CREATING A DIGITAL TAPE DIAGRAM

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Abstract

The authors describe how tape diagrams created in GeoGebra can be used to foster critical thinking about multiplicative relationships in proportional situations. They describe multiple iterations of applet development and the rationale for various modifications.

Keywords: Tape diagrams, mathematical action technology, multiplicative relationships

1 INTRODUCTION

The National Council of Teachers of Mathematics (NCTM) states that "effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving" (2014, p. 24). While representations offer students access to mathematical concepts and enhance their understanding of those concepts, they also become tools for communication around shared understanding. A Tape Diagram, which is "a drawing that looks like a segment of tape, used to illustrate number relationships" (p. 87), is a particular representation featured prominently in the Common Core State Standards for Mathematics (CCSSI, 2010). The authors of *Progressions for the Common Core State Standards in Mathematics* (2011) illustrate how tape diagrams could be helpful in solving a proportion problem (see Figure 1).

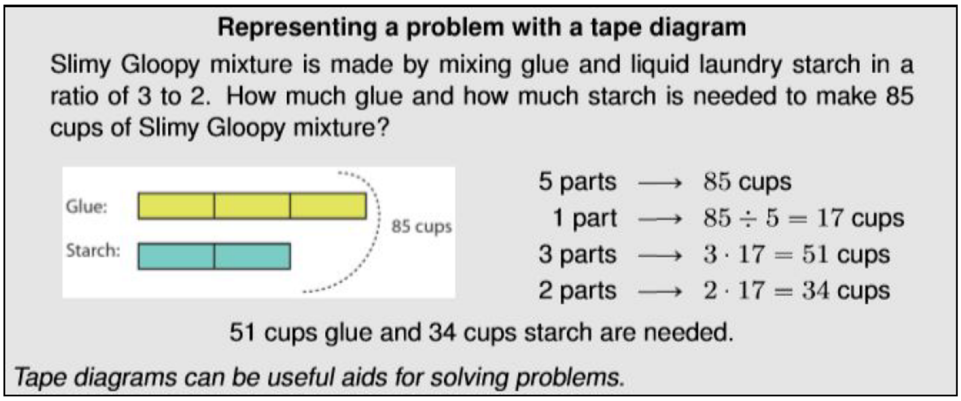

Figure 1. An example task solved with static tape diagram (Common Core Standards Writing Team, 2011, p. 7).

Tape diagrams are typically introduced in second grade and revisited throughout the later grades. According to CCSSM, sixth graders are expected to "use proportional, ratio, and rate reasoning to solve real-world and mathematical problems" (CCSSI, 2010).

Research suggests that tape diagrams can develop students visualization of equivalent ratios and foster multiplicative thinking inherent in proportional situations (see e.g., Beckmann 2004, Murata 2008, and Hoven & Garelick 2007). However, as students engage in tape diagram solutions with pencil and paper, the static nature of the tool may inhibit exploration and understanding. In such situations, "mathematical action technologies" (Dick & Hollebrands, 2011) i.e., technologies such as dynamic geometry environments in which users interact with digital mathematical objects such as observing results of making changes to one representation on other linked representations, "allow students to explore mathematical ideas and observe, make, and test conjectures about mathematical relationships" more efficiently (NCTM, 2014, p. 79). In this paper, we share our construction of a digital tape diagram, a mathematical action technology that empowers students to explore proportional relationships.

In explaining the importance of design in advanced digital technologies, Drijvers (2015) argues that "the criterion for appropriate design is that it enhances the co-emergence of technical mastery to use the digital technology for solving mathematical tasks, and the genesis of mental schemes that include the conceptual understanding of the mathematics at stake" (p. 15). Our design was informed by this criterion, and we discuss how our design decisions provide students with "affordance bearers" (Brown, 2015) for the mathematical actions they can take with the sketch. For Brown, the meaning of affordance bearer (attributed to Scarantino, 2003) is the particular aspect of a technology that allows the potential affordance of the technology to bear fruit. She gives the example of ZoomFit on a graphing calculator, which only takes into account the domain, not range, of a graph and therefore may or may not be an affordance bearer depending on the graph in consideration. In a companion piece (Meagher, Edwards, and Özgün-Koca, 2019), we explore ways in which students and teachers used the finished applet to solve a variety of proportional reasoning tasks. However, in this article, the emphasis is on the development of the applet itself rather than discussing its classroom use.

2 An Early Version

A GeoGebra Book with various iterations of our sketches is available at `https://ggbm.at/qkhfxrep`. In our first attempt, `Tapev0`, users enter values for Part 1 and Part 2 to specify the ratio of the length of the "top" tape to the "bottom" tape; Point `C` controls the lengths of the tapes. Controlling the tape diagram via Point `C` provides the user with a means of dynamic interaction. Visualizing this multiplicative relationship in a dynamic setting fosters multiplicative thinking as the user observes that the areas grow and shrink in proportion—with the ratio remaining fixed as `C` is dragged. We constructed the tapes as polygons and controlled their appearance using the ratio of values entered in input boxes Part 1 and Part 2 and the location of `C`, a draggable point along the x-axis (see Figure 2). Syntax associated with the construction of these tapes is provided in Figure 3.

`Poly1`, the bottom tape, and `Poly2`, the top-most tape, are both defined using the `Polygon` command. Both definitions define the polygons using the locations of the 4 vertices, with two coordinates of each shape defined with the `DynamicCoordinates` function. The definition of `Poly2` makes use of `p`, a value defined as the ratio of the value of input box Part 1 divided by the value of input box Part 2. The `DynamicCoordinates` command takes three parameters, namely `(<Point>, <x-Coordinate>, <y-Coordinate>)`, and dynamically updates vertices of both tapes when `C` is changed. The vertices of the top tape (i.e., `Poly2`) are updated when ratio `p` is changed.

