Using Action–Consequence–Reflection GeoGebra Activities To Make Math Stick

Karen D. Campe

Abstract

In this article, the author discusses the Action-Consequence-Reflection cycle for promoting conceptual understanding with technology in the mathematics classroom, along with some cognitive science research support. Several GeoGebra activities are presented that capitalize on this process, covering common curricular topics in secondary math. Throughout, the author shares strategies for classroom implementation that encourage reflection and enhance student learning.
Keywords: technology, algebra, geometry, precalculus, conceptual understanding

1 Introduction

GeoGebra, like other dynamic mathematical technologies, is a worthy platform for developing understanding in the secondary mathematics classroom. GeoGebra brings accuracy, speed, the power of visualization, and opportunities for exploration and discovery to the learning environment (Dick & Hollebrands, 2011; King & Schattschneider, 1997; Waits & Demana 1998). With its integrated multiple representations, or "views," GeoGebra can be characterized as a "mathematical action technology" that performs mathematical tasks and dynamically responds to user inputs and manipulations (Dick & Hollebrands, 2011, p. xii). There is a distinction between using technology as a tool for performing mathematical operations and using it as a tool for developing understanding via action–consequence scenarios, and it is this latter role that has real promise for student learning (Dick & Hollebrands, 2011).

GeoGebra enables users to participate in the Action–Consequence–Reflection cycle (Campe, 2011; Dick, Burrill & Gill, 2007; Dick & Hollebrands, 2011) which has these components:

- Performing a purposeful mathematically meaningful action (on a graph, geometric figure, etc.)
- Observing a mathematical consequence
- Reflecting on the result and reason about the underlying mathematical concepts.

Arguably, reflection is the most important component of the cycle because it helps make the mathematics "stick" for students.

The process of reflection combines several cognitive strategies—retrieval, elaboration, generation, calibration, and self-explanation—that have been shown to make student learning deeper and more durable (Barton, 2018; Brown, Roediger, & McDaniel, 2014; Dunlosky, Rawson, Marsh, Nathan, & Willingham, 2013). Reflection "adds layers to learning and strengthens skills" and is "the act of taking a few minutes to review what has been learned . . . and asking yourself questions" (Brown, et al., 2014, p. 209). When students reflect and reason on the mathematical concepts they are learning, they deepen their understanding, correct errors, and increase their ability to transfer the knowledge to different situations, improving performance and retention (Chi, 2000; Collins, Brown & Newman, 1989; Dunlosky et al., 2013). Thus, we must plan for the reflection stage of the cycle as essential to teaching with technology tools, and not stop after designing the technology-based task.

In this paper, I describe four categories of Action–Consequence–Reflection activities and provide interactive GeoGebra examples and student worksheets for each. They demonstrate how to leverage the Action–Consequence–Reflection Cycle to promote enduring conceptual understanding for students.

2 Exploring Graphs and Sliders

When students graph a function with technology and then make changes to the equation, they can observe the corresponding consequences on the function's graph. A slider automates this process and allows students to view algebraic and graphical changes together on the same screen.

2.1 Power Functions

In the *Power Functions* activity [`https://ggbm.at/GzQQNNr4`], students use a slider to control the value of the exponent of the function $f(x) = x^n$ while a checkbox toggles between positive and negative leading coefficients. As students interact with the applet, they notice the contrasting shapes of even- and odd-powered functions, and how the sign of the leading coefficient affects the graph. The construction prompts students to uncover reasons for these patterns, setting the stage for study of polynomial graphs, a numerical investigation of exponentiation on large and small numbers, or an introduction to limits.

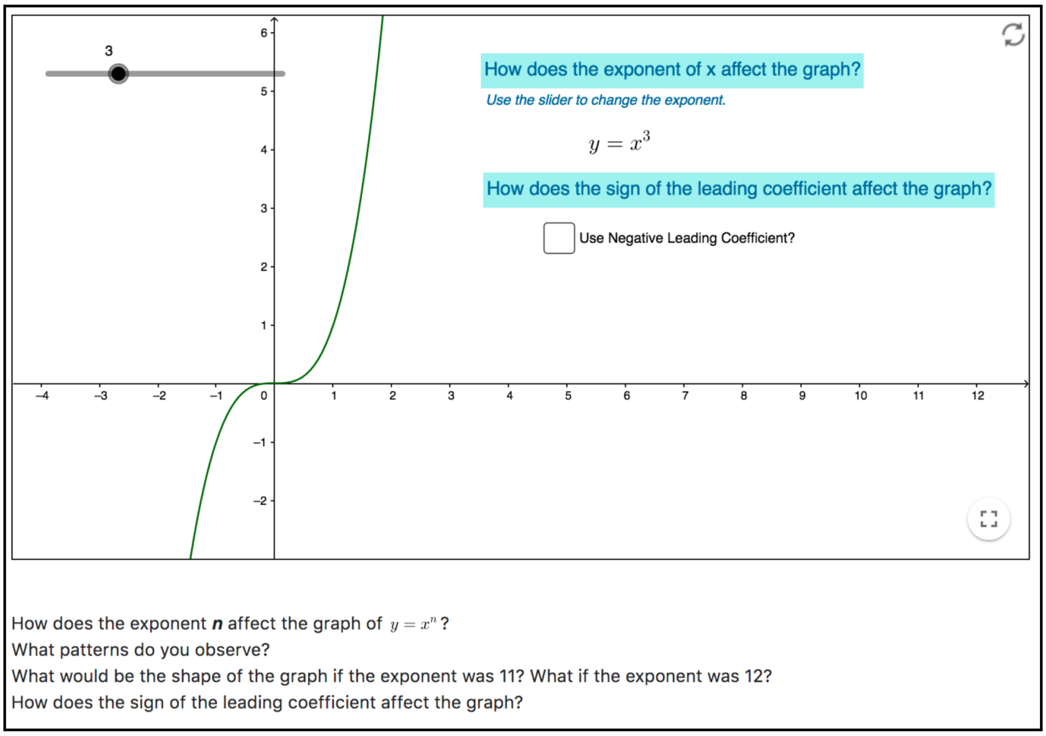

Figure 1. *Power Functions* applet (available at `https://ggbm.at/GzQQNNr4`).

