Dynamic Illustration of Some Fibonacci Identities

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Abstract: In the following paper, the authors illustrate several algebraic Fibonacci identities using visual capabilities of GeoGebra Dynamic Mathematics Software. Specifically, we explore the classic "missing unit" puzzle (Lloyd, 1858), generalized Fibonacci numbers, and Fibonacci and Lucas vectors.

Keywords: Fibonacci polynomial, Fibonacci and Lucas vectors, Cassini's identity

1. INTRODUCTION

Leonardo Fibonacci, also known as Leonardo Pisano was one of the most outstanding mathematicians of the European middle ages. Some of his famous books include the following.

- *Liber Abaci*, which explains the major contributions to algebra by al-Khowarizmi and Abu Kamil.
- *Flos* (Blossom or Flower), an exploration of Number Theory.
- *Liber Quadratorum* (Book of square numbers), also Number Theory text
- *Practica Geometriae* (Practice of Geometry). In this text, Fibonacci used algebra to solve geometric problems and vice-versa, a radical approach for Europe in the day.

Picking up the thread from *Practica Geometriae*, we have explain some of Fibonacci's algebraic identities visually using the dynamic mathematics software, GeoGebra.

2. SAM LLOYD'S PUZZLE

Let us consider a problem posed by William Hooper in 1774 in his *Rational Recreations*. It was later made famous by Sam Lloyd Sr. who presented it to the American Chess Congress in 1858. From then on it became synonymous with Sam Lloyd. The problem is as follows : An 8x8 square is cut into four pieces and rearranged to form a rectangle with length 13 and breadth 5 units, resulting in a gain of 1 square unit in area. The problem is best explained by Cassini's identity.

Consider the matrix $M = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$

Since the elements of this matrix are the members of the sequence of Fibonacci Polynomials, we shall refer to it as a Fibonacci Matrix. The determinant of the matrix is (-1). Hence,

$$\begin{aligned} det(M^n) &= (detM)^n \\ &= (-1)^n \end{aligned} \tag{1}$$

It can be easily proved by induction that

$$M^n = \begin{bmatrix} f_{n+1}(x) & f_n(x) \\ f_n(x) & f_{n-1}(x) \end{bmatrix}$$

Thus,

$$f_{n+1}(x)f_{n-1}(x) - (f_n(x))^2 = (-1)^n \tag{2}$$

This is known as Cassini's identity. Now, taking $x = 1$, we obtain the corresponding Fibonacci numbers. Thus,

$$F_{n-1}(x)F_{n+1} - F_n^2 = (-1)^n \tag{3}$$

Here, a GeoGebra sketch can be used to illustrate this relationship. We created an applet, `geo-paradox.ggb`, to construct a visual representation of the paradox. Each triangular and trapezoidal piece in Figure 1 is detachable and can be dragged, turned and rearranged as shown in the Figure 2.

The trapezoids and triangles don't fill the 5×13, even though they may appear to at first glance. The area of the square rectangle is one unit larger than the non-square rectangle, $5 \times 13 - 8^2 = 1$. This is a special case of (3) which is further generalized in the next section.

3. GENERALIZED FIBONACCI NUMBERS

Let us consider the sequence $\{G_n\}$ where $G_1 = a$, $G_2 = b$, and $G_n = G_{n-1}+G_{n-2}$, $n > 3$. The ensuing sequence a, b, $(a+b)$, $(a+2b)$, $\ldots$ is called a Generalized Fibonacci Sequence (GFS). On taking a closer look, we find that

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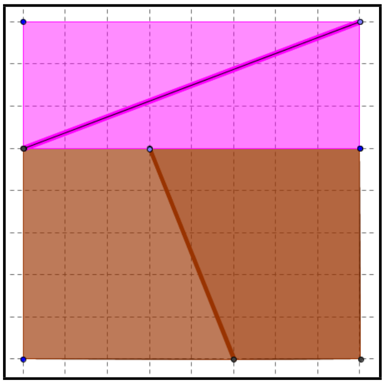

