TRANSFORMATIONS AND COMPLEX NUMBERS

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Abstract

In this paper, we use complex-number operations to carry out transformations of points and graphs of functions and establish connections between geometry and algebra in the high-school curriculum. We use dynamic geometry software to visualize the geometric effect of these algebraic operations and connect complex-number operations to translations, rotations, and dilations.

Keywords: GeoGebra; Common Core; Complex Numbers; Algebra; Geometry; Transformations

1 Introduction

Students typically first encounter complex numbers when solving quadratic equations, that is equations of the form $ax^2 + bx + c = 0$ where a, b, and c are real numbers with $a \neq 0$. For instance, applying the quadratic formula, $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$, to $3x^2 + x + 5 = 0$, they determine that the discriminant, $b^2 - 4ac = -59$ is negative. Many times, students are told "they cannot take the square root of a negative number." Later on, students learn that it is *possible* to take the square root of a negative number and, in particular, $i = \sqrt{-1}$, or equivalently $i^2 = -1$. We call i an imaginary or complex number. A complex number is of the form $z = a + bi$, where a and b are real numbers and i satisfies $i^2 = -1$.

The Complex Number System appears in the Common Core State Standards for Mathematics (CCSS-M) in *Number and Quantity, High School*. Students learn to perform complex number operations and represent addition, subtraction, multiplication, and conjugation of complex numbers geometrically on the complex plane (CCSS. Math.Content.HSN.CN.B.5). In this paper, we use these complex-number operations to carry out transformations of points and graphs of functions. We seek to establish connections between geometry and algebra in the high-school classroom. During our discussion, we use the dynamic geometry software, GeoGebra (Hohenwarter, 2002) to visualize the geometric effect of these algebraic operations. We start by providing some historical comments on complex numbers. Next, we connect some complex-number operations with their corresponding geometric transformation. We conclude the paper with reflections on our work with teachers and students.

2 Historical Notes on Complex Numbers

Many have written about the solution of the cubic equation and our historical notes stem from Dunham's work (1990). Historically, Gerolamo Cardano's (1501-1576) work on the solution of the general cubic equation helped to give complex numbers a legitimate place. Cardano solved the general cubic equation, $x^3 + px^2 + qx + r = 0$ in 1545 by using the substitution $z = x - \frac{p}{3}$ to eliminate the quadratic term, thus obtaining a "depressed" cubic, $z^3 + bz + c = 0$. Next, he applied the formula obtained from Niccolo Fontana (1500-1557) to solve the depressed cubic. Scipione del Ferro (1465-1526) had also derived this formula 30 years earlier but had not published it. Both Fontana and del Ferro showed that one of the solutions of $x^3 + bx + c = 0$ is given as

$$x = \sqrt[3]{-\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}} - \sqrt[3]{\frac{c}{2} + \sqrt{\frac{c^2}{4} + \frac{b^3}{27}}}$$

In particular, when Cardano solved polynomials such as $x^3 - 6x + 4 = 0$ we note that the solution given by the formula is $x = \sqrt[3]{-2 + \sqrt{-4}} - \sqrt[3]{2 + \sqrt{-4}}$. One can easily check that $x = 2$ is a solution and use algebra to show that the two other solutions are $x = -1 \pm \sqrt{3}$. Thus the number

$$x = \sqrt[3]{-2 + \sqrt{-4}} - \sqrt[3]{2 + \sqrt{-4}}$$

corresponds to one of these solutions. As a matter of fact, $x = \sqrt[3]{-2 + \sqrt{-4}} - \sqrt[3]{2 + \sqrt{-4}} = 2$.

Thus, as Cardano would have put it, 2 was "disguised" as $x = \sqrt[3]{-2 + \sqrt{-4}} - \sqrt[3]{2 + \sqrt{-4}}$. Yet it would be another two centuries before Euler, Gauss, and Cauchy made it evident that complex numbers were an important and vital part of the mathematical landscape. In the next sections, we discuss how complex-number operations are related to geometric transformations. We start with complex-number addition and their relationship to translations.

3 Review of Complex Number Operations

We recall that a complex number, $z = a + bi$, has two parts the *real part*, a, which is denoted by $\mathrm{Re}(z)$, and the *imaginary part*, b, which is denoted by $\mathrm{Im}(z)$. When we add or subtract complex numbers, say $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, we add the real part of z_1 to the real part of z_2 and the imaginary part of z_1 to the imaginary part of z_2. In other words, $z_1 + z_2 = (a_1 + a_2) + (b_1 + b_2)i$. This is very similar to how we add vectors, which is component-wise. That is the sum of vectors $\binom{a_1}{b_1}$ and $\binom{a_2}{b_2}$ is defined to be

$$\binom{a_1}{b_1} + \binom{a_2}{b_2} = \binom{a_1 + a_2}{b_1 + b_2}.$$

Recall that subtraction of complex numbers is defined similarly, that is $z_1 - z_2 = (a_1 - a_2) + (b_1 - b_2)i$, which is again analogous to how we subtract vectors. This is why it is possible to identify the set of complex numbers to the set of vectors on a plane. Hence we can identify the complex number $z_1 = a_1 + b_1 i$ to the vector $\binom{a_1}{b_1}$.