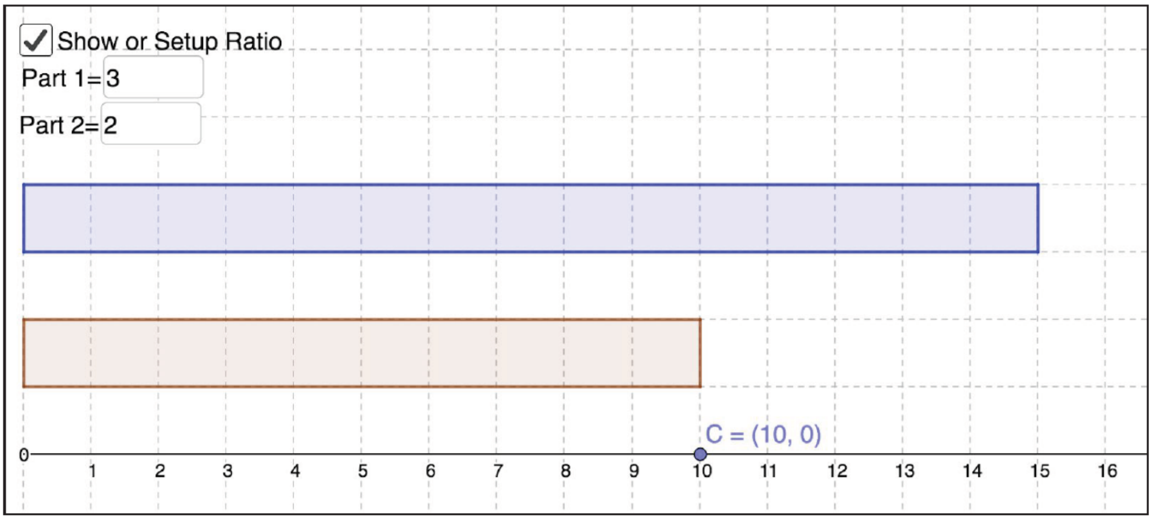

Figure 2. Early tape diagram sketch, `Tapev0`.

$$\text{Poly1} = \text{Polygon}((0,1),(0,2),\text{DynamicCoordinates}(C,x(C),2),\text{DynamicCoordinates}(C,x(C),1))$$

$$\text{Poly2} = \text{Polygon}\left((0,3),(0,4),\text{DynamicCoordinates}\left(C,\frac{x(C)}{p},4\right),\text{DynamicCoordinates}\left(C,\frac{x(C)}{p},3\right)\right)$$

Figure 3. Syntax for constructing rectangular tapes shown in Figure 2.

2.1 Limitations

Although our initial applet provided students with opportunities to engage with the tape diagram in new ways, the design had significant limitations that became apparent as we began to solve tasks. For instance, a user cannot determine the length of the top tape. Moreover, the tapes assume non-integer lengths—a source of confusion. As a team, we decided that more purposeful labeling was needed. This is an early example of thinking in terms of affordance bearers. The technology creates the top tape accurately, but not in a way that can support student learning. By far, however, the feature that evoked the most concern was the width limitation of the sketch. For instance, when attempting to solve the Slimy Gloopy task (see Figure 1) with the applet, the ends of the tapes are not visible after the length of the bottom tape exceeds 14 units (see Figure 4).

3 A Second Attempt

Figure 5 shows a revised sketch, `Tapev0.5`, which provides pictorial and symbolic representations of various ratio relationships, with lengths of each tape labeled more clearly. For instance, the default settings of the revised applet encourage the user to connect the ratio $3 : 2$ to the ratio of tape areas, $45 : 30$, through inspection of numerical representations within the sketch. In addition, parts of the tapes were represented as congruent grid rectangles and as numeric areas when the `Show Box Value` checkbox was clicked (see Figure 6).

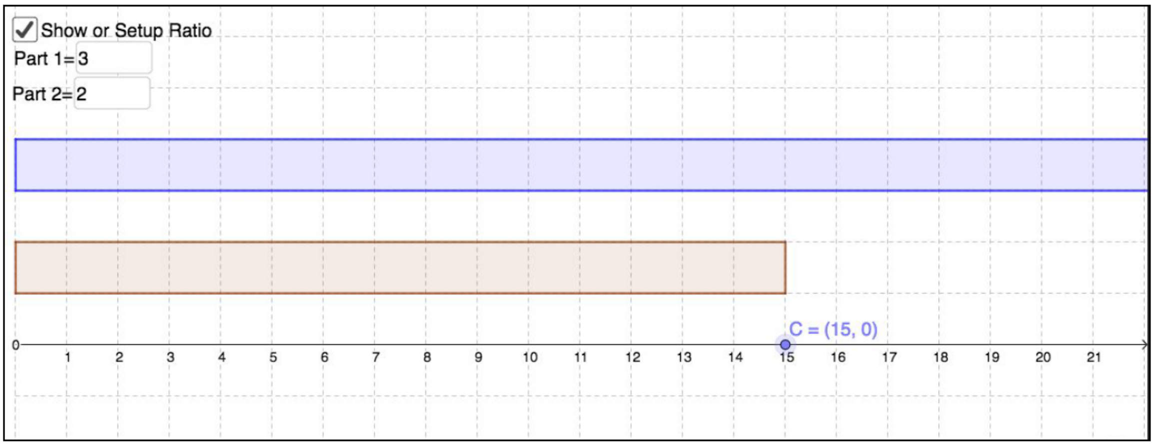

Figure 4. The width of the initial applet does not accommodate long tape lengths.

