triangle Construction*

In this exploration we investigate other components of triangle $\triangle BAC$ in the construction in Exploration 1. Continuing with the construction from Exploration 1 and as in Figure 2, consider the interiors of the three quadrilaterals V, VI, and VII, each of which share a side with the inner triangle. Measure the area of each quadrilateral. Calculate the ratio of the area of each of the quadrilaterals to the area of the outer triangle. Do the same for the three triangles on the corners, II, III and IV, each of which share a vertex with the inner triangle.

Question 2.2

(a). What can you say about the area of the inner quadrilaterals compared to the area of the outer triangle $\triangle BAC$?

(b). What can you say about the area of the corner triangles compared to the area of the outer triangle $\triangle BAC$?

(c). Do your answers to a. and b. agree with your answer to Question 2.1?

See `https://www.geogebra.org/m/vnffpxtv` for a student activity and `https://www.geogebra.org/m/mnruvmq8` for the completed construction. Based on the drag test, and inspecting the area measures, the areas of the quadrilaterals are all the same. The ratio of the area of any of the quadrilaterals to the area of the outer triangle is 5 to 21. The areas of all the triangles on the corners are the same as well, and the ratio of the area of any of the corner triangles to the area of the outer triangle is 1 to 21. A proof of these ratios is given in the answer to Question 4.5 in Section 4. Also, we have

$$\frac{\text{area}(I)}{\text{area}(\triangle BAC)} = \frac{\text{area}(\triangle BAC) - 3\,\text{area}(II) - 3\,\text{area}(IV)}{\text{area}(\triangle BAC)} = 1 - 3\left(\frac{5}{21}\right) - 3\left(\frac{1}{21}\right) = \frac{1}{7}$$

which should agree with your conjecture in the answer to Question 2.1.

2.3 *Exploration 3: Picture Proof of Ratio of Areas*

In this exploration, we provide a "picture proof" through a GeoGebra construction of the answer to Question 2.1. This picture proof, as given by Johnston (1992), will consist of rotating and dividing pieces of the outer triangle to show that it does, in fact, consist of seven triangles of equal areas to the inner triangle. Continuing with the GeoGebra construction from Exploration 1, complete the following steps.

1. Use the polygon tool to make sure $\triangle EDF$ and $\triangle BAC$ are constructed.
2. Hide line segments $\overline{CA'}$, $\overline{BC'}$ and $\overline{AB'}$ and trisection points A', B' and C'.
3. Construct midpoints on segments $\overline{BA}$, $\overline{AC}$ and $\overline{CB}$. Label these midpoints X, Y and Z, respectively.
4. As in Figure 3 below, construct corner triangles $\triangle BXE$, $\triangle AYD$ and $\triangle CZF$, each of which has a side that is a line segment from a vertex of the inner triangle $\triangle EDF$ to the closest midpoint. See `https://www.geogebra.org/m/btsubjbp` for the construction.

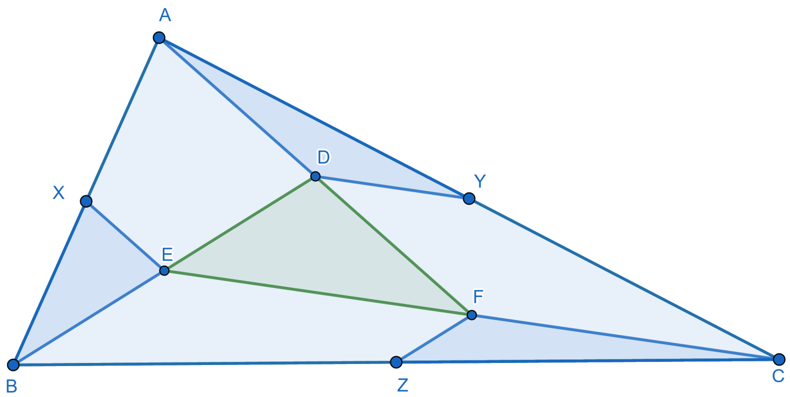

Figure 3. Construct Corner Triangles

5. On each side of the outer triangle, rotate the corner triangle 180 degrees about the midpoint of that side clockwise. For example, rotate triangle $\triangle BXE$ about point X clockwise. Hide any duplicate point names that are created. See `https://www.geogebra.org/m/vhjkzxme` for the construction.
6. Hide each corner triangle and the midpoints of the sides of the outer triangle.
7. Use the polygon tool to construct each of the quadrilaterals on the corners and make each a different color.
8. In each quadrilateral construct a diagonal as a dashed segment with an endpoint outside the outer triangle. See `https://www.geogebra.org/m/v6ypbk7p` for the construction.

As in Figure 4, the construction results in seven triangles, the inner triangle, and six triangles in the quadrilaterals.

Question 2.3

1. *Measure the lengths of all three sides of the inner triangle. Do the same for the triangles formed inside the quadrilaterals.*

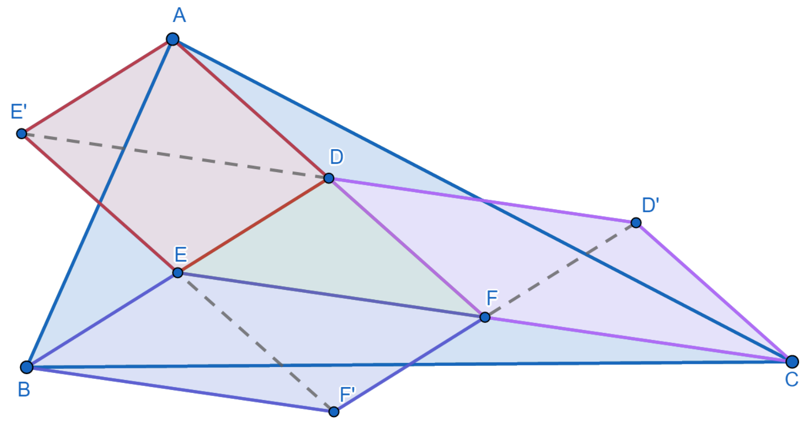

Figure 4. Picture Proof Construction

2. *What can you conclude about each of the seven triangles? How does this demonstrate your conjecture from Exploration 1?*