To multiply complex numbers $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$ we recall that we distribute as when we multiply binomials, and then use the fact that $i^2 = -1$, to collect like terms. That is,

$$\begin{aligned} z_1 \cdot z_2 &= (a_1 + b_1 i) \cdot (a_2 + b_2 i) \\ &= a_1 a_2 + a_1 b_2 i + b_1 a_2 i + b_1 b_2 i^2 \\ &= a_1 a_2 - b_1 b_2 + (a_1 b_2 + b_1 a_2) i \end{aligned}$$

For completeness we recall that when we ask students to divide complex numbers we ask them to write the quotient as a complex number. This means that when we divide $z_1 = a_1 + b_1 i$ by $z_2 = a_2 + b_2 i$

we multiply the ratio $\frac{z_1}{z_2}$ by $\frac{\overline{z_2}}{\overline{z_2}}$, where $\overline{z_2} = a_2 - b_2 i$ is called the *complex conjugate* of z_2. This means that

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{a_1 + b_1 i}{a_2 + b_2 i} \cdot \frac{a_2 - b_2 i}{a_2 - b_2 i} \\ &= \frac{a_1 a_2 + b_1 b_2 + (a_2 b_1 - a_1 b_2) i}{a_2^2 + b_2^2}.\end{aligned}$$

Readers can relate this approach to the "rationalize the denominator" technique.

4 Connecting Addition and Subtraction with Translations in the Complex Plane

Students usually understand that to translate a point in the plane horizontally, they need to add a number to, or subtract a number from the x-coordinate of the point. Similarly, to translate a point in the plane vertically, students will add or subtract a number from the y-coordinate. This idea parallels the geometric interpretation of addition and subtraction of complex numbers. We begin by exploring how the operations of addition and subtraction are represented in the complex plane. Next, we discuss a more general case of transformations of a plane.

Take two complex numbers, say $z_1 = 1 + 2i$ and $z_2 = 4 + i$ (Note: To type z_1 in GeoGebra we enter `z_1` in the Input box at the bottom of the screen.) The Algebra View displays them in symbols while the Graphics Views shows them as points (Figure 1).

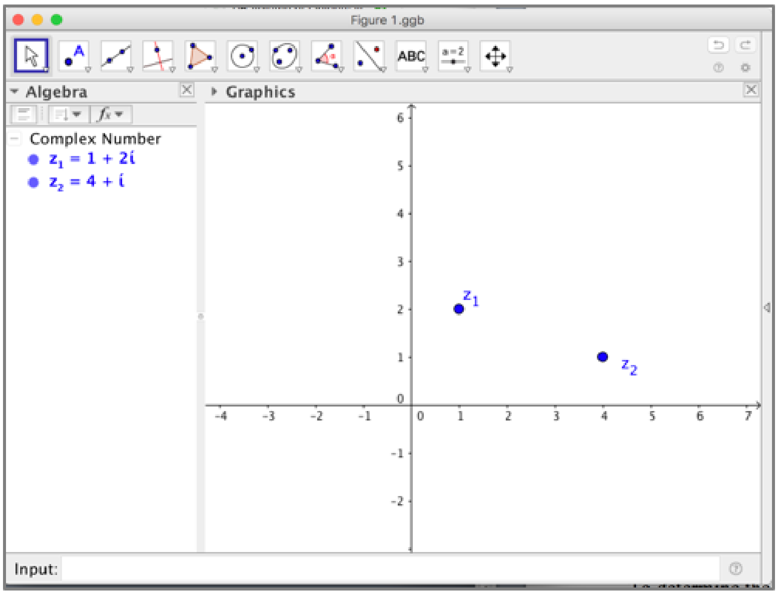

Figure 1. Representing complex numbers on the plane.

To determine the sum of the two complex numbers z_1 and z_2, we enter $z_3 = z_1 + z_2$ in the Input box. The software automatically displays the sum z_3 with a label on the complex plane (Figure 2).

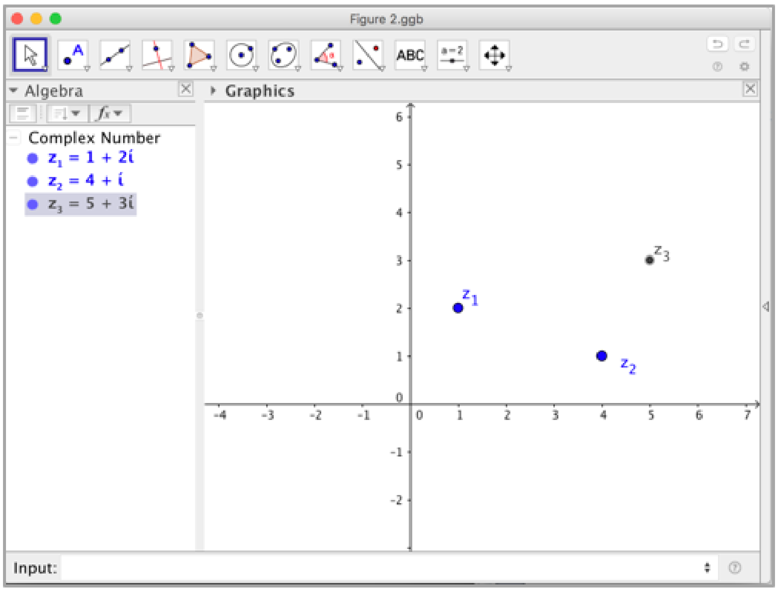

Figure 2. The sum of z_1 and z_2 represented in the complex plane.