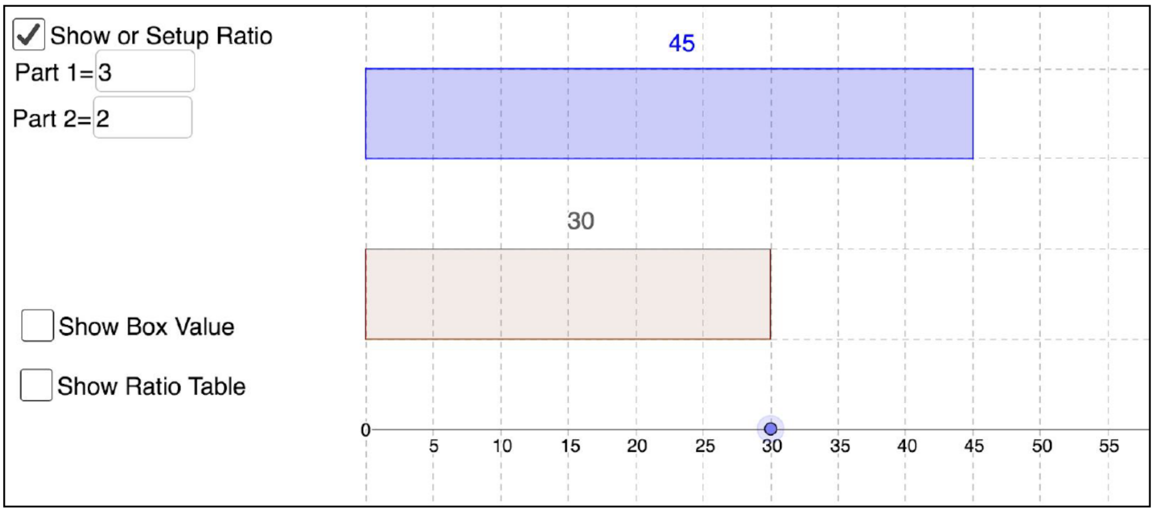

Figure 5. First revision of the tape diagram sketch, `Tapev0.5`.

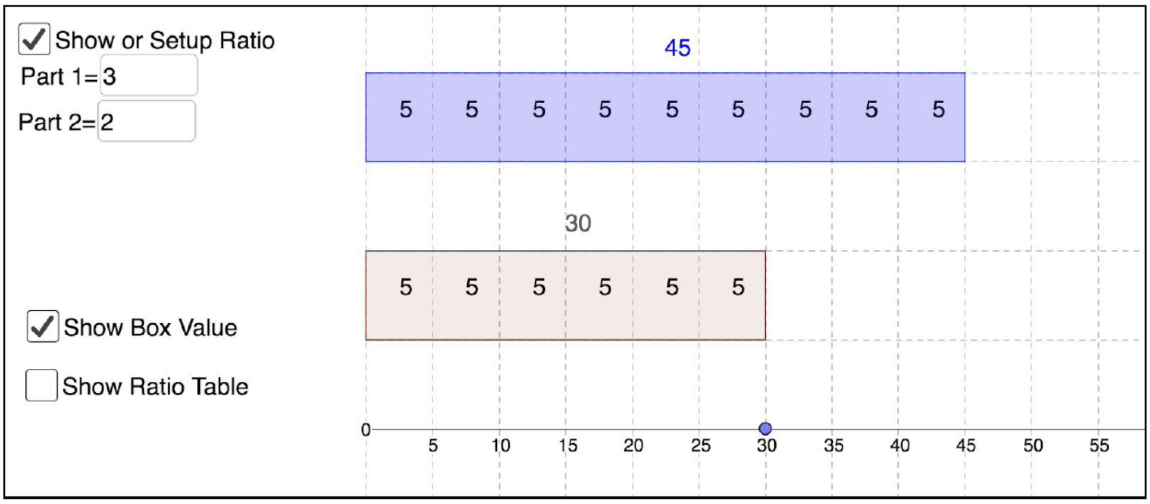

Figure 6. Sketch `Tapev0.5` with area regions labeled.

Our intent with the revised applet was to encourage users to examine various measures on the screen, looking for relationships between the area ratio (e.g., $45 : 30$) and the ratio Part1:Part2 ($3 : 2$). For instance, exploring Figure 6, one may recognize $45 : 30$ and $3 : 2$ as equivalent ratios. Others may see that $3 \times (3 \times 5) = 45$ and $2 \times (3 \times 5) = 30$ and that there are three 5's in the bottom tape for every two 5's in the top tape–although admittedly, this would be difficult for most middle or secondary level students to visualize with the applet alone.

3.1 Labeling Individual Regions of Tapes

When students click `Show Box Value`, individual regions of the tapes are labeled with their areas. The labels of the top and bottom tapes are generated as sequences of text, using a combination of `Sequence` and `Text` commands shown in Figure 7.

$$\text{bottomlabels} = \text{Sequence}\left(\text{Text}\left(\text{stepX}, \left(\text{i} \cdot \text{stepX} - \frac{\text{stepX}}{2}, 1.5\right)\right), \text{i}, 1, \frac{\text{poly1}}{\text{stepX}}\right)$$

$$\text{toplabels} = \text{Sequence}\left(\text{Text}\left(\text{stepX}, \left(\text{i} \cdot \text{stepX} - \frac{\text{stepX}}{2}, 3.5\right)\right), \text{i}, 1, \frac{\text{poly2}}{\text{stepX}}\right)$$

Figure 7. Definition of labels for individual regions.

