EXPLORING EUCLIDEAN AND TAXICAB GEOMETRY WITH DYNAMIC MATHEMATICS SOFTWARE

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Abstract

Technology has changed the nature of mathematics learning and instructional practices (Andreasen and Haciomeroglu, 2013; Edwards, 2015; Haciomeroglu, Bu, Schoen, and Hohenwarter, 2011). Dynamic and interactive technology enriches students' learning opportunities and shifts the focus of instruction to understanding and student-centered learning by providing a means of modeling mathematical relationships (Bu and Henson, 2016; Haciomeroglu, Bu, Schoen, and Hohenwarter, 2011). The authors connect Euclidean and non-Euclidean geometries through an exploration of rich tasks of Taxicab geometry, sharing methods for organizing and presenting tasks to enhance students' understanding of geometry concepts.

Keywords: Taxicab Geometry, non-Euclidean geometry, Manhattan distance

1 Taxicab Geometry

Taxicab geometry is a non-Euclidean geometry in which distance is determined as a taxicab drives in an ideal city with a rectangular grid of streets, not as the crow flies as in Euclidean geometry. For instance, in the Euclidean geometry, the distance between the points $O(0,0)$ and $P(3,3)$ is $3\sqrt{2}$ ($d = \sqrt{((3-0)^2+(3-0)^2)}$), and in the Taxicab geometry, the distance is 6 units ($d = |3-0|+|3-0|$). In the Euclidean geometry, there is only one line with the distance of $3\sqrt{2}$ units from O to P. However, in the Taxicab geometry, there are many routes to connect the given points O and P. As seen in Figure 1, the routes (i) $O \to A \to P$, (ii) $O \to B \to P$, and (iii) $O \to C \to D \to E \to F \to G \to P$ add up to 6 units. Even if an arbitrary point along the side of the square is chosen to take smaller steps, the distance (e.g., $O \to H \to I \to J \to D \to K \to L \to M \to F \to N \to Q \to R \to P$) remains the same.

The Taxicab geometry differs from Euclidean geometry in just one axiom–Side-Angle-Side Similarity (SAS) axiom (Krause, 1986). Consider the isosceles right triangles $\triangle ABC$ and $\triangle BDC$ shown in Figure 2. Assuming that angles in the Taxicab geometry are measured the way that angles are measured in the Euclidean geometry, $\triangle ABC$ is an isosceles right triangle with two legs of length 4 and hypothenuse of length 8, whereas $\triangle BDC$ is also an isosceles right triangle with legs of length 4 but with hypothenuse that is also length 4. These two triangles satisfy the SAS theorem but are not similar in Taxicab geometry. Moreover, The triangle $\triangle BDC$ is an equilateral triangle but is not equiangular (Malkevitch, 2007).