We now observe the geometry of adding two complex numbers. In Figure 3 we formed a quadrilateral by connecting the four complex numbers,z_0, z_1, z_2, and z_3, where z_0 is the complex number $0+0i$.

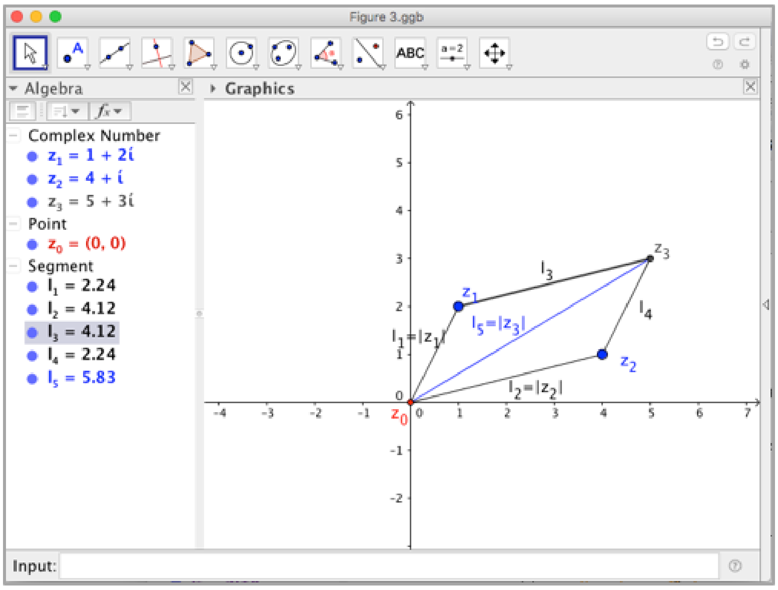

Figure 3. Algebraic and geometric sum of complex numbers.

Readers can recognize that the sum of complex numbers is related to vector addition, where the sum of the vectors corresponding to $\binom{1}{2}$ and $\binom{4}{1}$ is the vector $\binom{5}{3}$ and the quadrilateral is a parallelogram, as we shall see later. Thus we can see how complex numbers provide a geometric connection to vector addition since $(1 + 2i) + (4 + i) = (1 + 4) + (2i + i) = 5 + 3i$. Figure 3 also shows in the Algebra panel the length of each segment of the quadrilateral connecting two complex numbers. As we can see l_1 represents the length between z_0 and z_1, l_2 represents the length between z_0 and z_2, l_3 represents the length between z_1 and z_3, and l_4 represents the length between z_2 and z_3. We observe that the opposite sides of the quadrilateral are congruent, that is $l_1 = l_4$ and $l_2 = l_3$ thus the quadrilateral is a parallelogram. We label the length of each segment in the Graphics panel using modulus notation. The modulus of a complex number $a + bi$ is denoted by $|a + bi|$ and defined as $|a + bi| = \sqrt{a^2 + b^2}$, that is it can be viewed as the distance of the complex number to the origin. The importance of the modulus will be discussed in the section on multiplication and division of complex numbers.

Similarly, to determine the difference of the same complex numbers z_1 and z_2, we enter in the input box, $z_4 = z_1 - z_2$. The software automatically displays the difference with a label on the complex plane and we used a different color (right click on the object and scroll down to Properties – this will open a box that will allow you to color the object) when drawing the segment from the origin to z_4 to distinguish this segment from the others (Figure 4). We also computed the difference $z_5 = z_2 - z_1$. This segment is also distinguished with a different color. Note that $l_6 = |z_4|$ and $l_7 = |z_5|$.

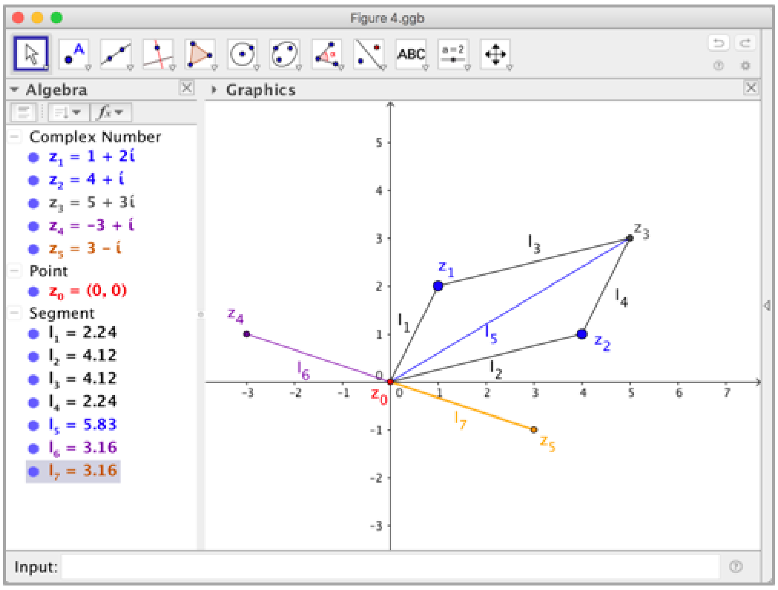

Figure 4. Algebraic and geometric difference of complex numbers.