Here, `bottomlabels` command tells GeoGebra to write the value of `stepX` (namely, 5) in each of the 6 rectangular regions within the bottom-most tape. Similarly, the `toplabels` command tells GeoGebra to write the value of `stepX` in each of the $45/5 = 9$ rectangular regions within the top-most tape. The `Sequence` command creates a sequence of objects. The command takes four parameters, namely `(<Expression>, <Variable i>, <Start Value a>, <End Value b>)`. Consider, for instance, the definition of `bottomlabels`. The first parameter passed to the `Sequence` function for `bottomlabels` is

$$Text(stepX, (i \cdot stepX - \tfrac{stepX}{2}, 1.5))$$

The `Text` command takes the value of `stepX` (i.e., the space between labeled coordinates on the x-axis), converts it to text, then places the resultant text in the Graphics view at position `(i*stepX-stepX/2, 1.5)`. Note that when `i=1` and `stepX=5`, the text "5" is placed at location (1*5-5/2, 1.5) = (2.5, 1.5). The remaining parameters passed to the Sequence command control how many times (and where) `stepX` is placed in the Graphics view. These parameters identify `i` as the counting variable and tell GeoGebra to let values of `i` range from 1 to `poly1/stepX` (e.g., from 1 to $30/5 = 6$).

3.2 Constructing a Ratio Table

To address concerns regarding tape lengths that extend off the screen, we constructed a table of values that is shown when the user checks the `Show Ratio Table` box (see Figure 8). Here, the ratio table starts with the tape diagram values.

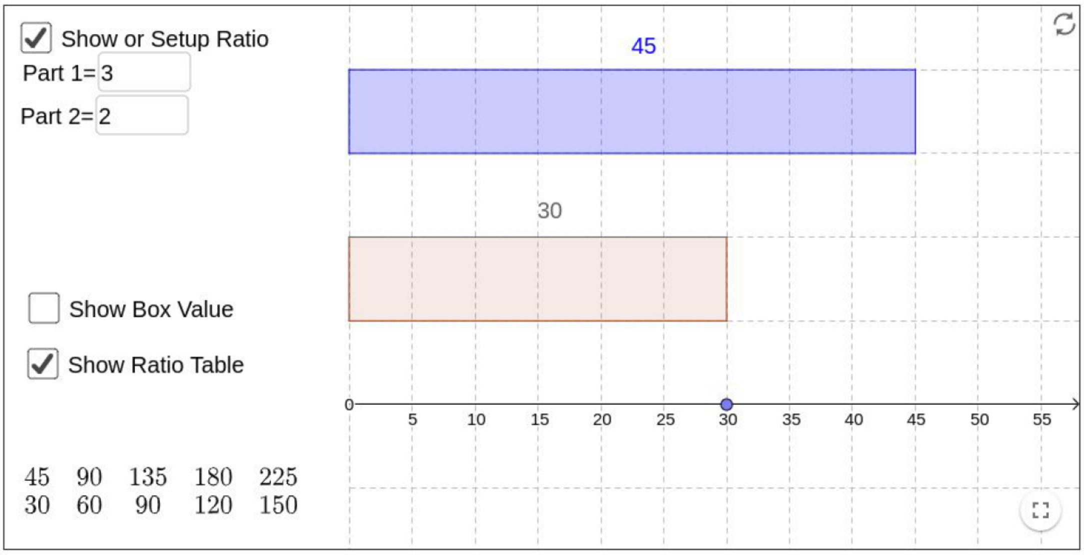

Figure 8. Tape diagram revision with ratio table visible.

This table was created by adding a text box to the sketch and entering a LaTeX array command in the box, as shown in Figure 9. Note that we defined the variables `A1-E1` and `A2-E2` beforehand from the GeoGebra Input Bar using the definitions `A1=poly1, B1=poly1*2, C1=poly1*3, D1=poly1*4, E1= poly1*5; A2=poly2, B2=poly2*2, C2=poly2*3, D2=poly2*4, E2=poly2*5`.

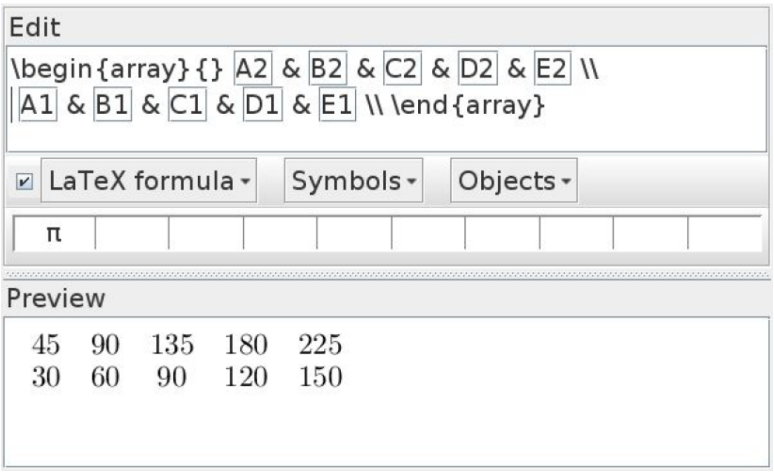

Figure 9. Ratio table textbox with LaTeX array command.

The LaTeX `array` command is used to format text in a rectangular array—for instance, to construct a matrix or table of values. Individual cells of the array are delimited with an ampersand (i.e., `&`), with rows of the array delimited by a double backslash (i.e., \\). (Editor's note: For more information about the formatting of arrays, see documentation at `https://www.overleaf.com/learn/latex/Tables`. For information about using LaTeX in GeoGebra see Quinlan (2013)). We constructed a ratio table in our first revision to provide access to equivalent ratios with larger values (due to limitations of extending the tape diagram on the x-axis) and to provide another dynamic representation for the proportion to foster multiplicative thinking.