The reflection questions provided within the applet and worksheet (see Appendix) are deliberately open-ended to encourage creative thought, because students may notice other graph features ancillary to the main goal of examining end behavior. Students may make observations about points in common to all the graphs, behavior of these functions between $x = 0$ and $x = 1$, or what happens to the graph when the exponent is zero. Teachers can pursue avenues of questioning and discussion that focus students' attention on patterns of end behavior and other phenomena.

2.2 *Function Transformations*

In the *Function Transformations* activity [`https://ggbm.at/YNqVpHQS`], students manipulate parameters a, h, and k to transform $y = f(x)$ into $y = a \cdot f(x-h) + k$ with a variety of different parent functions. Students may work on one parameter at a time or can freely explore. Teachers can use probing questions to build and clarify the concept that each parameter has a particular effect on the parent function.

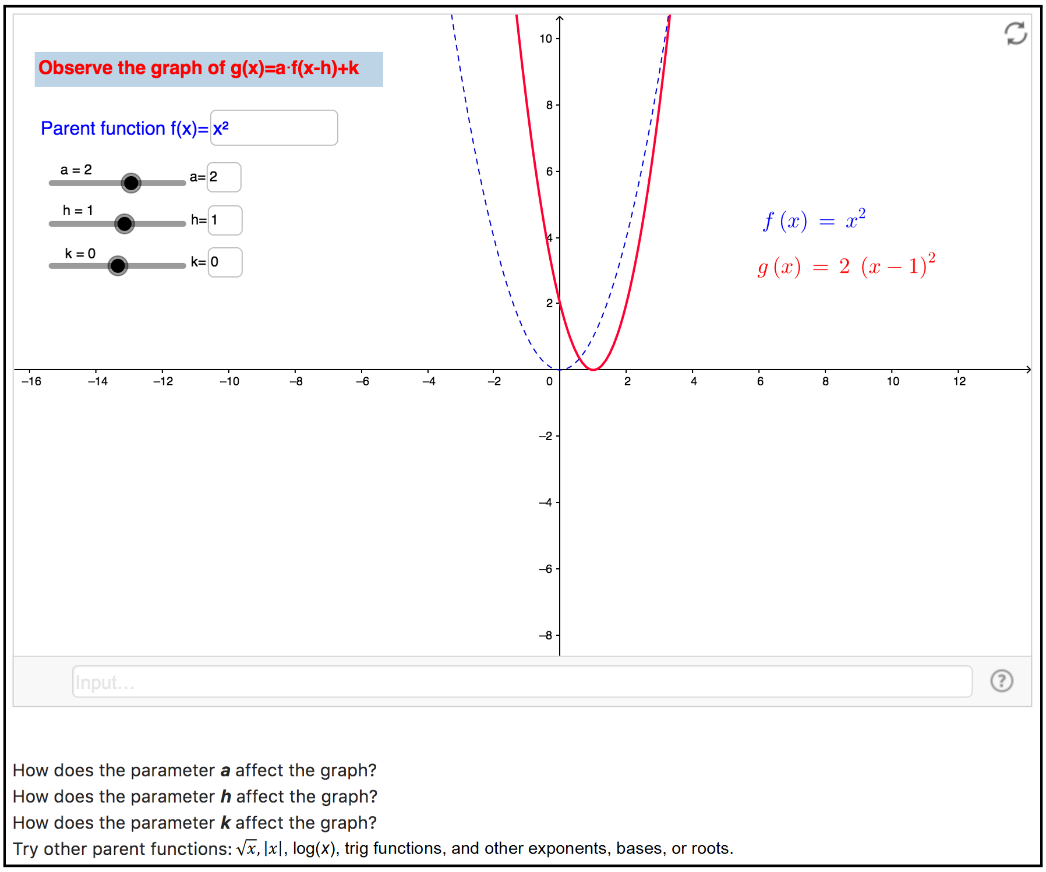

Figure 2. *Function Transformations* applet (available at `https://ggbm.at/YNqVpHQS`)

Potential extension questions for students to examine include:

- What happens if $a = -1$?
- If you transform a graph from its parent function with pencil and paper, does the order of transformations matter?
- Consider the parameter b in $y = f(b \cdot x)$; can you replace b with some value of a in $y = a \cdot f(x)$ so that the resulting transformations the same? How is a related to b?

To optimize learning, teachers should require students to actively record and discuss their observations from the technology setting. Students may use their notebooks or an electronic document, or teachers can provide a lab sheet to make notes and sketches, suggest conjectures, make predictions, and answer targeted questions. Even if technology access is limited to demonstrating the process on a teacher computer projected to the class, support students' active involvement through the lab sheet and class discussion. The activity should engage students in *doing* math, not simply *viewing* math.

3 Interactive Visualizers

Visual tools help make abstract math concepts more concrete for all students, not just those who struggle. GeoGebra provides an interactive environment for exploring challenging concepts, leveraging the power of visualization to deepen understanding.