We briefly comment that the difference of two complex numbers is related to the Triangle Midsegment Theorem to demonstrate one of many possible extensions that can be pursued using the geometry of complex numbers. Recall that the Triangle Midsegment Theorem states that the segment joining the midpoints of any two sides of a triangle will be parallel to the third side and half the length of the third side. Thus if we connect z_4 to z_1, z_5 to z_2, and z_1 to z_2, with segments l_8, l_9, and l_{10}, respectively, we observe that indeed the segments l_6, l_7, and l_{10} are congruent (Figure 5).

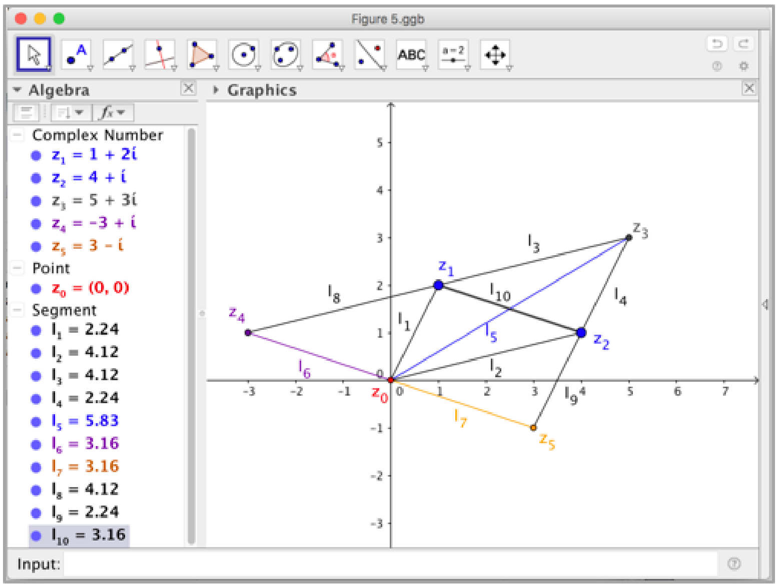

Figure 5. An illustration of the Triangle Midsegment Theorem with complex numbers.

One last note about subtraction – indeed the simplest way to think of subtraction of two complex numbers is that the opposite of the second complex number is being added to the first. In other words

$$z_4 = z_1 - z_2 = z_1 + (-z_2).$$

Note that this also coincides with how we explain vector subtraction, giving us yet another connection between algebra and geometry. We invite readers to drag complex numbers z_1 or z_2, and observe how the parallelogram changes. As you do so, reflect on the connections to vector addition and subtraction. In the next section, we use complex numbers to translate graphs of functions in the software.

5 Complex-Number Addition and Translations of a Graph

Remember that we can identify a complex number $a+bi$ to the point (a, b) in the complex plane. This will be a very useful representation as we continue exploring. On the real plane we graph a function, say $f(x) = x^2$, by plotting the points $(x, f(x))$. By doing this, we conceptualize the function as a mapping from the real numbers to the real numbers. The same graph with points $(x, f(x))$ plotted on the complex plane corresponds to a different interpretation: the inputs are the real part of the complex number $x + ix^2$ while the outputs are the imaginary part of the same complex number.

To translate the graph of the function we shift each of the points on the graph. For example, to shift one unit to the right and two units up, we add one unit to the x-value and two units to the y-value, thus obtaining points of the form $(x + 1, f(x) + 2)$. We can use the tools in the software to shift the graph of a function as described above. First, graph $f(x) = x^2$ by typing $f(x) = x^2$ in the Input box (see Figure 6). Then, construct a point A on the graph of $f(x)$ using the Point tool. Note that the software uses the notation $(x(A), y(A))$ to describe the point A on the graph of $f(x) = x^2$. Next, construct another point B on the plane. This point has coordinates $(x(B), y(B))$. Define in the Input box a point C as $C = (x(A) + x(B), y(A) + y(B))$. We drag the point A using the Move tool (pointer) and see the effect on the point C, the translation of A by the point B.