Fig 1: 8×8 rectangle constructed from trapezoids and triangles in GeoGebra

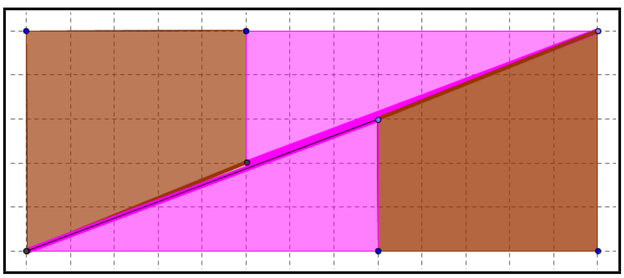

Fig 2: GeoGebra sketch of 5×13 rectangle constructed from identical trapezoids and triangles

$$G_n = aF_{n-2} + bF_{n-1} \tag{4}$$

Therefore

$$\begin{aligned}
\sqrt{5}G_n &= a\{\alpha^{n-2} - \beta^{n-2}\} + b\{\alpha^{n-1} - \beta^{n-1}\} \\
&= \alpha^n\{\frac{a}{\alpha^2} + \frac{b}{\alpha}\} - \beta^n\{\frac{a}{\beta^2} + \frac{b}{\beta}\} \\
&= \alpha^n\{a\beta^2 - b\beta\} - \beta^2\{a\alpha^2 - b\alpha\} \\
&= \alpha^n\{a + (a-b)\beta\} - \beta^n\{a + (a-b)\alpha\} \\
\therefore G_n &= \frac{c\alpha^n - d\beta^n}{\sqrt{5}} \\
&= \frac{c\alpha^n - d\beta^n}{\alpha - \beta}
\end{aligned}$$

Now

$$\begin{aligned}
c \cdot d &= \{a + (a-b)\beta\} \cdot \{a + (a-b)\alpha\} \\
&= a^2 + ab - b^2 \\
&= \mu \text{ (characteristic of the GFS)} \\
5\{G_{n+1}G_{n-1} - G_n^2\} &= \{c\alpha^{n+1} - d\beta^{n+1}\} \cdot \{c\alpha^{n-1} - \\
&\quad d\beta^{n-1}\} - \{c\alpha^n - d\beta^n\}^2 \\
&= -cd\{\alpha^{n+1}\beta^{n-1} + \alpha^{n-1}\beta^{n+1}\} \\
&\quad +2cd\{\alpha\beta\}^n \\
&= 5\mu(-1)^n \qquad (5) \\
G_{n+1}G_{n-1} - G_n^2 &= \mu(-1)^n \qquad (6)
\end{aligned}$$

With reference to the applet `samlloyd2.ggb`, when $\mu = 1$, the identity holds true for Fibonacci numbers as shown in Figure 3. When $\mu = 5$, the identity hods true for Lucas numbers as shown in Figure 4. Also, we notice that $\left\lfloor \frac{\alpha^{2n}}{\sqrt{5}} \right\rfloor = F_{2n}$ and $\left\lceil \frac{\alpha^{2n+1}}{\sqrt{5}} \right\rceil = F_{2n+1}$.

Thus, when the sliders are set to $a = 1$ and $b = \alpha$ in the applet `samlloyd.ggb`, we obtain the sequence $1, \alpha, (1 + \alpha), (2\alpha + 1), (3\alpha + 2), \dots G_n \dots$. If we choose any three consecutive terms $\{G_{n-1}, G_n$, and $G_{n+1}\}$, we find that

$$\begin{aligned}
\alpha^{n-1} \cdot \alpha^{n+1} &= \alpha^{2n} \\
G_{n-1}G_{n+1} &= G_n^2
\end{aligned} \tag{7}$$

We obtain this as the only case when the square and the rectangle have equal area and the thin parallelogram inside the rectangle vanishes completely showing the area of the parallelogram USWL as zero.