3.1 Domain and Range: Illustrated

In the applet *Domain and Range: Illustrated* [`https://ggbm.at/mpkdaeze`], students manipulate linear or quadratic functions on closed intervals with sliders on the bottom left of the window. The two "illustrator" sliders on the bottom right shade sections of the appropriate axes, reinforcing meanings of domain and range. Students can confirm their understanding by displaying a graph grid and the equation of the function. After exploring with the applet, they can practice this skill with additional static or dynamic examples. The activity can be extended by asking what happens to domain and range if the interval is not closed, and by having students create their own graphs from a given domain and range.

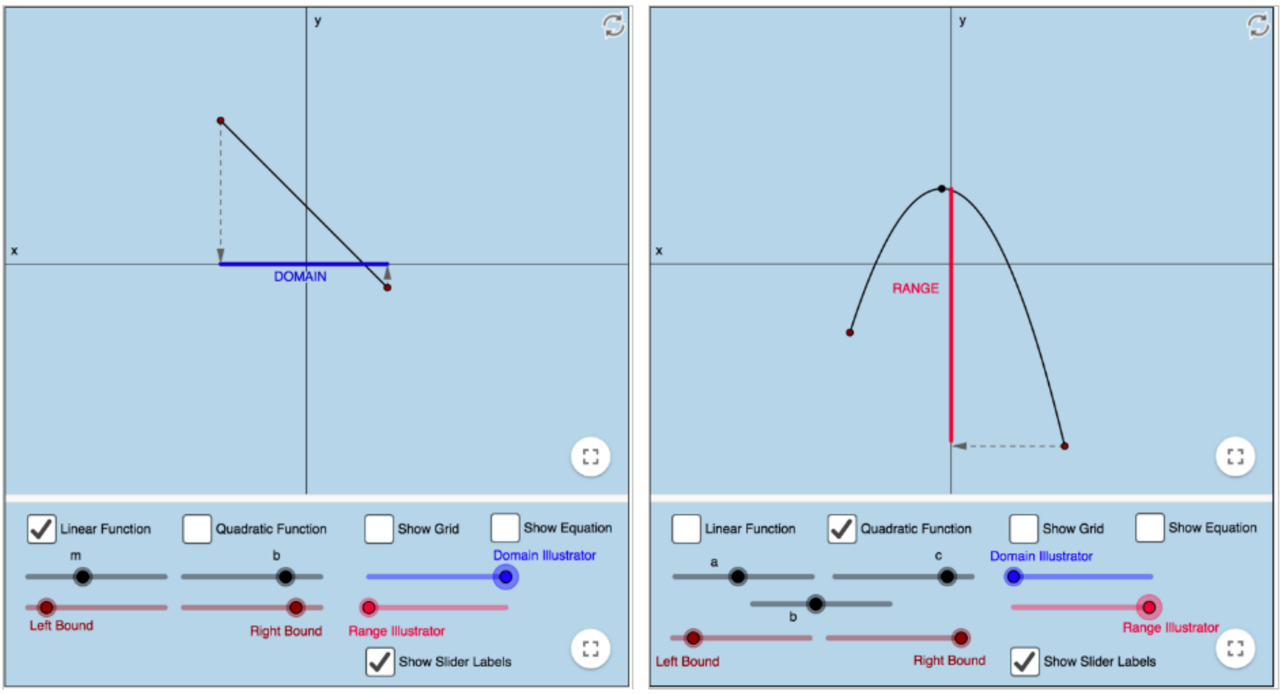

Figure 3. (Left) Domain of linear function illustrated; (Right) Range of quadratic function illustrated (applet available at `https://ggbm.at/mpkdaeze`).

Intended (and any unexpected) learnings can be summarized either individually or as a class to solidify the concepts and correct any misconceptions. Writing about observations in a log sheet or handout (see Appendix) provides students with opportunities to reflect on their learning through self-explanation and elaboration, two metacognitive strategies that help make the learning more durable. Overtly summarizing the findings of the investigation will enable students to develop a robust conceptual framework for the topic being studied, and make key connections to prior knowledge.

4 Understanding Structure

A significant task for students in Algebra 1, Algebra 2, and Pre-Calculus is to make generalizations about the structure of algebraic expressions and their relationships to the graphs of functions (Goldenberg, et al., 2015). When students use GeoGebra to explore function families, they build conceptual understanding of these structures and patterns, rather than simply memorizing algebraic and numerical methods for finding intercepts, asymptotes, or other key graph features.

4.1 Rational Functions

While studying the applet *Rational Functions* [`https://ggbm.at/sQy2z6kf`], students visualize functions and become familiar with patterns in their graphs. Students use interactive sliders to manipulate the parameters for two types of rational parent functions, transforming $y = \frac{1}{x}$ into $y = \frac{a}{x-h}$ and $y = \frac{1}{x^2}$ into $y = \frac{a}{(x-h)^2}$. The applet fosters exploration with different values for the parameters, including negative values of a.