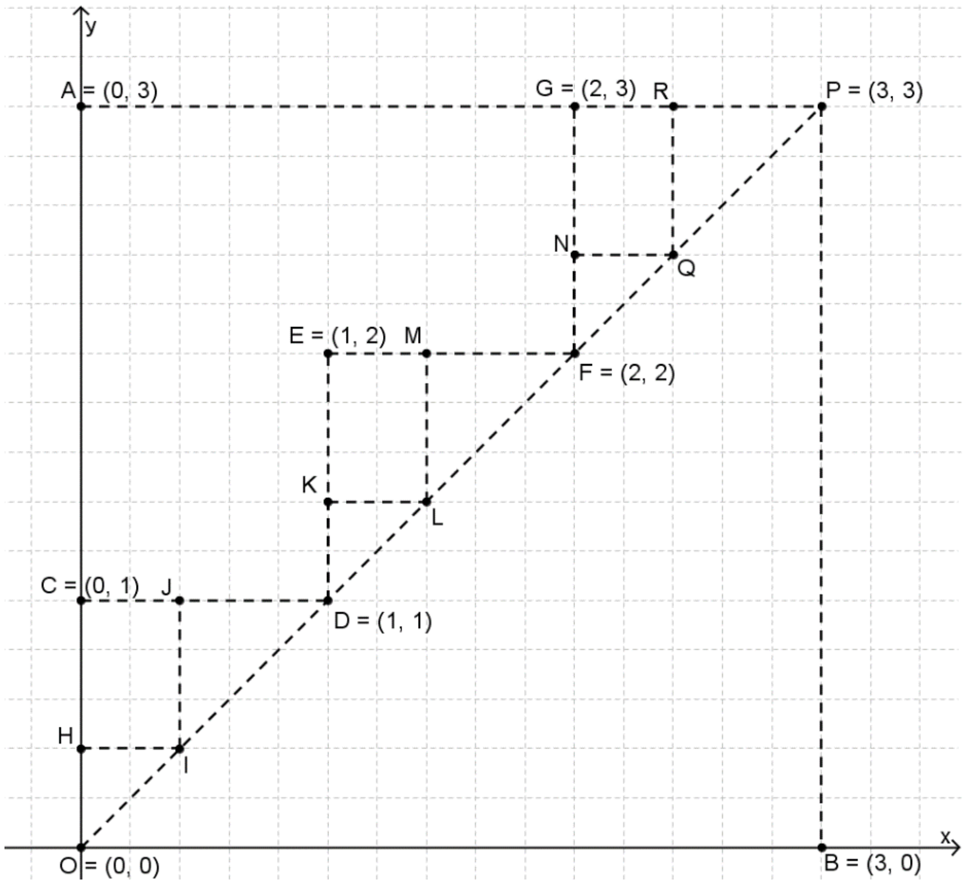


Figure 1. Euclidean and Taxicab distances.

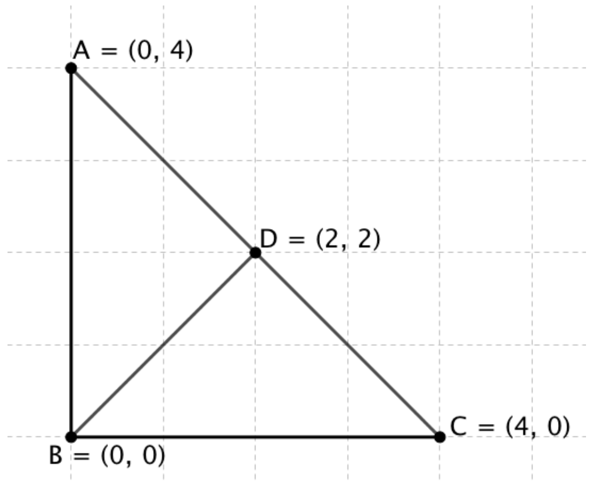


Figure 2. Side-Angle-Side (SAS) axiom.

For students who are familiar with Euclidean geometry and the coordinate plane, Taxicab geometry is easy to understand. For instance, a circle is the set of all points equidistant from a given point in both geometries. However, the set of all points that are equidistant from a point in Taxicab geometry resembles a square in Euclidean geometry due to the definition of distance. As seen in Figure 3, all points on the square are 2 "Taxicab" units from A.

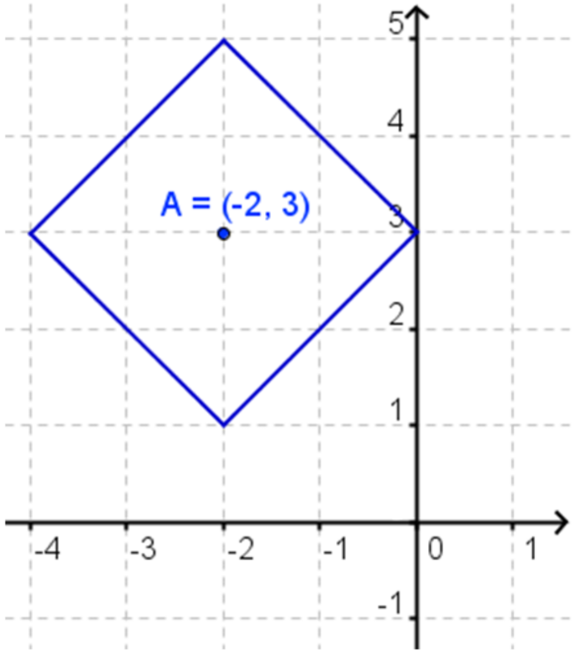


Figure 3. A Taxicab circle.