3. *What types of quadrilaterals have been formed?*

See `https://www.geogebra.org/m/nudrjufr` for the construction. Based on inspecting side lengths of the triangles during a drag test, these triangles appear to be congruent and thus have the same area. Because there are seven of them, which together make up the outer triangle, the ratio of the area of the inner triangle to the outer triangle appears to be 1 to 7. While conducting a drag test, the opposite side lengths of the quadrilaterals remain the same. As a result, the quadrilaterals appear to be parallelograms.

Also, see `https://www.geogebra.org/m/mpkmuvu4` (Flores, 2019a) for an interactive construction which allows you to control the angle of rotation of the corner triangles up to 180 degrees. Another interesting picture proof is given by Steinhaus (1999). In this construction, a line segment is drawn through each of the vertices of the inner triangle, parallel to the opposite side of the inner triangle. One of the two parts of each of the quadrilaterals is then rotated 180 degrees to form six triangles, each congruent to the inner triangle. See `https://www.geogebra.org/m/x5udzcnu` (Flores, 2019b) for an interactive construction which allows rotation of component parts up to 180 degrees.

3 Coordinate Geometry and the One-Seventh Triangle

In this section, for the case of an equilateral triangle, we will prove the conjecture in the answer to Question 2.1 as given by Warkentin (1992). We use the coordinate system, formulas from analytic geometry (such as the distance and slope formulas) and algebra. We will derive a formula which will confirm the ratio not only when vertices are connected to trisection points but which will also give the ratio when the vertices are connected to "n-section" points where $n \geq 2$.

Without loss of generality, consider an equilateral triangle with sides of length $2a$, with a vertex placed at the origin, and a side along the positive x-axis.

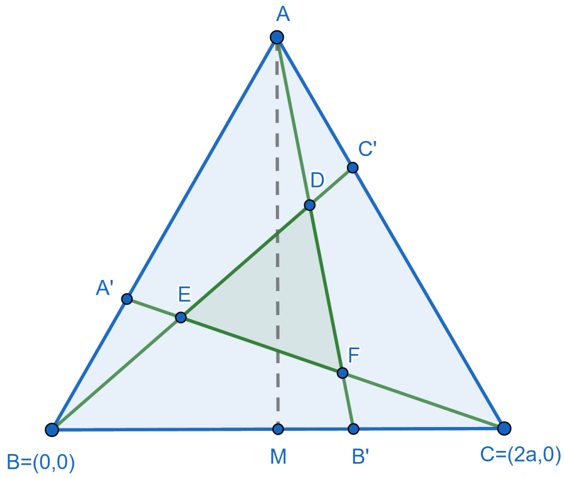

Figure 5. Equilateral Triangle

In Figure 5 assume that $\triangle BAC$ is equilateral and that

$$BA' = (1/n)BA,\ AC' = (1/n)AC \text{ and } CB' = (1/n)CB. \tag{2}$$

Question 3.1

(a). Find the coordinates of A, A', B' and C'.

(b). Find the equations of the lines $\overleftrightarrow{AB'}$, $\overleftrightarrow{CA'}$ and $\overleftrightarrow{BC'}$ in slope-intercept form.

(c). Find the coordinates of the intersections E, D and F.

To find the coordinates of A, drop a perpendicular to the midpoint, $M = (a, 0)$, of $\overline{BC}$. Then $\triangle MAC$ is a 30-60-90 triangle with base of length $MC = a$ and height of $MA = \sqrt{3}a$. Thus, we have $A = (a, \sqrt{3}a)$. C' is the point $(1/n)$th of the way from A to C. Thus, we have

$$C' = (1 - 1/n)(a, \sqrt{3}a) + (1/n)(2a, 0) = \left(\frac{na + a}{n}, \frac{na\sqrt{3} - a\sqrt{3}}{n}\right).$$

Similarly, we have

$$A' = (1 - 1/n)(0, 0) + (1/n)(a, a\sqrt{3}) = \left(\frac{a}{n}, \frac{a\sqrt{3}}{n}\right) \text{ and}$$
$$B' = (1 - 1/n)(2a, 0) + (1/n)(0, 0) = \left(\frac{2an - 2a}{n}, 0\right).$$

To find the equation of $\overleftrightarrow{AB'}$ first find its slope

$$m = \frac{\sqrt{3}a - 0}{a - (2an - 2a)/n} = \frac{\sqrt{3}n}{2 - n}.$$

Then plug the coordinates of either A or B' into $y = mx + b$ and solve for b. Plugging in the coordinates of B' gives

$$0 = \frac{\sqrt{3}n}{2-n}\frac{2an-2a}{n} + b \implies b = \frac{2a\sqrt{3}(1-n)}{2-n}.$$

Thus, the equation of $\overleftrightarrow{AB'}$ is

$$y = \frac{\sqrt{3}n}{2-n}x + \frac{2a\sqrt{3}(1-n)}{2-n}.$$

Similarly, solving for the equations of $\overleftrightarrow{CA'}$ and $\overleftrightarrow{BC'}$ gives, respectively,

$$y = \frac{\sqrt{3}}{1-2n}x - \frac{2a\sqrt{3}}{1-2n} \quad \text{and} \quad y = \frac{\sqrt{3}(n-1)}{n+1}x. \tag{3}$$