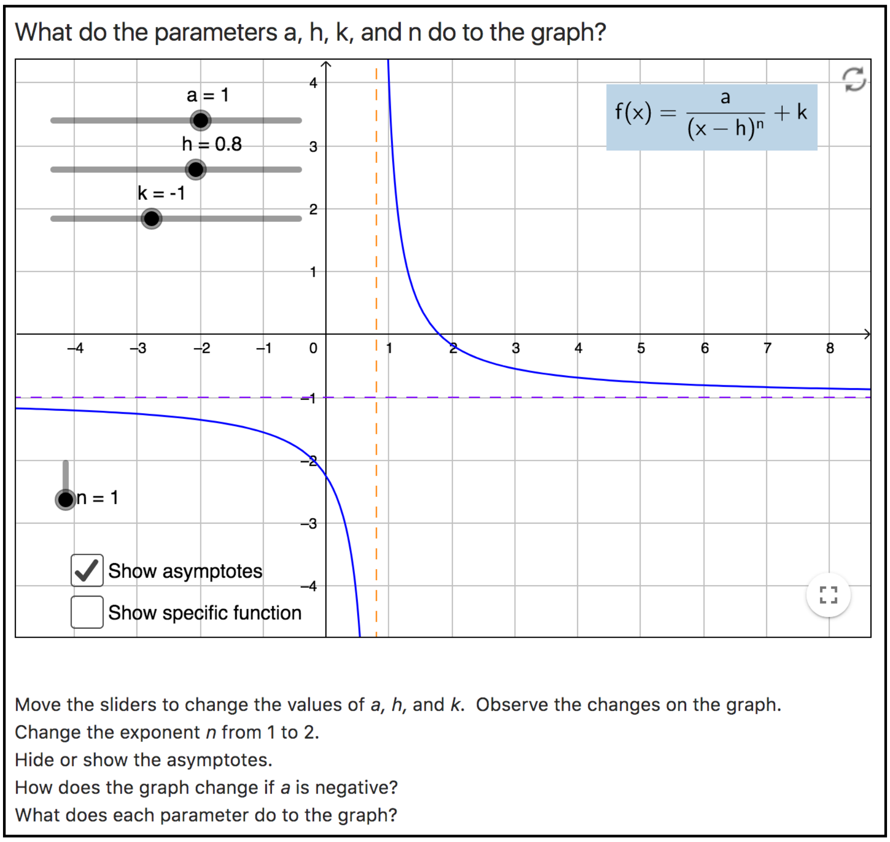

Figure 4. *Rational Functions* applet (available at `https://ggbm.at/sQy2z6kf`).

As students investigate the functions with the applet, the accompanying handout (see Appendix) guides them through these stages: (1) *Explore the graphs* of target functions on an appropriate window; (2) *Record conjectures* about the roles of the parameters and how they change the shape of the graph; (3) *Make predictions* about what a given function will look like and verify with the graphing technology and/or provide a function for a given graph; and (4) *Generalize* the observed relationships by formulating rules and strategies. Students can also apply this knowledge to more complicated functions, contemplating "what if...?" questions about additional factors in the numerator and denominator and other rational function scenarios.

5 Investigating Invariants

An invariant is something about a mathematical situation—a measurement, calculation, shape, or location—that stays the same while other parts of the situation change (Cuoco, Goldenberg, & Mark, 1996; Driscoll, 2007; Sinclair, Pimm & Skelin, 2012). Particularly important in geometry, invariants are fruitful targets for GeoGebra explorations.

5.1 Interior & Exterior Angles

In the *Interior & Exterior Angles of Triangle* activity [`https://ggbm.at/z4uthpnr`], students investigate relationships among interior and exterior angles of a dynamic triangle, discovering and testing their own hypotheses. Students drag a vertex, then record several data sets in a chart and make conjectures prior to confirming sums with the checkboxes.

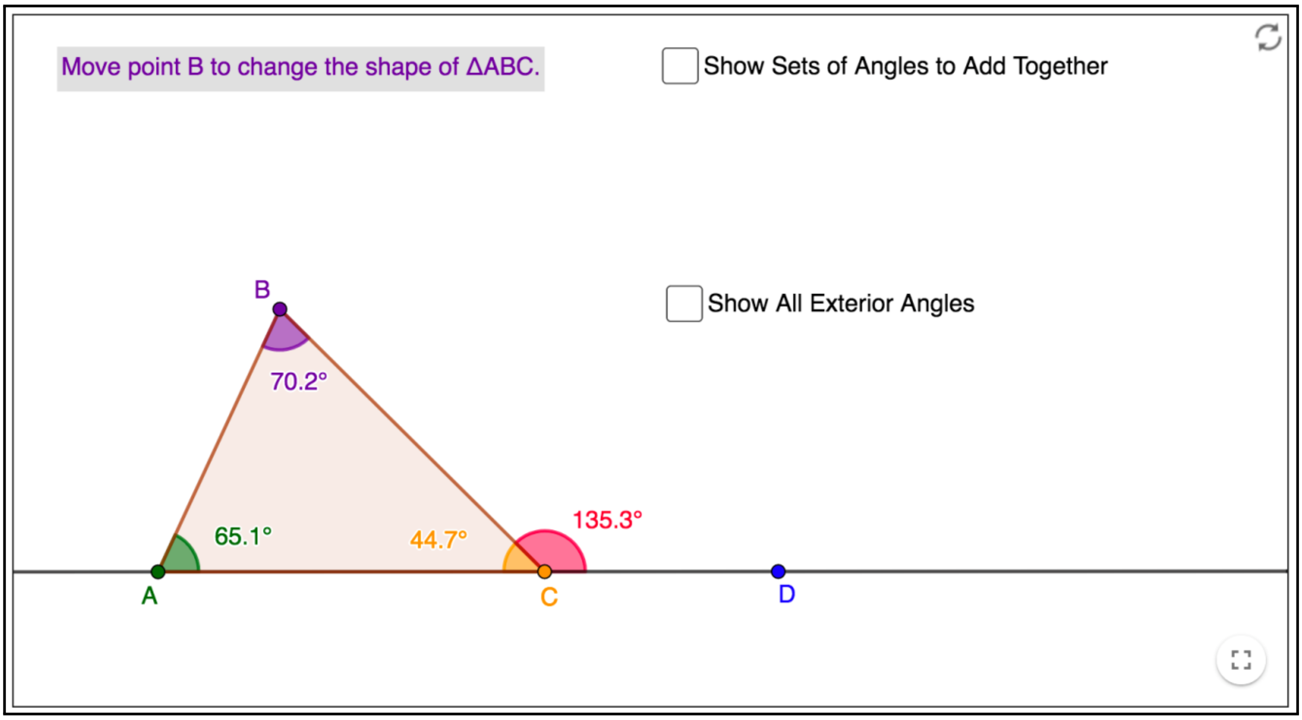

Figure 5. *Interior & Exterior Angles* applet (available at `https://ggbm.at/z4uthpnr`).