3.3 Limitations

When attempting to solve the Slimy Gloopy task (see Figure 1) with the revised version of the applet, we were not able to determine a combination of tapes that summed to 85. As shown in Figure 10, the applet skips to values immediately below and above the desired configuration. Moreover, we were only able to see equivalent ratios of areas larger than or equal to the areas of the tapes depicted in the sketch. Again, the table was not acting as an affordance bearer with full potential to support student learning. In the next section, we share a later revision designed to overcome such limitations.

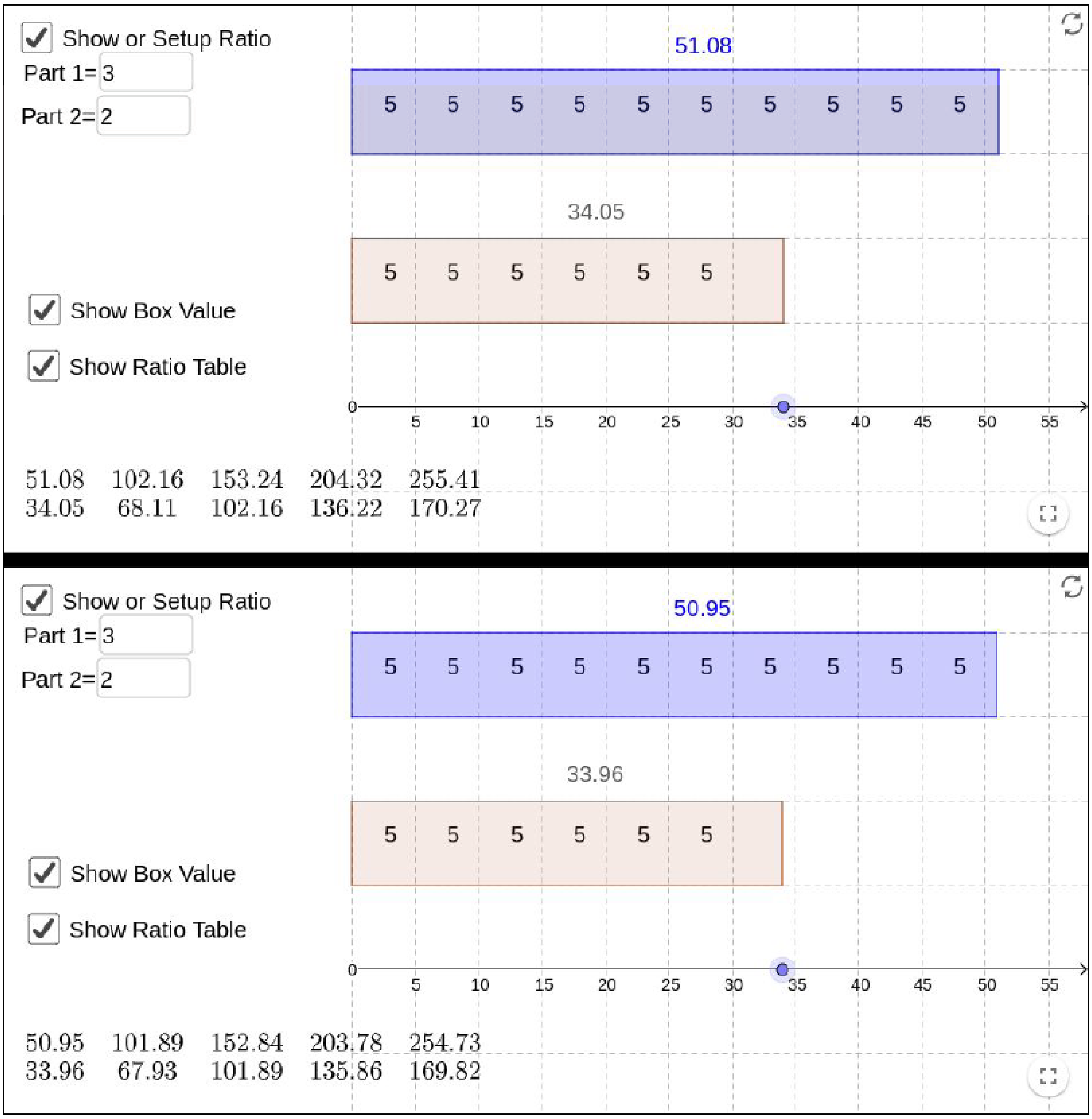

Figure 10. The applet skips over solutions to the Slimy Gloopy task.

4 A Later Version

Figure 11 highlights a more recent version of our applet, `Tapev15`, one constructed after approximately 15 revisions. To declutter the screen, we replaced the $x - y$ coordinate grid with a smaller one superimposed on the tapes, and we moved the tapes closer together, eliminating the vertical gap between them for easier visual comparison. Moreover, we eliminated Point `C`, opting to control tape length with a slider, k, instead. Slider k controls the scale of the tapes: ka is the width (and thus the area) of the top tape; kb is the width (and thus the area) of the bottom tape. This is reflected in the definition of the top and bottom tapes as polygons provided in Figure 12.

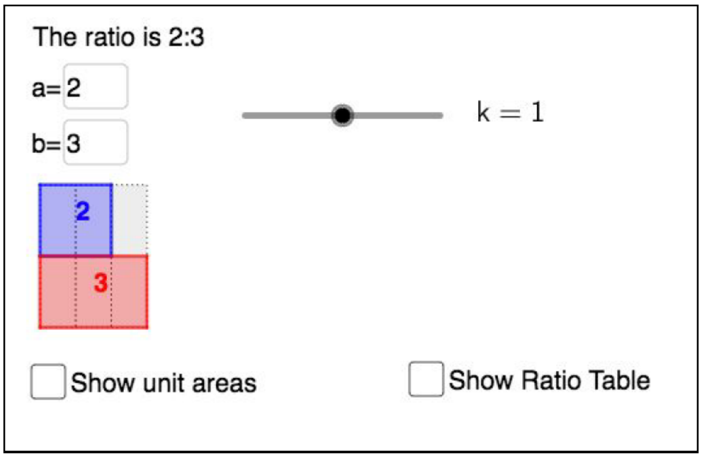

Figure 11. A more recent revision, `Tapev15`.