Consider the following problem taken from *Taxicab Geometry* (Krause, 1986, p. 14). The reader may wish to create representations for the task and reflect on his or her own thinking processes before reading our explanations:

> **The Apartment Problem.** Alice and Bruno are looking for an apartment in Ideal City. Alice works as an acrobat at amusement park $A = (-2, 3)$. Bruno works as a bread tester in bakery $B = (1, -4)$. Being ecologically aware, they walk wherever they go. They have decided their apartment should be located so that the distance Alice has to walk to work plus the distance Bruno has to walk to work is as small as possible. Where should they look for an apartment?

Examples of possible routes between two workplaces (e.g., $A \to C \to B$, $A \to E \to F \to G \to H \to B$, and $A \to D \to B$) are shown in Figure 4. As seen in the figure, the Taxicab distance between the points A and B is 10 units; thus, the sum of the distance of the apartment from A and the distance of the apartment from B cannot not be smaller than 10 units. Before attempting to solve The Apartment Problem with students, we suggest solving related tasks with distances provided. Tasks of this sort will help students more fully understand The Apartment Problem and help them solve it.

To attack The Apartment Problem, let's begin by assuming that Alice and Bruno want to walk the same Taxicab distance to work. In other words, they want to rent an apartment, which is 5 units from A and 5 units from B. If the set of all points at 5 taxi-distance units from A and B are graphed, two Taxicab circles are formed. This is illustrated in Figure 5.

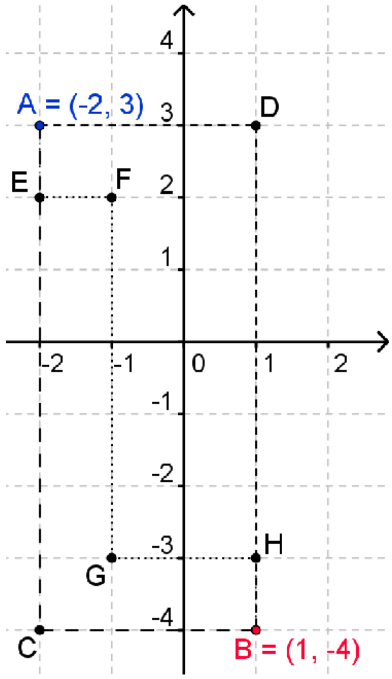


Figure 4. Taxicab distance between points A and B.