Figure 6 provides a condensed lab sheet designed for students to record data and findings gleaned from the *Interior & Exterior Angles* applet (see Appendix).

Record several sets of measurements in a chart like this one.

∠ABC	∠BCA	∠CAB	∠BCD

What do you notice about the angle measures as the shape of the triangle changes?

1. What is the sum of the 3 interior angles?
2. What other angle pairs would you like to add together?
3. How is exterior $\angle BCD$ related to the other angles?
4. What is the sum of the 3 exterior angles (one at each vertex)?

Figure 6. Lab sheet accompanying *Interior & Exterior Angles* applet.

The dynamic nature of the software ensures that multiple instances of the figure are created simply by dragging and moving an independent point. Although the activity provides numerical evidence for the geometric properties, this by itself does not constitute a proof of the theorems involved. The interactive environment can help students justify their conjectures, and provide groundwork to write deductive proofs.

5.2 Right Triangle Invariants

With the variety of views available in Geogebra Classic (algebra, graphing, geometry, spreadsheet), users can link multiple representations to further investigate a mathematical situation. If a constructed figure is measured in Geometry View, data can be collected into a Spreadsheet View to numerically verify invariant situations.

A right triangle is the setup in the *Right Triangle Invariants* activity [`https://ggbm.at/jnsHs8sK`], beginning with the readily apparent property that the two acute angles are complementary, having angle measures that add up to ninety degrees. This leads to the less obvious result that $sin(A) = cos(B)$, and students can use the definitions of the sine and cosine ratios along with a labeled diagram to justify this. The activity can be further extended to verify the Pythagorean Identity $sin^2(x) + cos^2(x) = 1$ and can also illuminate other properties that were previously merely presented to students for memorization.

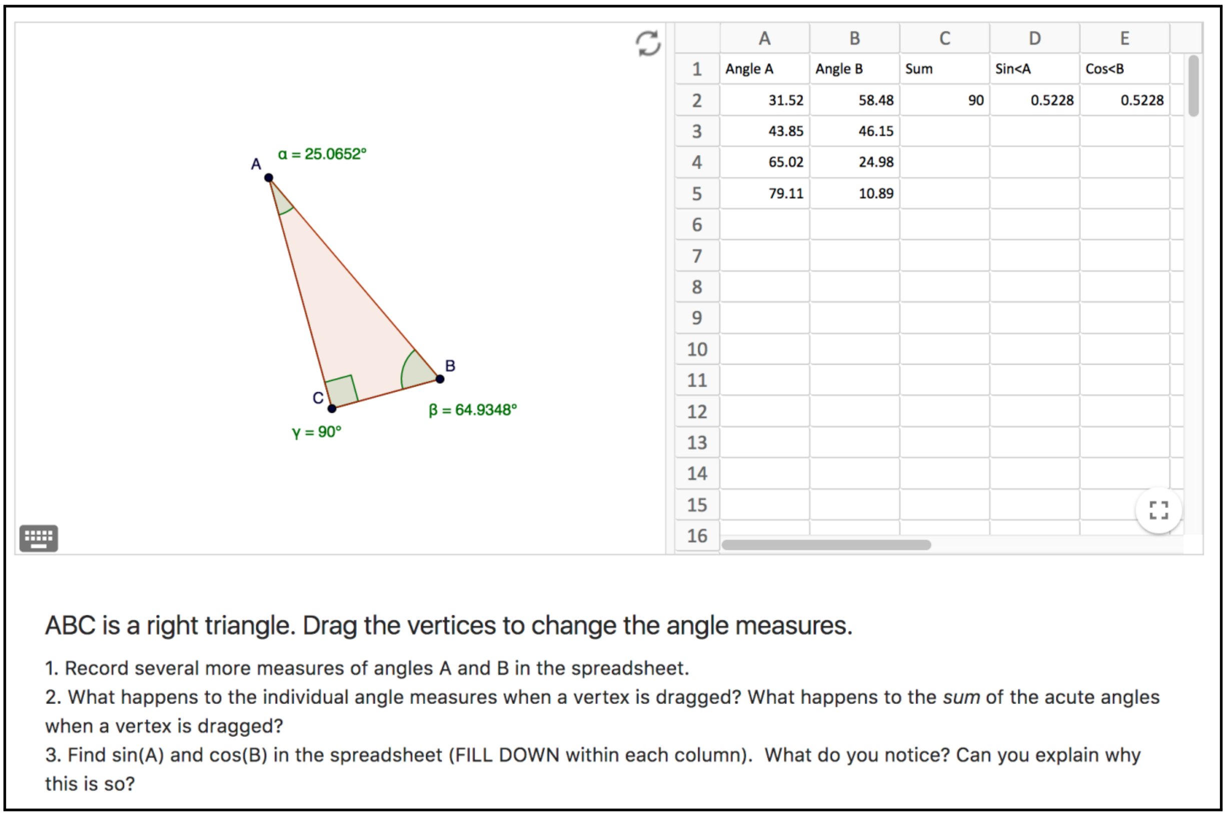

Figure 7. *Right Triangle Invariants* (available at `https://ggbm.at/jnsHs8sK`).