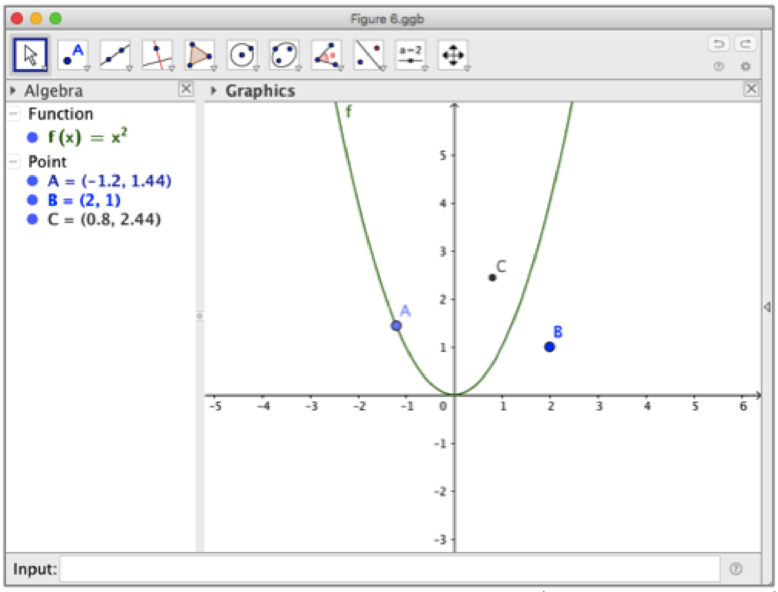

Figure 6. Constructing a translation of point A on the graph of $f(x)$.

Next, we use another dynamic geometry software tool, Locus (on the 4^{th} button from the left), to produce the result of shifting the whole graph. The software provides directions on how to use this tool (see Figure 7). We first select the point C, then point A, and the software provides the resulting graph (see Figure 8). We invite readers to drag the entire graph of $f(x)$, or the point B, using the Move tool and consider the horizontal and vertical translations given by the coordinates of the point B . Furthermore, we can easily redefine $f(x)$ by simply typing another function in the Input Box. For instance, try a "depressed" cubic such as $f(x) = x^3 - x - 1$. Notice that it has one real solution and two complex solutions. We now shift our focus to discuss the connections between the operations of multiplication and division of complex numbers with rotations and dilations of the plane.

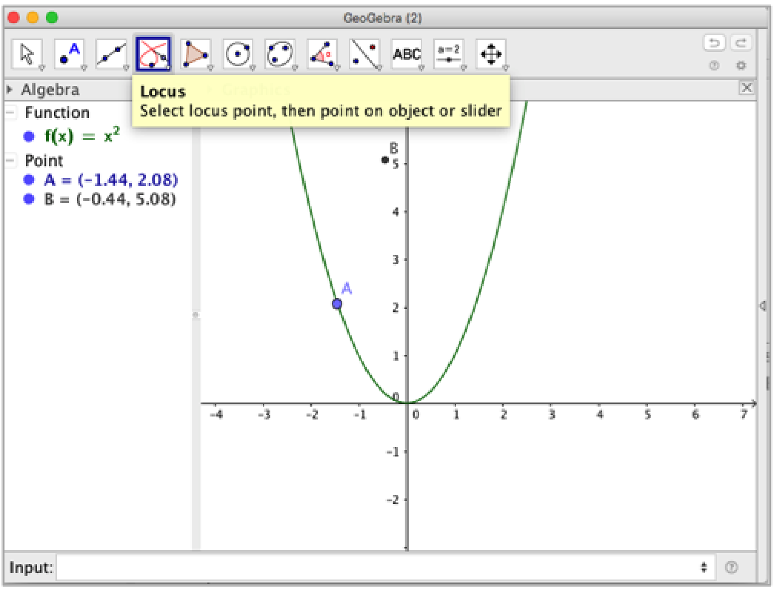

Figure 7. Locus command on GeoGebra.

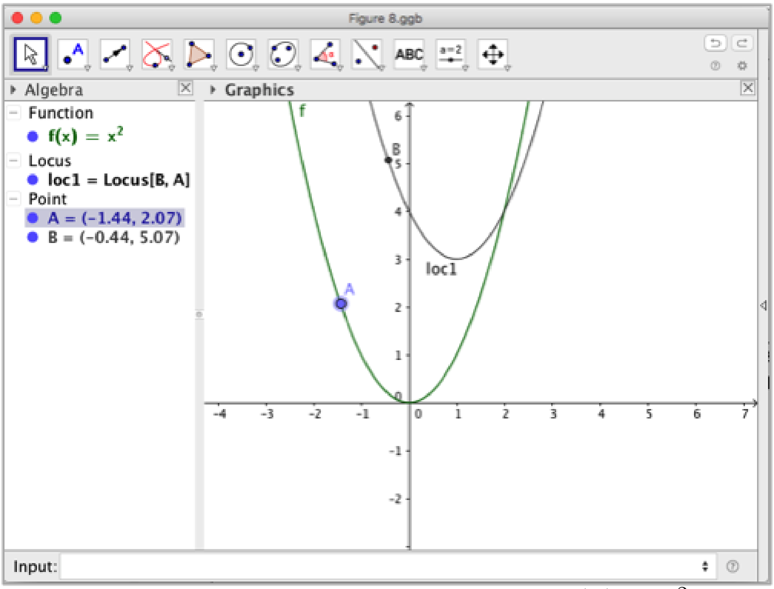

Figure 8. Translation of the graph of $f(x) = x^2$.