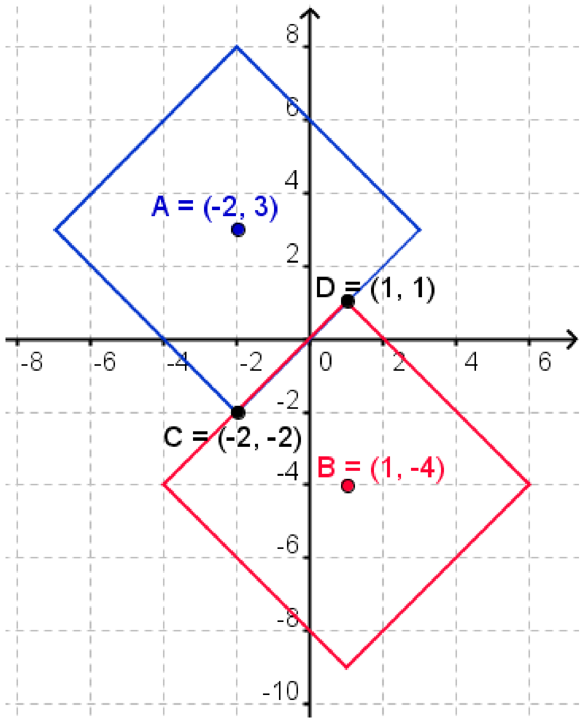
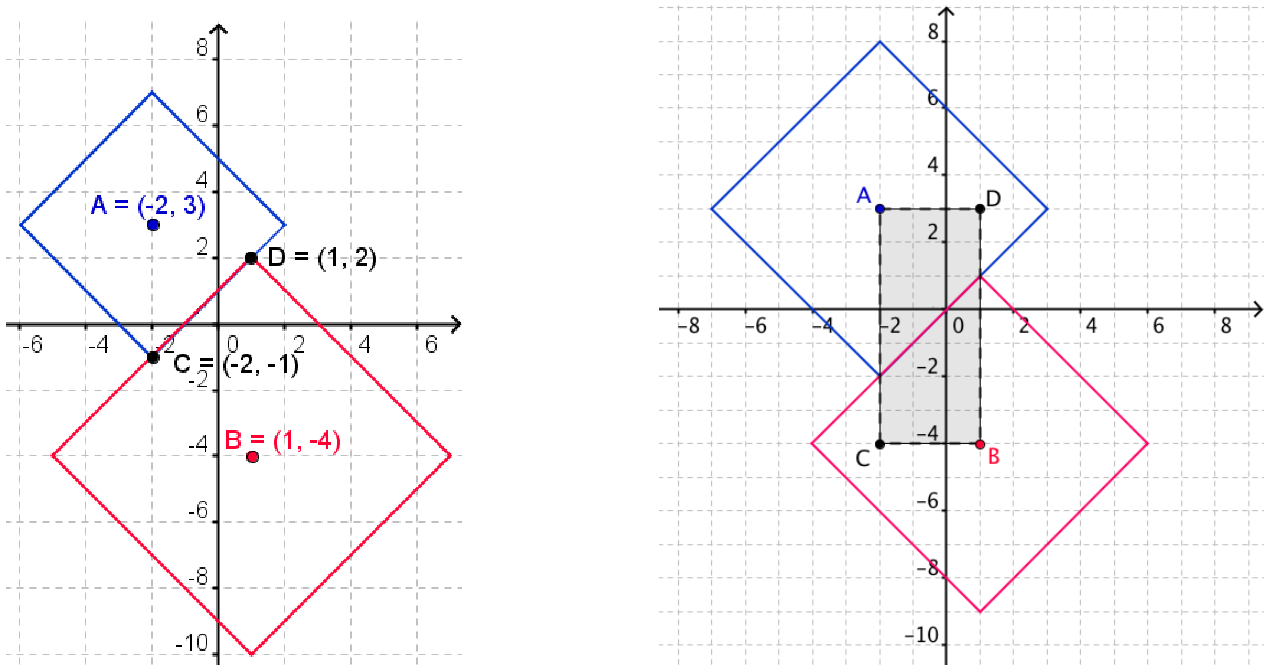


Figure 5. Alice and Bruno's Taxicab circles.

As seen in Figure 5, the points $C(-2,-2)$ and $D(1,1)$ are 5 units from both work places. The important question to consider, however, is "Are the points of the line segment CD also 5 units from A and B?" In other words, can they rent an apartment at any point on the line segment? The answer is, yes! In fact, they can rent an apartment at any point on the line segment connecting the points C and D. Every point on that line segment is 5 units from A and 5 units from B, therefore, the sum of the distances from any point on the line segment to the points A and B is 10 units.

Since this first scenario only included living an equal distance between the work locations, one still must consider what would happen if they decided to rent an apartment closer to Alice's workplace or closer to Bruno's workplace. For instance, say they want to rent an apartment which is 4 units from A and 6 units from B. In this case, they should look for an apartment between the points $C(-2,-1)$ and $D(1,2)$ (see Figure 6a).



(a) Closer to Alice's workplace.