As E is the intersection of $\overleftrightarrow{CA'}$ and $\overleftrightarrow{BC'}$, using (3) and solving for the x-coordinate of E,

$$\begin{aligned}&\frac{\sqrt{3}}{1-2n}x - \frac{2a\sqrt{3}}{1-2n} = \frac{\sqrt{3}(n-1)}{n+1}x \implies [(n-1)(1-2n)-(n+1)]x = -2a(n+1)\\ &\implies x = \frac{-2a(n+1)}{(n-1)(1-2n)-(n+1)} = \frac{a(n+1)}{n^2-n+1}.\end{aligned}$$

Plugging this x-coordinate into the equation of $\overleftrightarrow{BC'}$ in (3) gives the y-coordinate of E and we have

$$E = \left(\frac{a(n+1)}{n^2-n+1}, \frac{\sqrt{3}a(n-1)}{n^2-n+1}\right). \tag{4}$$

Finding the coordinates of D and F in the way that we found the coordinates of E gives

$$D = \left(\frac{a(n-1)(n+1)}{n^2-n+1}, \frac{\sqrt{3}a(n-1)^2}{n^2-n+1}\right) \text{ and } F = \left(\frac{a(2n^2-4n+3)}{n^2-n+1}, \frac{\sqrt{3}a}{n^2-n+1}\right). \tag{5}$$

Question 3.2

(a). *Find ED, DF and FE. What can you say about $\triangle EDF$?*

(b). *Show*

$$\frac{\text{area}(\triangle EDF)}{\text{area}(\triangle BAC)} = \frac{(n-2)^2}{n^2-n+1} \tag{6}$$

by using your result from part (a) to relate the area formulas of $\triangle BAC$ and $\triangle EDF$.

(c). *Find*

$$\frac{\text{area}(\triangle EDF)}{\text{area}(\triangle BAC)}$$

for $n = 3, n = 4$ and $n = 5$.

Using the distance formula, our assumption that $n-2>0$, $n^2-n+1>0$ for all n and (4) and (5) we have

$$\begin{aligned} ED &= \sqrt{\left(\frac{a(n-1)(n+1)-a(n+1)}{n^2-n+1}\right)^2+\left(\frac{a\sqrt{3}(n-1)^2-a\sqrt{3}(n-1)}{n^2-n+1}\right)^2} \\ &= \frac{a(n-2)}{n^2-n+1}\sqrt{(n+1)^2+\left(\sqrt{3}(n-1)\right)^2} = \frac{2a(n-2)}{n^2-n+1}\sqrt{n^2-n+1} \\ &= \frac{2a(n-2)}{\sqrt{n^2-n+1}}. \end{aligned} \tag{7}$$

Again, using the distance formula

$$DF = FE = \frac{2a(n-2)}{\sqrt{n^2-n+1}}. \tag{8}$$

Therefore, $\triangle EDF$ is an equilateral triangle, similar to equilateral triangle $\triangle BAC$. Letting a_1 be the altitude of $\triangle EDF$ and a_2 be altitude of $\triangle BAC$ we then have

$$\frac{a_2}{a_1} = \frac{AB}{ED} \implies a_2 = a_1\frac{AB}{ED}.$$

Thus, using (7) we have

$$\begin{aligned} \frac{\text{area}(\triangle EDF)}{\text{area}(\triangle BAC)} &= \frac{(1/2)EDa_1}{(1/2)ABa_2} = \frac{EDa_1}{AB[a_1(AB/ED)]} = \frac{ED^2}{AB^2} \\ &= \frac{\left[2a(n-2)/\sqrt{n^2-n+1}\right]^2}{(2a)^2} = \frac{(n-2)^2}{n^2-n+1}. \end{aligned} \tag{9}$$

A construction demonstrating this formula can be found here `https://www.geogebra.org/m/bduegj56`. Computer algebra using GeoGebra CAS to find (4), (5), (7), (8), and (9) can be found here `https://www.geogebra.org/m/qhpceapj`.

We have the ratios

$$\frac{(3-2)^2}{3^2-3+1} = \frac{1}{7} \text{ for } n=3,\ \frac{(4-2)^2}{4^2-4+1} = \frac{4}{13} \text{ for } n=4, \text{ and } \frac{(5-2)^2}{5^2-5+1} = \frac{3}{7} \text{ for } n=5. \tag{10}$$

The first equality agrees with the answer we found to Question 2.1. Also, we assumed that $n>2$. But if $n=2$ the sides of $\triangle BAC$ are bisected and $\overline{BC'}$, $\overline{CA'}$ and $\overline{AB'}$ meet at a single point called *the centroid*. This is consistent with our formula as

$$\frac{(2-2)^2}{2^2-2+1} = 0 \tag{11}$$

which is the area of $\triangle EDF$ as it collapses to a single point. Thus, our formula holds for $n \geq 2$.

Question 3.3

(a). Show that only for $n = 3$

$$\frac{\text{area}(\triangle EDF)}{\text{area}(\triangle BAC)} = \frac{(n-2)^2}{\sqrt{n^2-n+1}} = \frac{1}{m} \tag{12}$$

for some positive integer m, and thus, only for $n = 3$ is area$(\triangle BAC)$ *a positive integer multiple of* area$(\triangle EDF)$.

(b). Given your answer from part (a), for what values of n is a construction such as in Section 2.3 possible? Why?

(c). What happens as $n \to \infty$?

Because every unit fraction, i.e., every fraction of the form $1/m$ for some positive integer m, is less than or equal to 1/2 consider

$$\frac{\text{area}(\triangle EDF)}{\text{area}(\triangle BAC)} = \frac{(n-2)^2}{n^2-n+1} \leq \frac{1}{2} \iff n^2 - 7n + 7 \leq 0 \iff 1.21 \leq n \leq 5.79.$$

where the last inequality comes from solving the corresponding quadratic equation in the previous inequality and by choosing three test points (or by considering the graph of the corresponding upward turning parabola).