BottomTape = Polygon((0, 0), (b · k, 0), (b · k, 1), (0, 1))
TopTape = Polygon((0, 1), (a · k, 1), (a · k, 2), (0, 2))

Figure 12. Definition of tapes using user-entered ratio elements, a and b, and slider, k.

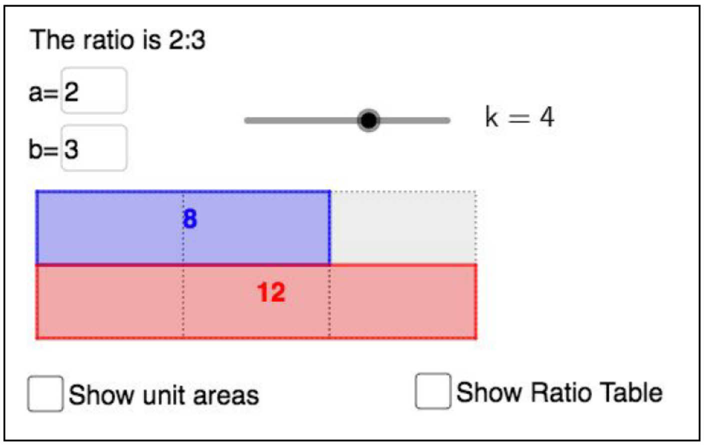

Figure 13. Multiplicative relationships in the revised applet.

In the revised sketch, users see the value of the ratio $a : b$ and the value of slider k at all times, making the relationship between the user-entered ratio and the ratios of tape areas more apparent. For instance, in Figure 13, users see that $2 \times 4 = 8$ and $3 \times 4 = 12$. The revised sketch emphasizes that there are two areas in the upper tape for every 3 areas in the lower tape.

We replaced the text "Show Box Value" with "Show unit areas" to avoid confusion with the word "box" (typically used in reference to 3D objects) and to more precisely convey the functionality of the checkbox. Although the functionality remains unchanged, the label connects its purpose more meaningfully to the vocabulary that promotes meaningful discussion of proportion in our classrooms.

4.1 Revising the Ratio Table

One of the most significant revisions that we implemented was updating the ratio table functionality. Recall that our original ratio table contained the first five multiples of the top and bottom tape areas (i.e., `1*area, 2*area, 3*area, 4*area, and 5*area`). This definition fixed the step size between entries in the table to equal the areas of the tapes. Moreover, values smaller than the tape areas were not included in the table. These changes were accomplished by constructing a new input box, step, and two new functions, `A(n)` and `B(n)`, used to define values in the first and second rows of the ratio table, respectively (see Figure 14). The syntax for these functions are provided in Figure 15.

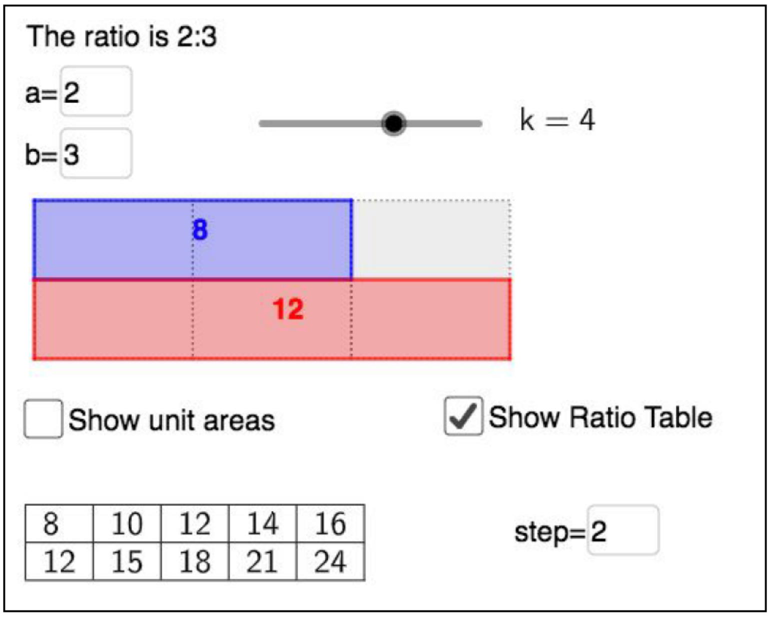

Figure 14. Updated ratio table functionality.

$$\mathrm{A(n)} = \mathrm{TopTape} + (\mathrm{n} - 1)\mathrm{step}$$

$$\mathrm{B(n)} = \mathrm{BottomTape} + (\mathrm{n} - 1)\mathrm{step} \cdot \frac{\mathrm{b}}{\mathrm{a}}$$

Figure 15. Definition of functions `A(n)` and `B(n)`.

Functions `A(n)` and `B(n)` define the first entries in each row as the areas of the top and bottom tapes. Since these areas are defined as multiples of k (as shown in Figure 11), fractional areas are possible. Note that `step` defines the increment size of the first row of the ratio table and `step*(b/a)` defines the increment size of the second row. Figure 16 shows the result of dragging slider k to be a fractional value (note: constructing a slider with a non-fixed increment is an interesting exercise). We recommend that you download the sketch at `https://ggbm.at/drsdhvvj` and reverse engineer its construction.