6 Connecting Multiplication with Transformations in the Complex Plane

We start our discussion with a particular situation – consider the complex number $z_1 = 1 + 2i$ and multiply it by the complex number $z_2 = 0 + i$. We enter both of them in the Input box using the notation we introduced in the previous sections. To obtain the product of these two complex numbers we type `z_3=z_1*z_2` in the Input box. See the resulting product in Figure 9. Note the relationship between z_1 and z_3: it appears that z_3 is the point obtained by rotating z_1 90° counterclockwise about the origin. If you need extra data to make this conclusion, drag the complex number z_1 and observe the position of z_3. We suggest you consider the coordinates of both z_1 and z_3.

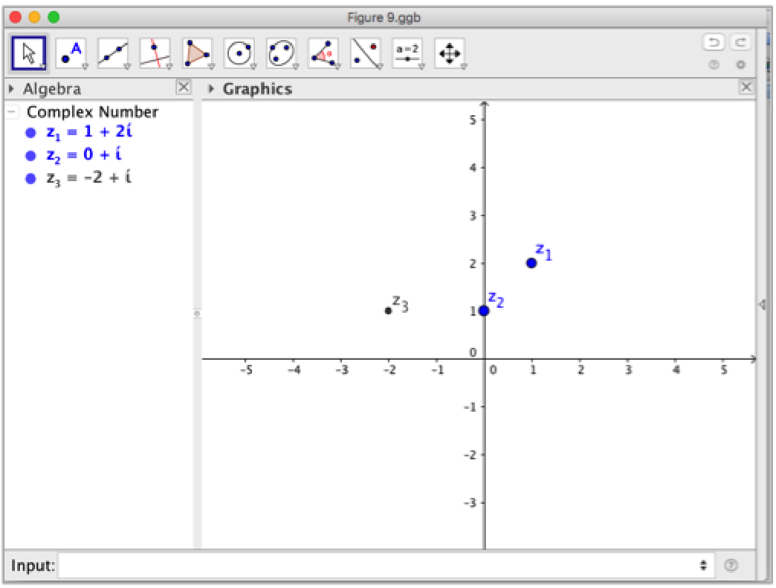

Figure 9. Multiplication of a complex number by i.

We now focus on changing z_2. Consider the effect of multiplying by $z_2 = 0 + 2i$ instead. In other words, we translate z_2 one unit up vertically. In this case, z_3 is the product of z_1 by $2i$. Before continuing consider the following question *What is similar and what is different when multiplying by 2i than when multiplying by i*? You are on the right track if you thought that z_3 is again a $90°$ counterclockwise rotation of z_1 about the origin. However, somehow z_3 has also shifted away from the origin (see Figure 10). How far is z_3 from the origin and how can we determine this using what we know about z_2? If we compare the moduli of z_1 and z_3, we note that $|z_3| = 2 \cdot |z_1|$. Before continuing, drag z_1 to points where you can focus on its coordinates. Next, consider the question *Would a similar situation occur if we continue moving z_2 along the y-axis*? Try it!

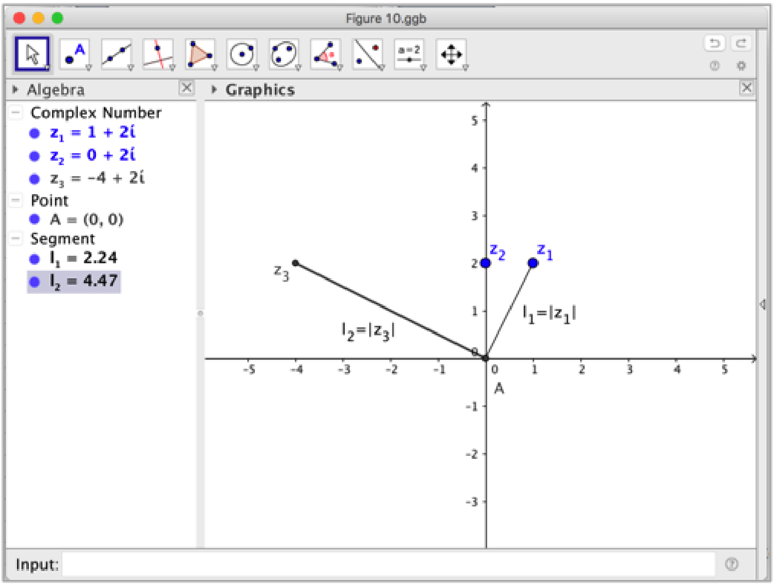

Figure 10. Multiplication of a complex number by $2i$.

We again note that z_3 is a $90°$ counterclockwise rotation of z_1 about the origin and that its modulus changes by a factor of $|z_2|$. Let's move z_2 again – this time to the complex number $1 + i$. If we consider the modulus of z_2 (construct the segment from the origin to z_2), the software displays about 1.44. *Can you guess what this number is*? Let's compute the following ratio: $\frac{|z_3|}{|z_1|}$ (see Figure 11), by entering `ratio = l_2/l_1` in the Input box, where $l_2 = |z_3|$ and $l_1 = |z_1|$. Some readers may observe that

$$\frac{|z_3|}{|z_1|} \approx |z_2| = \sqrt{2}.$$

Is this true? If we move z_2 about the complex plane, is it always true that $\frac{|z_3|}{|z_1|} = |z_2|$? This is a great opportunity to recall the polar representation of a complex number $z = a + bi$. Namely, z can be represented as $z = r(\cos\theta + i\sin\theta)$, where $r = \sqrt{a^2 + b^2}$ and θ is the angle of the counterclockwise rotation about the origin from the point $(r, 0)$ to the point (a, b). The polar representation is particularly useful to compute the product of two complex numbers. The product of the complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ is given by:

$$\begin{aligned} z_1 \cdot z_2 &= [r_1(\cos\theta_1 + i\sin\theta_1)] \cdot [r_2(\cos\theta_2 + i\sin\theta_2)] \\ &= r_1 \cdot r_2[\cos(\theta_1 + \theta_2) + i\sin\theta_1 + \theta_2)]. \end{aligned}$$