(b) Taxicab ellipse

Figure 6

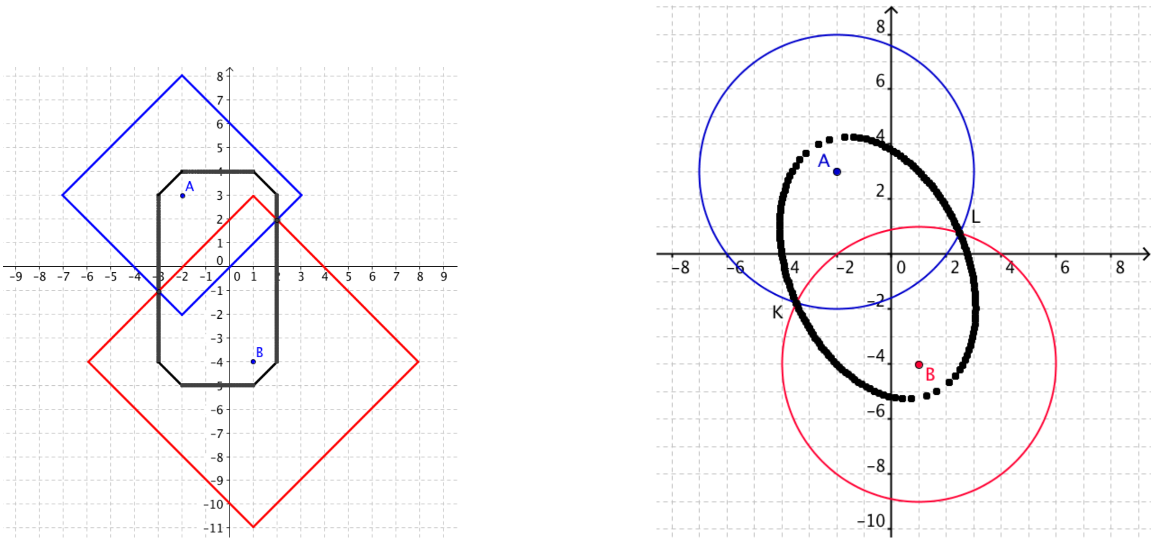
What if the apartment is 3 units from A and 7 units from B? Can you begin to visualize the solution without drawing the Taxicab circles? What if Alice walks farther than Bruno? Do you see a pattern? What does the solution set look like?

Typically, it's difficult for high school students to visualize two changing objects. For this reason, we used GeoGebra to design an interactive worksheet (available at `http://bit.ly/taxicab-sketch`) that enables students to draw and modify two Taxicab circles simultaneously with a slider tool, providing an opportunity to explore all variations of the problem. When the intersecting sides of the Taxicab circles were traced for varying distances from A and B (all with a total of 10 units), the shaded rectangle ABCD was formed (see Figure 6b). This is referred to as a Taxicab ellipse, or the figure formed by all points where the sum of the distance to two points, the foci, is the same. In this example, the foci were in a diagonal position in relation to each other, and the sum of the distance to the two points A and B was the same as the distance between the points (namely, 10 units).

1.1 Questions for Further Thought

Using the GeoGebra sketch (see `http://bit.ly/taxicab-sketch`), consider the following in Taxicab geometry.

- What happens if the points A and B (i.e., foci) are still in a diagonal position, but the sum of distances from the points is greater than the distance between the points?
- What if the points are in a vertical (or horizontal) position but the sum of distances from the points A and B (or foci) is equal to the distance between the points?
- What if the points are in a vertical (or horizontal) position but the sum of distances from the points A and B (or foci) is greater than the distance between the points?



(a) Another Taxicab ellipse.

(b) Euclidean ellipse.