6 Planning for Reflection

In all of the activities described in this paper, explicit reflection questions are a crucial component, not just an addendum to the GeoGebra applets. Teachers can foster reflection by asking students questions that will promote sense-making and push their thinking forward, with prompts such as these (Dick & Hollebrands, 2011; Martin et al., 2009):

- What will happen if . . . ? [prediction]
- What must change to make . . . happen? [cause and effect]
- How is . . . related to . . . ? [compare and contrast]
- When will . . . be true? [conjecture, testing, generalization]
- Why do you think this happens? [justification]
- How do you know? [evidence and proof]

These questions are the essence of the sense-making process; without them, even the most compelling action-consequence situation will be ineffective at moving students toward deep learning and long-term understanding (Brown, et al., 2014; Dick & Hollebrands, 2011). Teacher guidance using focusing questions, discussion of mathematical thinking, and clear lesson summaries, are much more effective at facilitating learning than assuming students will be able to “get it” just because they used a technology tool in the lesson (Barton, 2018; Herbel-Eisenmann & Breyfogle, 2005). It is critical for teachers to plan for reflection, making it a part of classroom practice and one more important “habit of mind” for our mathematics students (Cuoco, et al., 1996).

If we leave reflection to chance, we take the risk that students might not pay attention to the mathematical objectives we intended for the learning episode, hindering their progress and retention. Without reflective questions, students might recall that they used the GeoGebra apps, but not master or remember the mathematics we aimed to teach. The simple reflection routine, *"What do you observe? What changes? What stays the same?"* opens the door for deep mathematical understanding. GeoGebra technology has great power to enhance learning—maximize this power with the Action-Consequence-Reflection cycle and make the math stick for your students' success.

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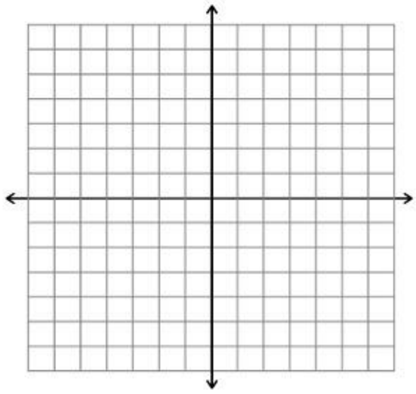
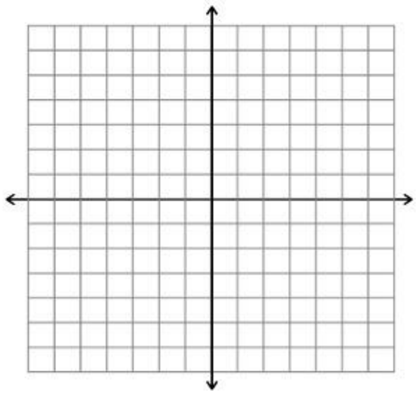
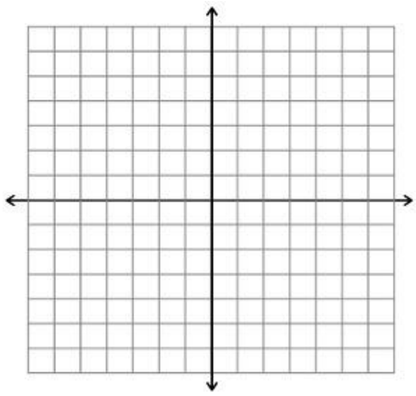
Karen D. Campe, `karen.campe@gmail.com`, has fifteen years' experience teaching secondary math and pre-service teachers, and has provided technology professional development since 1998 at conferences, workshops, and webinars. Karen is passionate about using all types of technology to enhance student understanding in mathematics.

The author would like to thank Tim Brzezinski, Marie Nabbout, and Steve Phelps for their assistance with some of the GeoGebra applets.

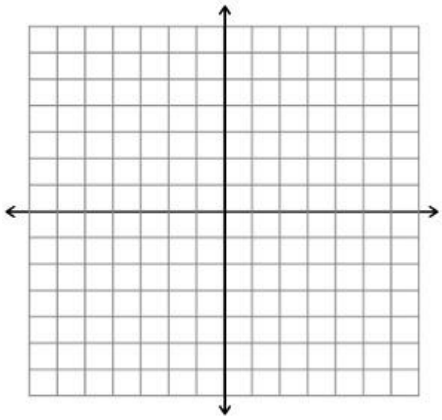
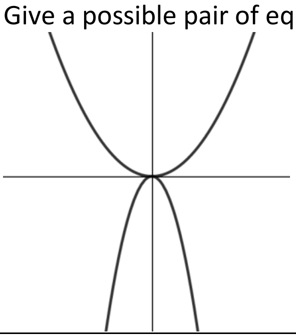
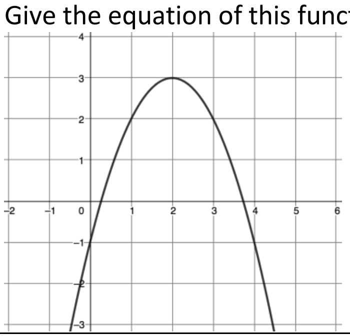
End Behavior Investigation

ggbm.at/GzQQNNr4

1	Use the slider to graph these functions: **Y1=** x^2 **Y2=** x^4 **Y3=** x^6 What do you observe?	
2	Use the slider to graph these functions: **Y1=** x^3 **Y2=** x^5 **Y3=** x^7 What do you observe?	
3	Graph these: **Y1=** x^2 **Y2=** $-x^2$ And then these **Y1=** x^3 **Y2=** $-x^3$ Explain.	
4	How does the degree of a polynomial affect the graph? What would x^{11} look like? What would x^{12} look like?	
5	How does the sign of the leading coefficient affect the graph?	
6	Go to www.geogebra.org/graphing and graph the functions for #1 above on one set of axes. Make a sketch. What do you notice about the graphs between x = 0 and x = 1? Can you explain why this is so?	