Therefore, for (12) to hold we must have $n = 2, 3, 4, 5$. But by (10) and (11), among these values of n, (12) only holds for $n = 3$ when $m = 7$. Thus, constructions such as the one in Section 2.3 are possible only when using trisection points.

However, similar constructions are possible for other values of n. For example, if $n = 4$ we know by (10) the ratio of the area of the inner to the outer triangle is 4 to 13. As in Figure 6 below, we can then divide the inner triangle into four congruent triangles and split the remaining part of the outer triangle into nine more triangles, each congruent to the inner four. The construction can be found here `https://www.geogebra.org/m/hq3snkae`. This construction was taken from (Warkentin, 1992) which also includes constructions for the cases when $n = 5$ and $n = 6$. For an interactive construction which allows rotation of the corner triangles up to 180 degrees see `https://www.geogebra.org/m/dfppga4m`.

We have

$$\lim_{n\to\infty} \frac{(n-2)^2}{n^2-n+1} = \lim_{n\to\infty} \frac{n^2-4n+4}{n^2-n+1} = \lim_{n\to\infty} \frac{1-4/n+4/n^2}{1-1/n+1/n^2} = 1.$$

This is consistent with the geometric interpretation of the formula because the inner triangle takes up more area and tends towards taking up the whole area of the outer triangle as $n \to \infty$.

4 Euclidean Geometry and the One-Seventh Triangle

In this section, we give a proof using Euclidean geometry, without additional mathematical tools implemented in Section 3 or Section 5. This proof relies on the fact that the area of a triangle equals one-half the length of its base times the length of the altitude and some algebra.

Assume that in Figure 7 we have (1) as in Section 2.1.

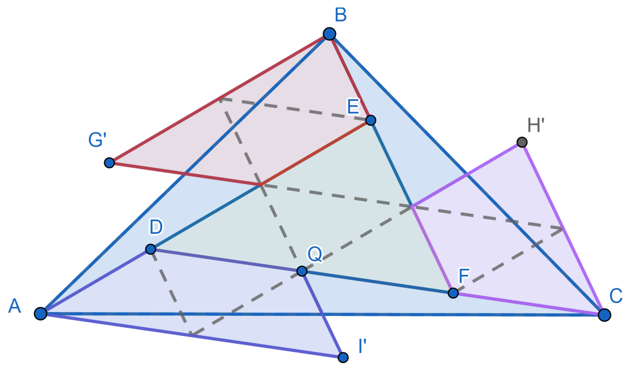

Figure 6. n=4 with 4/13 ratio

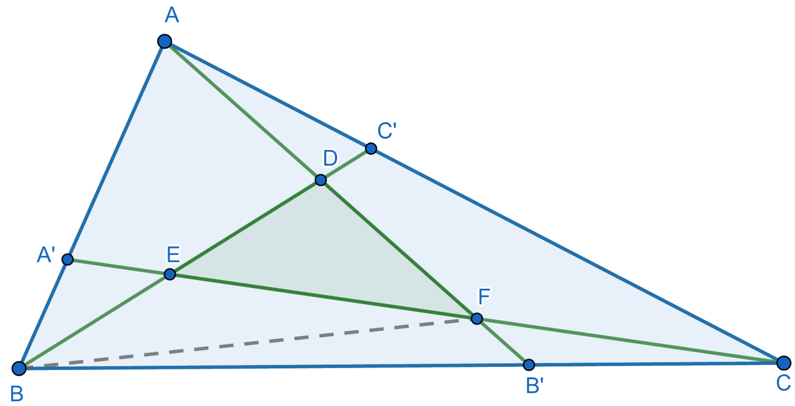

Figure 7. Diagram for Euclidean Proof

Question 4.1 *We know that the area of any triangle equals one-half the length of its base times the length of the altitude. Using this fact and* (1)*, what can you say about the relationship between* area$(\triangle BAB')$ *and* area$(\triangle B'AC)$ *and the relationship between* area$(\triangle BFB')$ *and* area$(\triangle B'FC)$*? Use this to show*

$$\text{area}(\triangle BAF) = 2\,\text{area}(\triangle FAC). \tag{13}$$

$\triangle BAB'$ and $\triangle B'AC$ share an altitude, the perpendicular line segment drawn from A to $\overline{BC}$. Let a_1 be the length of this altitude. Using (1) we have

$$B'B = CB - CB' = CB - (1/3)CB = (2/3)CB.$$

Thus,

$$\text{area}(\triangle BAB') = (1/2)a_1 B'B = a_1(1/3)CB = a_1(CB') = 2\,\text{area}(\triangle B'AC).$$

Letting a_2 be the length of the altitude drawn from F to $\overline{BC}$ we see that

$$\text{area}(\triangle BFB') = (1/2)a_2 B'B = a_2(1/3)CB = a_2(CB') = 2\,\text{area}(\triangle B'FC).$$

Solving we then have

$$\begin{aligned}&\text{area}(\triangle BAB') - \text{area}(\triangle BFB') = 2[\text{area}(\triangle B'AC) - \text{area}(\triangle B'FC)]\\ &\Rightarrow \text{area}(BAF) = 2\,\text{area}(FAC).\end{aligned}$$

Question 4.2 *What can you say about the relationship between* area$(\triangle A'AC)$ *and* area$(\triangle BA'C)$ *and the relationship between* area$(\triangle A'AF)$ *and* area$(\triangle BA'F)$*? Use this to show*

$$\text{area}(\triangle FAC) = 2\,\text{area}(\triangle BFC). \tag{14}$$

$\triangle A'AC$ and $\triangle BA'C$ share an altitude and $\triangle A'AF$ and $\triangle BA'F$ share an altitude. Thus, using (1) as in Question 4.1, we see

$$\text{area}(\triangle A'AC) = 2\,\text{area}(\triangle BA'C) \text{ and } \text{area}(\triangle A'AF) = 2\,\text{area}(\triangle BA'F).$$