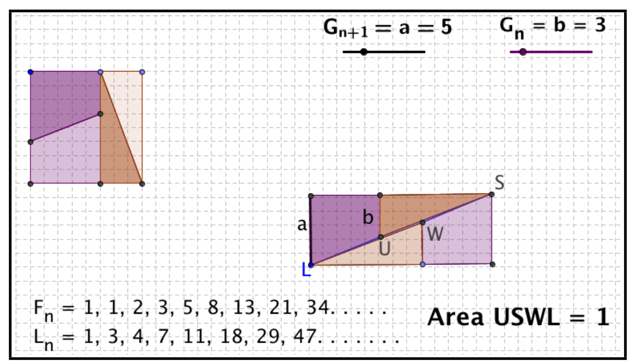

Fig 3: When a and b are consecutive Fibonacci numbers then area(USWL) =1

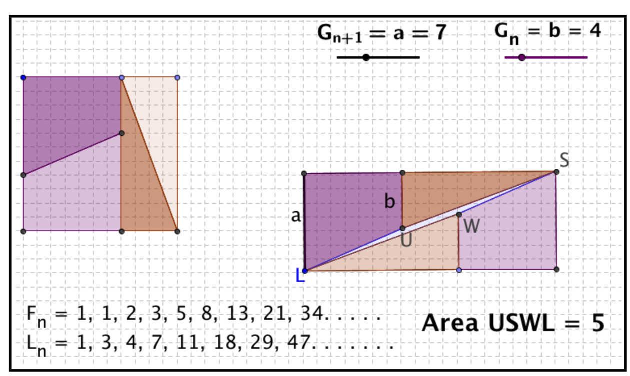

Fig 4: When a and b are consecutive Lucas numbers then area(USWL) = 5.

Let $\vec{l} = \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$ and $\vec{m} = \begin{bmatrix} L_{n+1} \\ L_n \end{bmatrix}$ be two vectors formed by Fibonacci and Lucas numbers. Also let $R = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

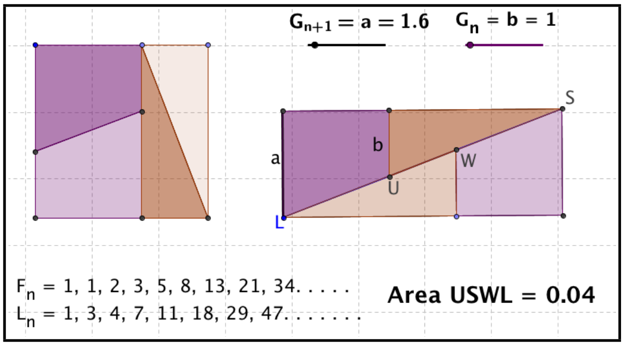

Fig 5: When $a \approx \phi$, the golden ratio, and $b = 1$ then area(USWL) ≈ 0 square units.

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix} = \begin{bmatrix} F_{n+1} + 2F_n \\ 2F_{n+1} - F_n \end{bmatrix} = \begin{bmatrix} L_{n+1} \\ L_n \end{bmatrix} \tag{8}$$

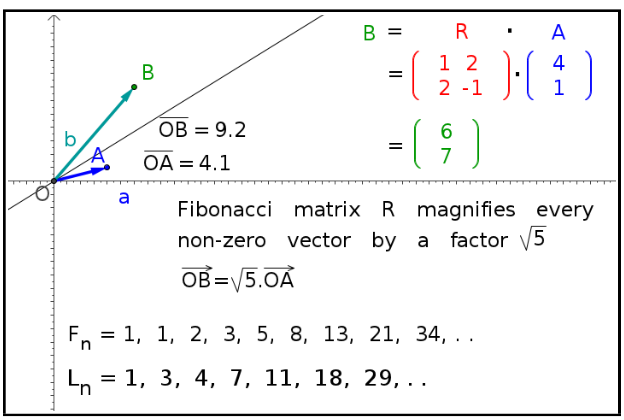

Fig 6: Fibonacci and Lucas vectors from `fibo-vec.ggb` sketch

With reference to the applet `fibo-vec.ggb`, we find that when a Fibonacci vector is multiplied by the R matrix we get a Lucas vector. Also we notice that when we take larger values of Fibonacci numbers as the components of the vector, the Fibonacci and the Lucas vectors become collinear and lie along a line through the origin with slope $\breve{}\beta$ or $\frac{1}{\alpha}$ where α is the golden ratio. This is apparent because $\lim_{x\to\infty} \frac{F_n}{F_{n+1}} = \frac{1}{\alpha}$

Also, we notice that the Fibonacci matrix R magnifies every non-zero vector by a factor $\sqrt{5}$. In this way, using the applets help us build conjectures regarding new identities.

The applets are available at `http://www.geogebra.org/en/upload/index.php?&direction=0&order=&directory=amitava`

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