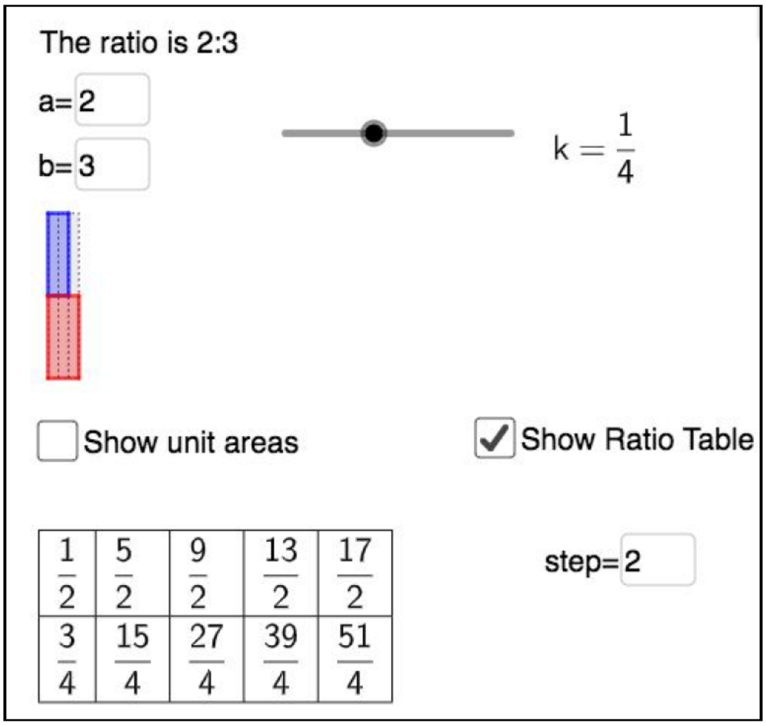

Figure 16. Slider and Ratio table with fractional values.

Since we wanted to represent values in the table as fractions rather than decimals, we used GeoGebra's `FractionText` command to convert numerical values to text. For instance, we defined `A1` in the table as `FractionText(A(1))`. The entire table was defined using GeoGebra's `TableText` tool (rather than a LaTeX array). Doing so allowed us to construct a table from text objects and to demarcate rows and columns of the table with horizontal and vertical line segments (see Figure 17).

$$\text{text1} = \text{TableText}\left(\begin{pmatrix} A1 & B1 & C1 & D1 & E1 \\ A2 & B2 & C2 & D2 & E2 \end{pmatrix}, \text{“|\_lllll”}\right)$$

Figure 17. GeoGebra's `TableText` command.

The horizontal and vertical bars passed as the right-most parameter generate the cell demarcations. This version of the applet enables students to generate a solution to the Slimy Gloopy task; namely, 51 cups of glue and 34 cups of starch, as shown in the fourth column of the table in Figure 18.

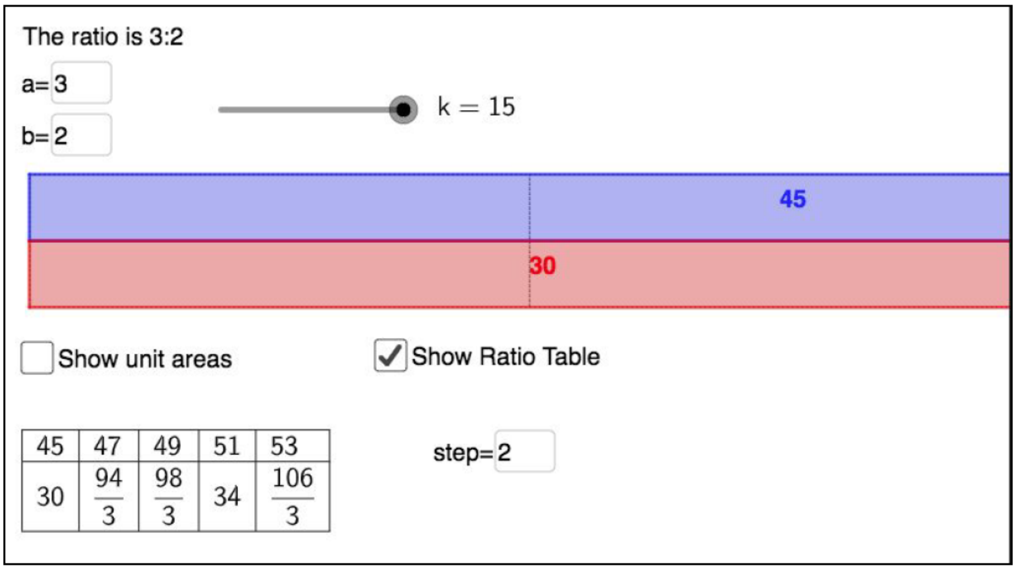

Figure 18. A solution to the Slimy Gloopy task within the applet.

5 Discussion

Informed by Drijvers' (2015) criterion of design, the trajectory outlined above is an attempt to hone and clarify the affordance bearers (Brown, 2015) that would best allow for learning of the underlying concept to occur. A simple example of this process is the change in language from `Show Box Value` to `Show unit areas`. As the applet developed, it became clear to us that "show box value" is (a) not a very meaningful phrase and (b) does not speak to the multiplicative nature of the conception we are trying to develop. "Show unit area" announces that we are talking about an area model, which is far more suggestive of the underlying multiplicative structure.

Another example is that, in the original design, the way for students to create an equivalent ratio was to drag a fixed point along the $x-$axis. Using the action technology (Dick & Hollebrands, 2011) in this way was not the best method for students to make the connection between the original ratio and equivalent ratios. The redesigns, which emphasized the multiplicative nature of the scaling that creates equivalent ratios by explicitly having the students control the scale factor, are much better affordance bearers since students have a much better chance of being able to use the technology to develop multiplicative reasoning.

In terms of developing multiplicative thinking, the scaling here is very important. One can think of the ratio in Figure 19 as 2 units of size 1 compared to 9 units of size 1.