Solving we have,

$$\begin{aligned} &\text{area}(\triangle A'AC) - \text{area}(\triangle A'AF) = 2[\text{area}(\triangle BA'C) - \text{area}(\triangle BA'F)] \\ &\Rightarrow \text{area}(\triangle FAC) = 2\,\text{area}(\triangle BFC). \end{aligned}$$

Question 4.3 *Use* (13) *and* (14) *to show*

$$\text{area}(\triangle FAC) = (2/7)\,\text{area}(\triangle BAC). \tag{15}$$

Notice that $\triangle BAF, \triangle FAC,$ and $\triangle BFC$ do not overlap and make up $\triangle BAC$.

Using (13) and (14) we have

$$\begin{aligned} &\text{area}(\triangle BAC) = \text{area}(\triangle BAF) + \text{area}(\triangle FAC) + \text{area}(\triangle BFC) \\ &= 2\,\text{area}(\triangle FAC) + \text{area}(\triangle FAC) + (1/2)\,\text{area}(\triangle FAC) \\ &= (7/2)\text{area}(\triangle FAC) \end{aligned}$$

which gives (15). Using similar arguments as above we also have

$$\begin{aligned} &\text{area}(\triangle BAD) = (2/7)\,\text{area}(\triangle BAC), \text{ and} \\ &\text{area}(\triangle BEC) = (2/7)\,\text{area}(\triangle BAC). \end{aligned} \tag{16}$$

Question 4.4 *Finally, use* (15) *and* (16) *to show that*

$$\text{area}(EDF) = (1/7)\,\text{area}(BAC).$$

We have

$$\begin{aligned} &\text{area}(\triangle EDF) = \text{area}(\triangle BAC) - [\text{area}(\triangle FAC) + \text{area}(\triangle BAD) + \text{area}(\triangle BEC)] \\ &= \text{area}(\triangle BAC) - (6/7)\,\text{area}(\triangle BAC) = (1/7)\,\text{area}(\triangle BAC). \end{aligned}$$

Question 4.5 *Use* (14) *and* (15) *to prove the ratios conjectured in your answer to Question 2.2.*

By (14) and (15) we have

$$\text{area}(\triangle BFC) = (1/2)[(2/7)\text{area}(\triangle BAC)] = (1/7)\text{area}(\triangle BAC).$$

Then with $CB' = (1/3)CB$, and since $\triangle BFC$ and $\triangle B'FC$ share an altitude,

$$\text{area}(\triangle B'FC) = (1/3)[(1/7)\text{area}(\triangle BAC)] = (1/21)\text{area}(\triangle BAC).$$

Using similar arguments as above we also have

$$\text{area}(\triangle DAC') = \text{area}(\triangle BA'E) = (1/21)\text{area}(\triangle BAC).$$

Thus, using the labels in Figure 2, we have

$$\text{area}(VII) = (2/7)\text{area}(\triangle BAC) - (1/21)\text{area}(\triangle BAC) = (5/21)\text{area}(\triangle BAC).$$

Similarly, we have

$$\text{area}(VI) = \text{area}(V) = (5/21)\text{area}(\triangle BAC).$$

5 LINEAR ALGEBRA AND THE ONE-SEVENTH TRIANGLE

In this section, we find the ratio of the area of the inner triangle to the area of the outer triangle using powerful tools of linear algebra. We first introduce some facts that we will need. We first introduce facts of linear algebra that we use to solve the problem in the case of a right triangle and then, finding the correct linear transformation, we solve the problem in general.

5.1 Linear Algebra Facts

1. Let P be a parallelogram which has the vectors (a_1, a_2) and (b_1, b_2) as adjacent sides and let

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - b_1 a_2,$$

   that is, D is the determinant of the matrix with columns $(a_1, a_2)^\top$ and $(b_1, b_2)^\top$. Then we have

$$\text{area}(P) = |D|.$$

   Thus, for a triangle T which has the vectors (a_1, a_2) and (b_1, b_2) as sides

$$\text{area}(T) = (1/2)|D|.$$

2. If S is a shape in $\mathbb{R}^2$ and L is a linear transformation of $\mathbb{R}^2$, then

$$\text{area}[L(S)] = |\det(L)|\,\text{area}(S)$$

   where $\det(L)$ is the determinant of L.

3. A linear transformation maps line segments to line segments and preserves the ratio of distances of points on line segments to endpoints.

Discussion and proofs of Facts 1 and 2 can be found in Farin & Hansford (2013) and at Khan Academy (2019). Fact 3 can be seen by considering, as in Figure 8, the image of line segment $\overline{p_1 p_2}$ under linear transformation L,

$$L[p_1 + t(p_2 - p_1)] = L(p_1) + t[L(p_2) - L(p_1)] \text{ for } 0 \leq t \leq 1.$$

5.2 Finding the ratio of areas using Linear Algebra

In Figure 9 (a) we have a right triangle with inner triangle $\triangle E_1 D_1 F_1$. In Figure 9 (b), assume (1) of Section 2.1 holds.