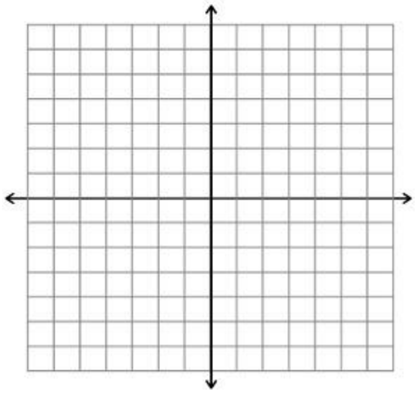
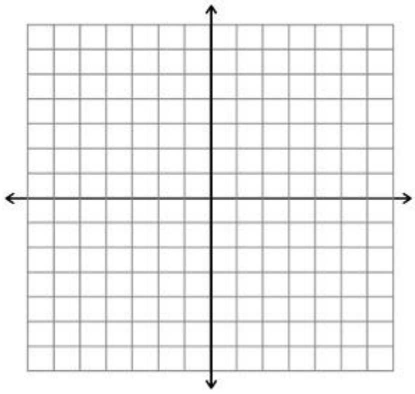
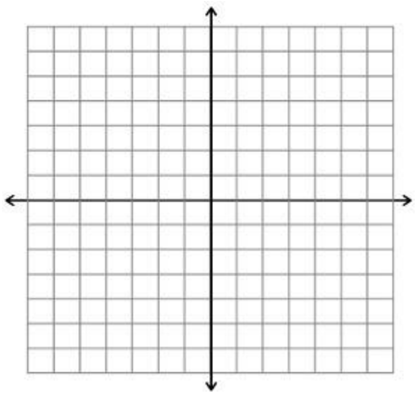
Function Transformations Investigation

ggbm.at/YNqVpHQS

1	Use the sliders to adjust the parameter ***A*** with the parent function $f(x) = x^2$. Describe what you observe. Does ***A*** affect the shape, location, or orientation of the graph?
2	Graph these functions **Y1=** x^2 **Y2=** $2x^2$ **Y3=** $5x^2$ **Y4=** $0.5x^2$ **Y5=** $-x^2$ Explain what happens when **A** gets larger, smaller (between 0 and 1), or negative: 
3	Now adjust parameters ***H*** and ***K*** while ***A*** stays the same. What do you observe? Do ***H*** and ***K*** affect the shape, location, or orientation of the graph? Where is the point (***H***, ***K***)?
4	Give a possible pair of equations for these functions:  Give the equation of this function: 
5	How is the graph of $g(x) = \mathbf{A}(x - \mathbf{H})^2 + \mathbf{K}$ related to $f(x) = x^2$? What is the same and different?
6	Try another parent function: $\sqrt{x}$, $\lvert x \rvert$, log(x), trig functions, and other exponents, bases, or roots. Experiment with different values for ***A***, ***H***, and ***K***. Notice the changes they result in for the graph of each function. Explain what each parameter does to the graph: **A**: **H**: **K**:

Domain and Range Investigation

ggbm.at/mpkdaeze

1	Choose a Linear Function and adjust the 4 sliders on the left. Make a sketch. Use the Domain Illustrator to visualize the domain. Use the Range Illustrator to visualize the range. What do you observe?	
2	Check the box for **Show Grid** for the linear function. What is the domain? _____ $\le x \le$ _____ What is the range? _____ $\le y \le$ _____	
3	Choose a Quadratic Function and adjust the 5 sliders on the left. Make a sketch. Use the Domain Illustrator to visualize the domain. Use the Range Illustrator to visualize the range. What do you observe? How is this similar and different from the linear function?	
4	Check the box for **Show Grid** for the quadratic function. What is the domain? _____ $\le x \le$ _____ What is the range? _____ $\le y \le$ _____	
5	How do the black sliders adjust the graph of a linear function? A quadratic function? How do the red sliders adjust the graph of the functions?	
6	Experiment with the sliders to create a function with the given domain and range. Check the box **Show Equation** to confirm. Make a sketch. Domain: $-4 \le x \le 2$ Range: $0 \le y \le 3$. Equation: $f(x) =$	
7	Design your own domain and range requirements and find a function that meets them. Domain: _____ $\le x \le$ _____ Range: _____ $\le y \le$ _____ Equation:	

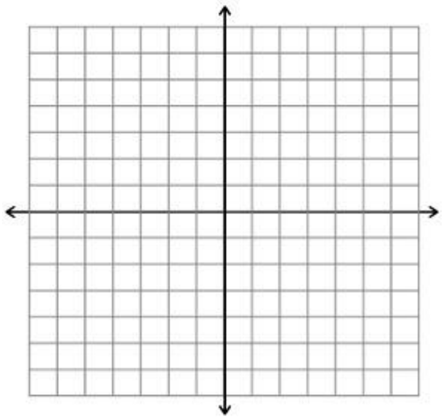
Rational Functions Investigation ggbm.at/sQy2z6kf

1 Use the sliders to change the values of ***H*** and ***K***. Observe the changes on the graph.

Make a sketch of one of your functions:

f(*x*)=

Where is the point (***H***, ***K***)?



2 Now adjust parameter ***A*** while ***H*** and ***K*** stay the same. What do you observe?

Explain what happens to the graph as ***A*** increases? As ***A*** decreases?