In Figure 11 we have that $z_3 = z_1 \cdot z_2$, which is why the moduli of the product, z_3, is a dilation of the moduli of z_1 by a factor of r_2, and the angle of the rotation from z_1 to z_3 about the origin is θ_2.

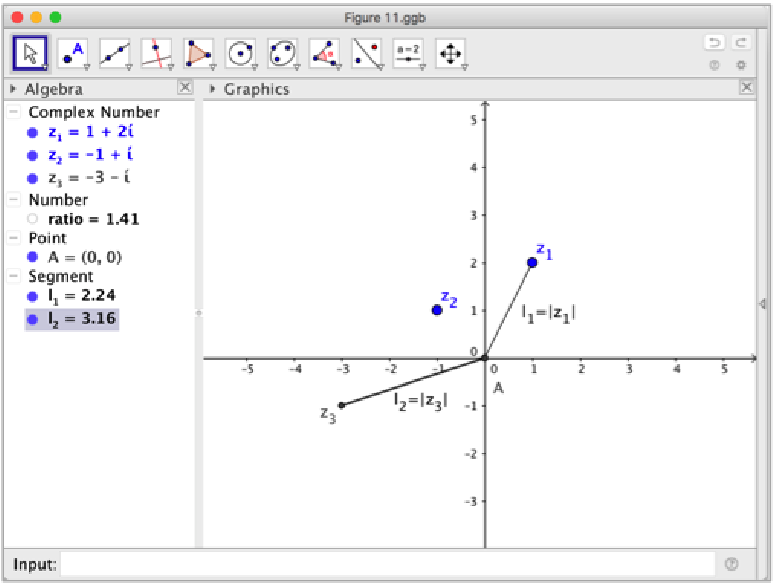

Figure 11. Multiplication of a complex number by $1 + i$.

In our examples above, we can provide a mathematical justification of the $90°$ counterclockwise rotation about the origin and why the modulus of z_1 changes by a factor of $|z_2|$. Note that we can reverse the roles played by z_1 and z_2 in the above discussion due to the commutative property of multiplication. In the next section, we use complex number multiplication to rotate graphs. Again, we use the power of the locus tool to produce the resulting graph.

7 Rotation of a Graph using a Complex Number

Imagine that we want to rotate the graph of the function $f(x) = x^2$ by an angle of $45°$ counterclockwise about the origin. We now understand enough so that our initial idea might be to multiply the complex number $z = x + iy$ by the complex number $z_1 = 1 + i$ since the argument of the complex number z_1 is $45°$. Consider a complex number A with coordinates $(x(A), y(A))$, and multiply it by the complex number $1 + i$ as follows:

$$\begin{aligned}[x(A) + iy(A)] \cdot [1 + i] &= x(A) + iy(A) + ix(A) - y(A)\\ &= [x(A) - y(A)] + i[x(A) + y(A)].\end{aligned}$$

Thus, to rotate a point A with coordinates $(x(A), y(A))$ on the graph of $f(x) = x^2$ by $45°$ counterclockwise about the origin, we just need to apply the transformation $\big(x(A) - y(A), x(A) + y(A)\big)$. We now provide directions to do it with the software. Construct first the graph of $f(x) = x^2$. Next, construct a point A on the graph of $f(x) = x^2$. Type in the Input box the following command: `B=(x(A)-y(A), x(A)+y(A))`. Now to rotate the graph of $f(x)$, we use the Locus tool, clicking on B first and then on A. The software then creates the "desired" parabola (see Figure 12). Readers may notice that points A and B are not equidistant to the origin.

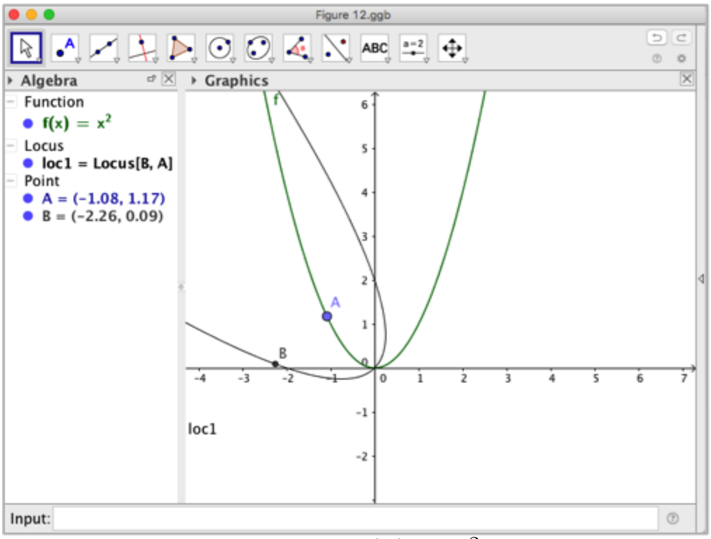

Figure 12. Initial attempt at rotating the graph of $f(x) = x^2$ by 45° counterclockwise using complex number multiplication.