Question 5.1 *The vector from $A_1 = (0, 1)$ to $(\frac{2}{3}, 0)$ is given by*

$$\left(\frac{2}{3}, 0\right) - (0, 1) = \left(\frac{2}{3}, -1\right)$$

and the vector from $(0, \frac{1}{3})$ to $C_1 = (1, 0)$ is given by

$$(1, 0) - \left(0, \frac{1}{3}\right) = \left(1, -\frac{1}{3}\right).$$

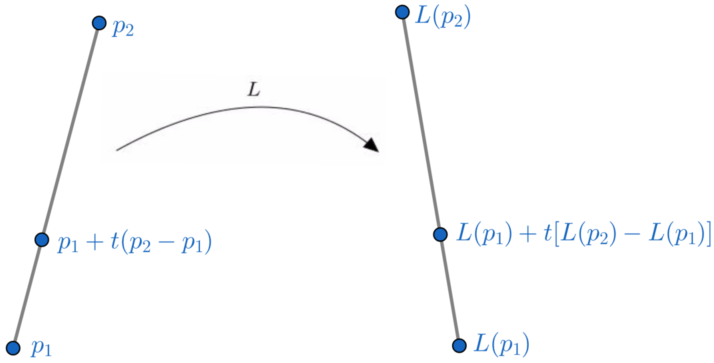

Figure 8. Linear Transformation of Line Segment

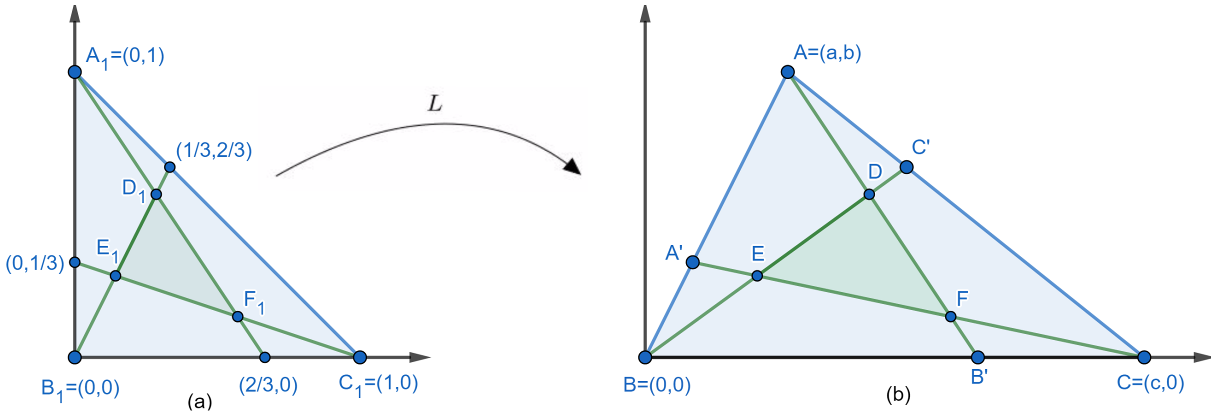

Figure 9. Linear Transformation of Construction in Right Triangle

Find the coordinates of intersection F_1 by solving

$$(0,1) + t\left(\frac{2}{3}, -1\right) = \left(0, \frac{1}{3}\right) + s\left(1, -\frac{1}{3}\right), \tag{17}$$

for s and t, that is, find "how far", as given by s and t, respectively, along each of the vectors we move to land on the intersection.

Equating coordinates in (17) and doing row operations,

$$\begin{aligned} (2/3)t - s &= 0 \\ t - (1/3)s &= 2/3 \end{aligned} \quad \longrightarrow \quad \begin{aligned} t - (3/2)s &= 0 \\ (7/6)s &= 2/3 \end{aligned} \quad \longrightarrow \quad \begin{aligned} t &= 6/7 \\ s &= 4/7. \end{aligned}$$

Thus, we have

$$F_1 = (0,1) + (6/7)\left(\frac{2}{3}, -1\right) = \left(\frac{4}{7}, \frac{1}{7}\right)$$

Question 5.2 *As you did in Question 5.1, find the coordinates of E_1 and D_1.*

To find E_1 solve

$$t\left(\frac{1}{3}, \frac{2}{3}\right) = \left(0, \frac{1}{3}\right) + s\left(1, -\frac{1}{3}\right) \tag{18}$$

Equating coordinates in (17) and doing row operations,

$$\begin{array}{rcrcr} (1/3)t - s = 0 & \longrightarrow & t - 3s = 0 & \longrightarrow & t = 3/7 \\ (2/3)t + (1/3)s = 1/3 & & (7/3)s = 1/3 & & s = 1/7. \end{array}$$

Thus, we have

$$E_1 = (3/7)\left(\frac{1}{3}, \frac{2}{3}\right) = \left(\frac{1}{7}, \frac{2}{7}\right).$$

Solving for D_1 as we solved E_1 and F_1 we have

$$D_1 = \left(\frac{2}{7}, \frac{4}{7}\right).$$

Question 5.3 *Using Linear Algebra Fact 1 show*

$$\frac{\text{area}(\triangle E_1 D_1 F_1)}{\text{area}(\triangle B_1 A_1 C_1)} = 1/7. \tag{19}$$

Start by finding two vectors, v_1 and v_2, which form two sides of $\triangle E_1 D_1 F_1$.

The vectors from E_1 to F_1 and E_1 to D_1 are respectively,

$$v_1 = \left(\frac{4}{7}, \frac{1}{7}\right) - \left(\frac{1}{7}, \frac{2}{7}\right) = \left(\frac{3}{7}, -\frac{1}{7}\right) \text{ and } v_2 = \left(\frac{2}{7}, \frac{4}{7}\right) - \left(\frac{1}{7}, \frac{2}{7}\right) = \left(\frac{1}{7}, \frac{2}{7}\right).$$

Thus,

$$\text{area}(\triangle E_1 D_1 F_1) = (1/2)\left|\left(\frac{3}{7}\right)\left(\frac{2}{7}\right) - \left(-\frac{1}{7}\right)\left(\frac{1}{7}\right)\right| = 1/14$$

and as $\text{area}(\triangle A_1 B_1 C_1) = 1/2$, we have (19).