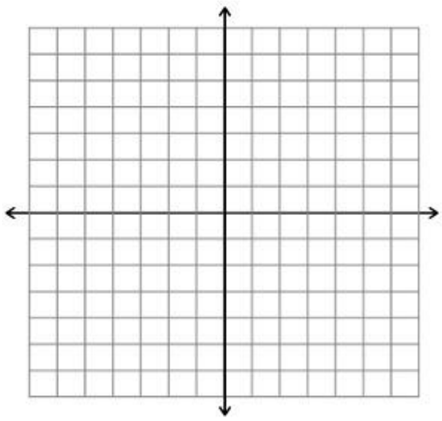
How does the graph change if ***A*** is negative?

3 Use the ***n*** slider to change the exponent from 1 to 2.

Make a sketch of one of your functions.

f(*x*)=

Compare this graph to the graph when the exponent =1. How are they the same and different?



5 Experiment with different values for ***A, H***, and ***K***. Notice the changes they cause for the graphs of $f(x)=\dfrac{a}{(x-h)^1}+k$ and $g(x)=\dfrac{a}{(x-h)^2}+k$. Explain what each parameter does to the graph:

A:

H:

K:

6 What do you think the graph of $h(x)=\dfrac{1}{(x-1)(x+1)}$ will look like compared with $j(x)=\dfrac{1}{(x-1)^2}$?

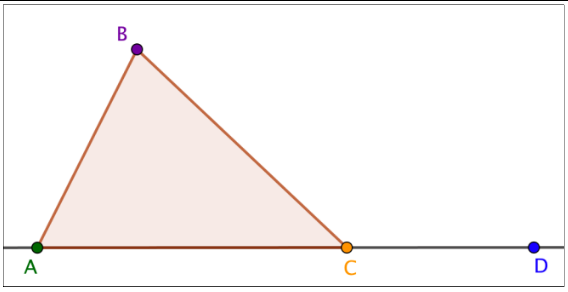
Go to www.geogebra.org/graphing and try different factors in the denominator. Describe your results.

Interior & Exterior Angles of a Triangle Investigation

ggbm.at/z4uthpnr

1 Move point B to change the triangle's shape.

What do you notice about the angle measures as the shape of the triangle changes?



2 Record the measures of the interior angles and exterior angle $\angle BCD$ of your triangle in the first row of this table.

Then move point B and record more sets of measurements (try to create different types of triangles).

$\angle ABC$	$\angle BCA$	$\angle CAB$	$\angle BCD$

3 What is the sum of the 3 interior angles of the triangle?

Does this sum change when the triangle changes shape?

4 What happens when other pairs of angles are added together? Check the boxes and then drag point B.

Sum of interior angles $\angle CAB$ + $\angle ABC$ = _________

Sum of interior $\angle BCA$ + exterior $\angle BCD$ = _________

Do these sums change when the triangle changes shape?

5 How is exterior $\angle BCD$ related to the other angles? Explain.

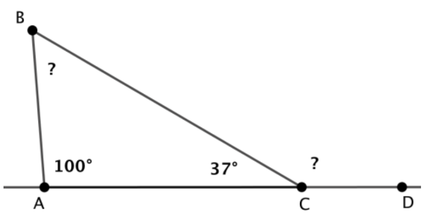
6 List the measures of the exterior angles of your triangle (one at each vertex).

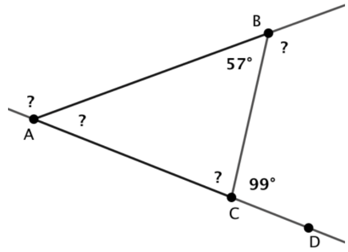
What is their sum?

Why do you think it is important to use one exterior angle at each vertex?

Exterior $\angle$ at vertex *A*	Exterior $\angle$ at vertex *B*	Exterior $\angle$ at vertex *C*

7 Find the missing angles in each diagram, then create & solve your own.





8 *Optional:* Prove one of the conclusions from this activity.

Right Triangle Investigation

ggbm.at/jnsHs8sK

1 Record several measures of angles A and B.

What do you notice about the angle measures and their sum?

What happens to the individual angle measures when a vertex is dragged?

What happens to the sum of the angle measures when a vertex is dragged?

Why do you think this happens?

Angle **A**	Angle **B**	**Sum**

2 Find sin(A) and cos(B) in the spreadsheet. (FILL DOWN within each column.)

What do you notice? Can you explain why this is so?

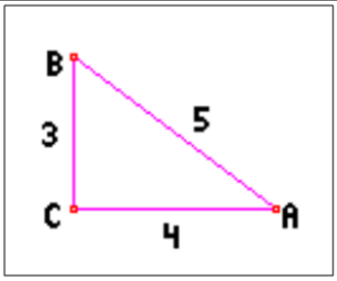
Sin(A)	**Cos(B)**

3 Look at this right triangle diagram. Write ratios for each of these:

Sin $\angle A$ = Sin $\angle B$ =

Cos $\angle A$ = Cos $\angle B$ =

How do the ratios relate to the table values above?



4 Summarize the relationship between sin(A) and cos(B) in words or with a mathematical rule.

5 *Optional*: Investigate the expression $\sin^2(A) + \cos^2(A)$.
Drag the left bar of the spreadsheet panel to reveal columns F and G of the spreadsheet.

FILL DOWN for columns F and G and examine the table.

Why do you think this result occurs?

Can you explain why this property is known as the "Pythagorean Identity"?
Hint: draw a right triangle with hypotenuse = 1.

D	F	G
Sin(A)	**Cos(A)**	**$\mathbf{Sin^2+Cos^2}$**