We recall that the point B is not only a rotation of point A, but also a dilation by a factor of $\sqrt{2} = |1+i|$. Thus our parabola is not only a rotation of $f(x) = x^2$ about the origin by 45° counterclockwise, but also a dilation by a factor of $\sqrt{2}$. *How do we address this issue*? We "normalize" our complex number z_1, that is we divide z_1 by $\sqrt{2}$, and use this new complex number to obtain the following transformation that is only a counterclockwise rotation about the origin by 45°:

$$\left(\frac{1}{\sqrt{2}}(x(A) - y(A)), \frac{1}{\sqrt{2}}(x(A) + y(A))\right)$$

Note that it is nearly impossible to tell the difference in the graphs in Figure 12 and Figure 13. If we look closely, we see that indeed there is a difference between the transformed graph in Figure 12 and the transformed graph in Figure 13. Note that the transformed graph in Figure 12 passes through the point $(0, 2)$ whereas the transformed graph in Figure 12 does not. Thus if we just want to rotate the graph by 45° we must normalize the number that we want to multiply by. This indeed can be a great way of introducing the idea of normalization before covering vectors. Again, we can easily redefine the function $f(x)$ in the Input box. Try $f(x) = \sin(x)$ first, and $f(x) = \frac{1}{x}$, next.

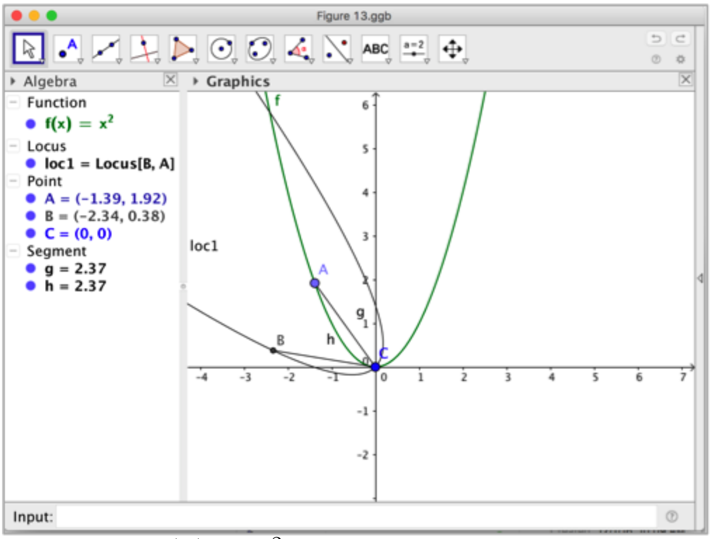

Figure 13. Rotating the graph of $f(x) = x^2$ by 45° counterclockwise using complex number multiplication.

8 Conclusions

In this paper, we have illustrated how dynamic geometry software can help us visualize and therefore connect complex numbers with their geometric interpretation. Specifically, we applied addition (and subtraction) of complex numbers to translate mathematical objects, and multiplication of complex numbers to rotate and dilate mathematical objects. We invite readers to investigate the effect of dividing by a complex number. Another set of transformations to consider are reflections – if addition and subtraction of complex numbers are related to translations and multiplication and division of complex numbers are related to dilations and rotations, *how are complex numbers related to reflections*? We suggest to start by considering how complex numbers are related to reflections about the x- and y-axis.

We have presented these ideas to current and future secondary teachers either in classes or in professional development workshops. In general, teachers are able to perform complex number operations symbolically but when we ask them to describe the geometry of the operations they are usually able to connect addition with translations but do not appear to be familiar with a geometric interpretation of multiplication and division. In particular, teachers recognize how empowering the dynamic software is to help them visualize and understand formulas they have used.

Of course, a question that has been asked of us in the past is why this connection is considered important in the standards. To many this connection seems to not have any real-world or pedagogical benefits. We beg to differ as research suggests that multiple representations of mathematical concepts can develop a deeper understanding of the concept. Yet unknown to most, there is an application in the real world that can also serve as an introduction to the concepts of matrices and vectors. Indeed, this connection between algebra and geometry has led to many developments in image processing and understanding how complex numbers can be represented geometrically can help in gaining a basic understanding of these developments. In a paper in progress, we discuss such a relationship between matrices, transformations, complex numbers, and applications. In the meantime, we hope that you feel compelled to try this approach in your classroom and see how students react when being able to visualize complex number operations with dynamic software.

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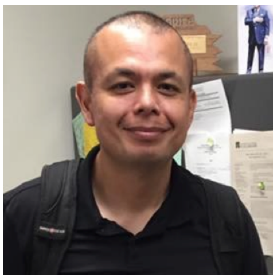

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