Question 5.4 *Find a linear transformation, L, that maps $(1, 0)$ to C and $(0, 1)$ to A.*

L has matrix representation in the standard basis vectors, (0,1) and (1,0), of

$$M = \begin{pmatrix} c & a \\ 0 & b \end{pmatrix}.$$

By Linear Algebra Fact 3, all of the elements of construction (a) in Figure 9 are mapped onto to corresponding elements of construction (b) in Figure 9 by L. For example,

$$\begin{aligned} L\left[\left(\frac{1}{3}, \frac{2}{3}\right)\right] &= \begin{pmatrix} c & a \\ 0 & b \end{pmatrix}\begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} = \begin{pmatrix} (1/3)c + (2/3)a \\ (2/3)b \end{pmatrix} = \begin{pmatrix} c \\ 0 \end{pmatrix} + (2/3)\left[\begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} c \\ 0 \end{pmatrix}\right] \\ &= C + (2/3)(A - C) = C'. \end{aligned}$$

In particular, we have $L(\triangle B_1 A_1 C_1) = \triangle BAC$ and $L(\triangle E_1 D_1 F_1) = \triangle EDF$. Thus, by Linear Algebra Fact 2 and (19) we have

$$\frac{\text{area}(\triangle EDF)}{\text{area}(\triangle BAC)} = \frac{\text{area}[L(\triangle E_1 D_1 F_1)]}{\text{area}[L(\triangle B_1 A_1 C_1)]} = \frac{\det(L)[\text{area}(\triangle E_1 D_1 F_1)]}{\det(L)[\text{area}(\triangle B_1 A_1 C_1)]} = 1/7.$$

For an interactive construction that maps the right triangle by a linear transformation to $\triangle BAC$ and in which you can specify a, b, and c see `https://www.geogebra.org/m/egfp2mj6`.

6 Final Remarks

The one-seventh area problem provides an opportunity for instructors to extend a geometric viewpoint beyond a geometry course as geometric reasoning and visualization complement algebraic thinking in linear algebra in line with (Zorn et al., 2015) (Content Recommendation 6, p. 12). Instructors can use the many problems presented in this paper to demonstrate applications of geometric constructions in GeoGebra, coordinate geometry, Euclidean geometry, and linear algebra and the connections among all of them.

Section 5.1 can be introduced even to students who have not yet completed linear algebra course work but can also be used to demonstrate the close connection between geometry and linear algebra for those who have. Among many questions that naturally arise, a student might try to apply linear algebra to the formula (6) in Question 3.2 to all triangles by finding the correct linear transformations and using its properties.

Many generalizations, variations and similar problems as the one-seventh area triangle problem exist and they can all be considered using the approaches of geometric construction, coordinate and Euclidean geometry and linear algebra as demonstrated in this paper.

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A Appendix: Worksheets

Worksheet 1 - Inner Triangle Construction

1. In GeoGebra, construct a dynamic triangle, $\triangle BAC$. Dilate point C about A with factor 1/3 to create C'. Repeat, dilating point B about C to create B' and point A about B to create A'. Points A', B', and C' are called trisection points. You should have the relationships

$$BA' = (1/3)BA,\ AC' = (1/3)AC \text{ and } CB' = (1/3)CB. \tag{1}$$

Join vertex A to the trisection point on the side opposite A. Repeat for vertices B and C. Your sketch should look something like the one below. Note that triangle $\triangle EDF$ is formed by the intersection points of segments AB', BC', and CA'.

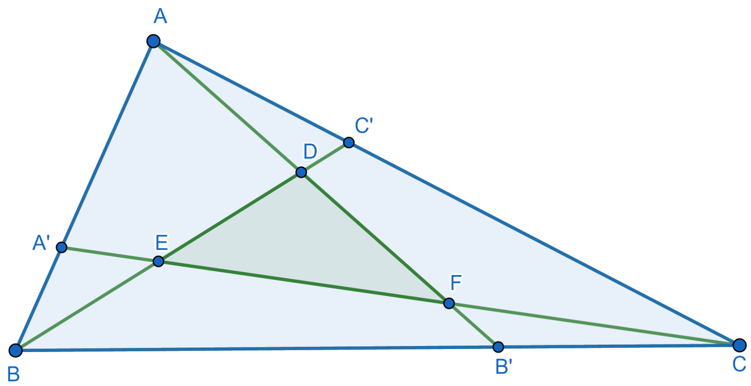

Figure 10. Construction of Inner Triangle

2. What relationships, if any, appear to exist between triangles $\triangle BAC$ and $\triangle EDF$? Justify your conjecture with evidence from your sketch. Include screenshots of your work along with written commentary.

3. Rigorously prove your conjecture using Euclidean geometry by answering the questions below.

Question 1 *The area of any triangle equals one-half the length of its base times the length of the altitude. Using this fact and* (1)*, what can you say about the relationship between* area($\triangle BAB'$) *and* area($\triangle B'AC$) *and the relationship between* area($\triangle BFB'$) *and* area($\triangle B'FC$)*? Use this to show*

$$\text{area}(\triangle BAF) = 2\,\text{area}(\triangle FAC). \tag{2}$$

Question 2 *What can you say about the relationship between* area($\triangle A'AC$) *and* area($\triangle BA'C$) *and the relationship between* area($\triangle A'AF$) *and* area($\triangle BA'F$)*? Use this to show*

$$\text{area}(\triangle FAC) = 2\,\text{area}(\triangle BFC). \tag{3}$$

Question 3 *Use* (2) *and* (3) *to show*

$$\text{area}(\triangle FAC) = (2/7)\,\text{area}(\triangle BAC). \tag{4}$$

Notice that $\triangle BAF$, $\triangle FAC$ and $\triangle BFC$ do not overlap and make up $\triangle BAC$.