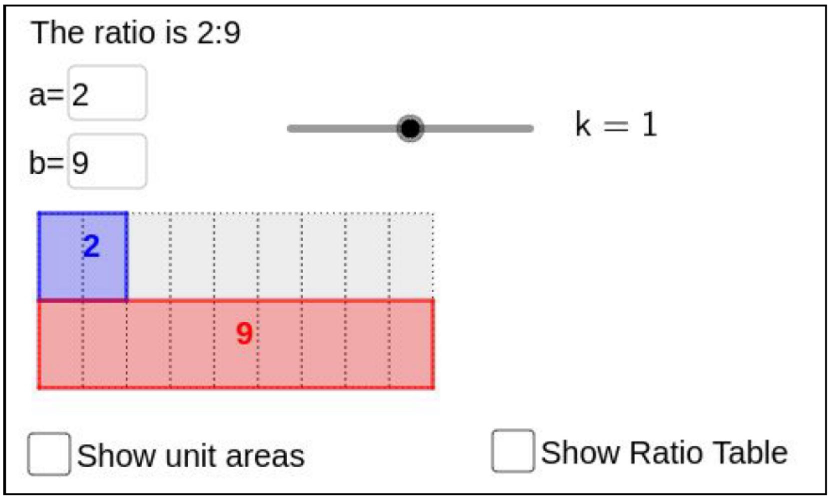

Figure 19. The ratio 2:9 as represented in the applet.

This is made explicit by clicking on the `Show Unit Areas` checkbox as depicted in Figure 20.

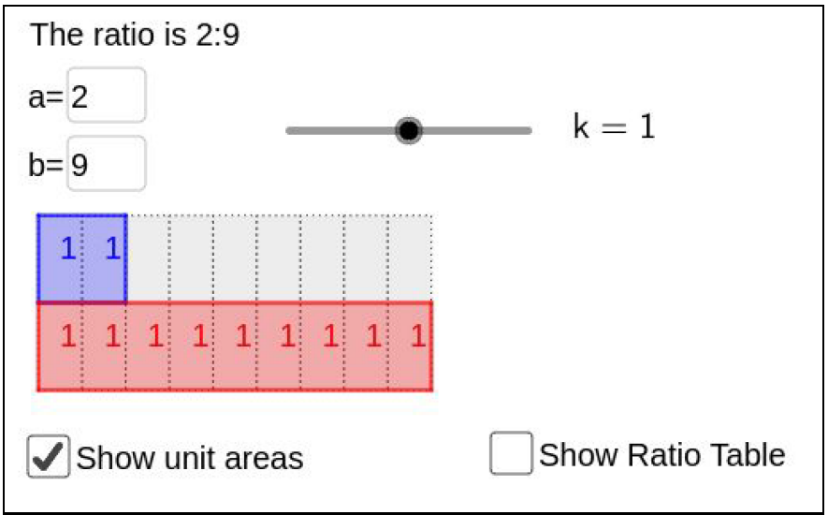

Figure 20. The `Show Unit Areas` option.

Dragging the slider and changing the value of k has the effect of dilating the unit size. For instance, in Figure 21, students still compare 2 units to 9 units; however, the unit has been dilated to 5. The ratio is embedded in the visualization, but the units are now of different size.

Thus `Show unit areas`, in terms of its name, is a significant enhancement as an affordance bearer over `Show Box Value`. In addition, the scale factor slider, in terms of its functionality, is a significant enhancement as an affordance bearer over the original dragging of a fixed point across the $x-$axis.

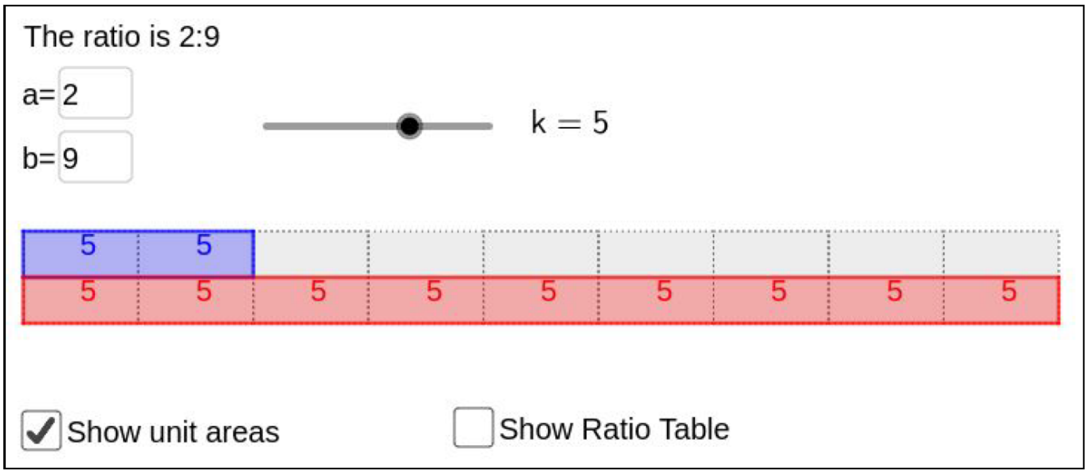

Figure 21. Dilated ratio, $5 \times 2 : 5 \times 9$.

6 Conclusion

The genesis of this paper was an attempt to design an electronic diagram that could be a mathematical action technology. As we used the diagram to solve particular tasks in ratios, we came to realize that some of the design decisions were not allowing for student learning of the underlying multiplicative nature of the structure and the concept. Various design decisions were made to provide appropriate affordance bearers within the technology. The final version of the applet allows students to engage with ratio tasks in a way that emphasizes the multiplicative nature of those tasks.

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