Question 4 *Using similar arguments you used to show* (4) *you have*

$$\text{area}(\triangle BAD) = (2/7)\,\text{area}(\triangle BAC) \text{ and } \text{area}(\triangle BEC) = (2/7)\,\text{area}(\triangle BAC). \tag{5}$$

Using (4) *and* (5) *determine the relationship between* area($\triangle BAC$) *and* area($\triangle EDF$).

Worksheet 2 - Other Components of the Triangle

1. In your sketch from Worksheet 1 there are three distinct quadrilaterals which share a side with inner triangle $\triangle DEF$. Modify this sketch, constructing these quadrilaterals as polygons. Your sketch should also include three distinct "corner" triangles that share a vertex with the inner triangle. Modify your GeoGebra sketch, constructing these "corner" triangles as polygons. In the figure below, the "corner" triangles are labeled as II, III, and IV and the quadrilaterals are labeled as V, VI, and VII.

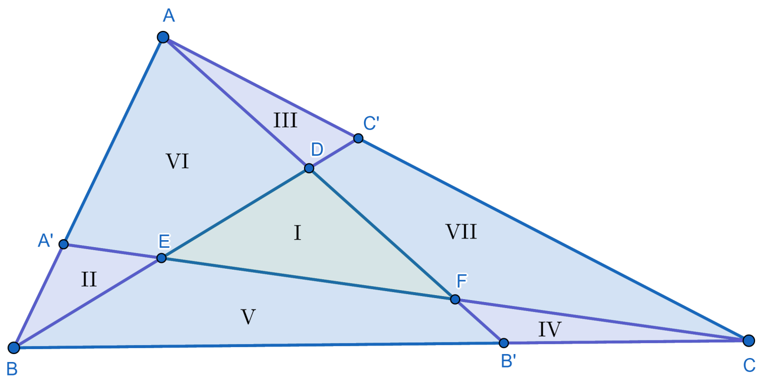

Figure 11. Other Components of Triangle

2. Calculate the ratio of the area of each of the "corner" triangles II, III and IV to the area of outer triangle, $\triangle BAC$. What do you notice? Justify your conjecture with evidence from your sketch. Include screenshots of your work along with written commentary.

3. Calculate the ratio of the area of each of the quadrilaterals, V, VI, and VII, to the area of outer triangle, $\triangle BAC$. What do you notice? Justify your conjecture with evidence from your sketch. Include screenshots of your work along with written commentary.

4. Rigorously prove your conjecture using Euclidean geometry by answering the questions below.

Question 1 *In Worksheet 1, you established the following relationships*

$$\text{area}(\triangle BAF) = 2\,\text{area}(\triangle FAC) \textit{ and } \text{area}(\triangle FAC) = (2/7)\,\text{area}(\triangle BAC).$$

Use these relationships and $CB' = (1/3)CB$ to prove your conjecture about the ratio of the area of triangle IV to the area of $\triangle BAC$? Can you make similar arguments about areas of triangles II and III?

Question 2 *Use your answer to the previous question to prove your conjecture about the ratio of the areas of V, VI, and VII to the area of $\triangle BAC$.*

B Appendix: Related Problem

A similar construction technique as seen in the picture proof of Section 2.3 can be used to make a conjecture about the ratio of the area of an inner parallelogram to the area of an outer parallelogram. Construct a parallelogram with midpoints on each side. Then, construct segments from the vertices of the parallelogram to the midpoints so the segments from opposite vertices do not intersect. The intersection points of these segments are the vertices of the inner parallelogram. Use the technique of rotating and dividing pieces of the outer parallelogram to create four quadrilaterals as shown in Figure 12.

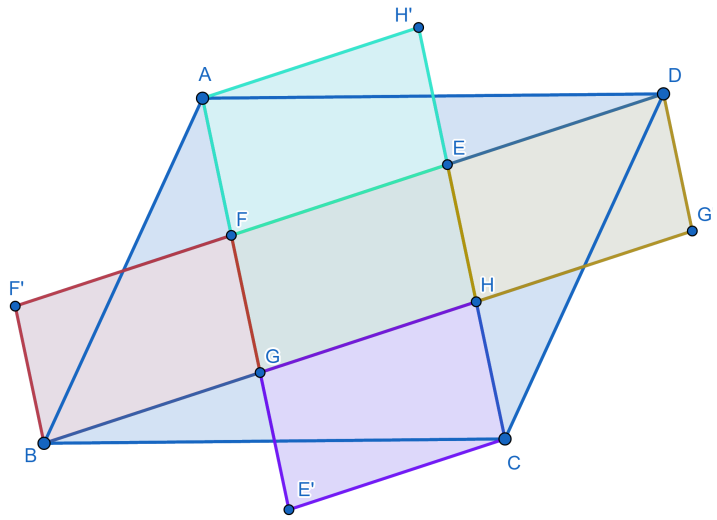

Figure 12. Parallelogram Construction

Question 1 *What conjecture can you make based on this construction? Measure areas and use the "drag test" as in Section 2.1 to confirm your conjecture. If instead of choosing midpoints you choose trisection points then what is the ratio of the outer to the inner parallelogram?*

The completed construction is here `https://www.geogebra.org/m/mqfhd5qs`. Upon inspection, there are four congruent parallelograms formed, each congruent to the inner parallelogram. Thus, the ratio of the area of the inner parallelogram to the area of the outer parallelogram is 1 to 5. Measuring areas and using the drag test gives a result that agrees with this. For an interactive construction which allows rotation of the corner triangles up to 180 degrees see `https://www.geogebra.org/m/zvfm6fhg`. If the construction uses trisection points, instead of midpoints, then the ratio of the area of the inner parallelogram to the area of the outer parallelogram is 1 to 13.