Figure 7

2 Suggestions for Teachers

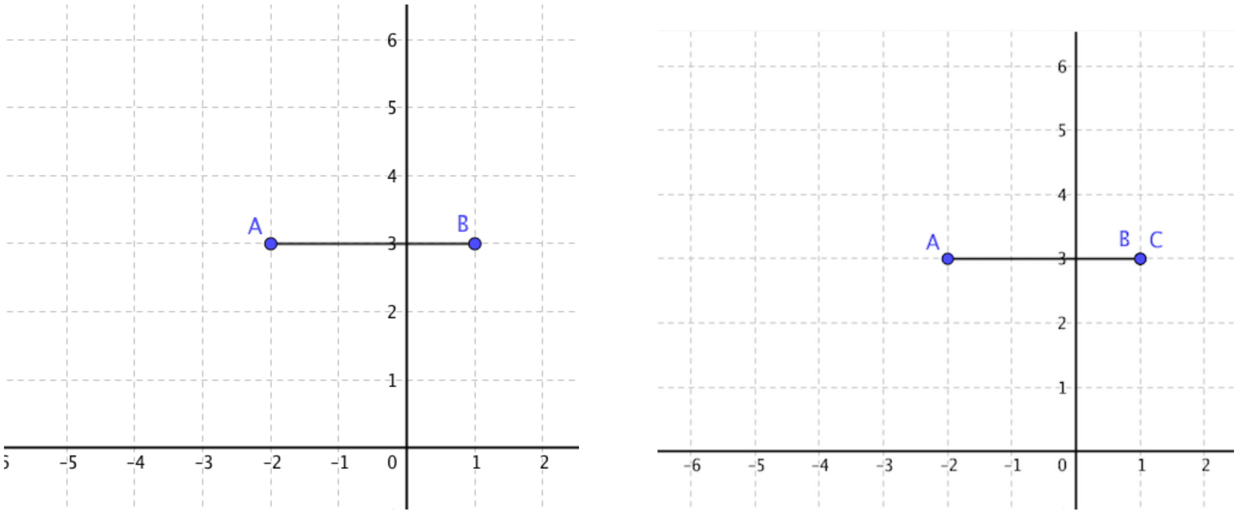
Teachers can pose additional challenges to develop students' understanding of both Taxicab and Euclidean ellipses (see Figures 7a and 7b). For example, given the solution of the problem in Taxicab geometry, students might be asked to use Euclidean geometry (or distance) to show what the solutions look like with two different definitions of distance—the Taxicab definition and the Euclidean definition.

Using the Taxicab geometry can provide avenues into understanding geometric reasoning which complements Euclidean geometry. Students can relate to geometry that is modeled by taxicabs, and we believe that they will deepen their understanding of Euclidean geometry as they explore geometric figures with the definition of distance used in Taxicab geometry. We recommend you try these tasks with your students and share your experiences with us.

We suggest that you begin the activity by calculating Taxicab and Euclidean distances and constructing Euclidean and Taxicab circles at a given point (see Tasks 1 and 2 in Student Tasks). This will allow students to examine the different definition of distance and see how it applies in each system.

In order to construct Taxicab circles with GeoGebra, you need to construct two segments (e.g., $\overline{AB}$

and $\overline{AC}$) with the length of 3 units at the point $A = (-2, 3)$ using the tool "Segment with Given Length" (see Figures 8 and 9).



(a) Segment $\overline{AB}$ **(b)** Segments $\overline{AB}$ and $\overline{AC}$

Figure 8

The segments $\overline{AB}$ and $\overline{AC}$ will be created on top of each other. Move the point $C = (1, 3)$ to the point $(-2, 0)$ so that the segments $\overline{AB}$ and $\overline{AC}$ will be perpendicular to each other (see Figure 9).

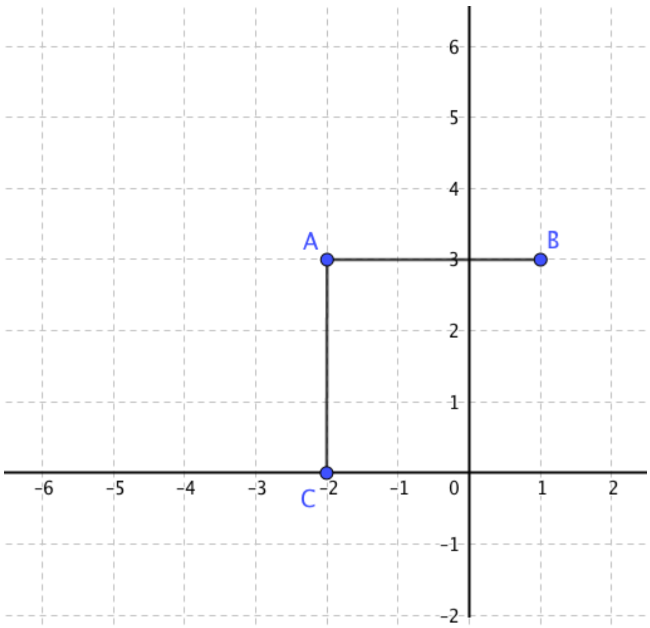
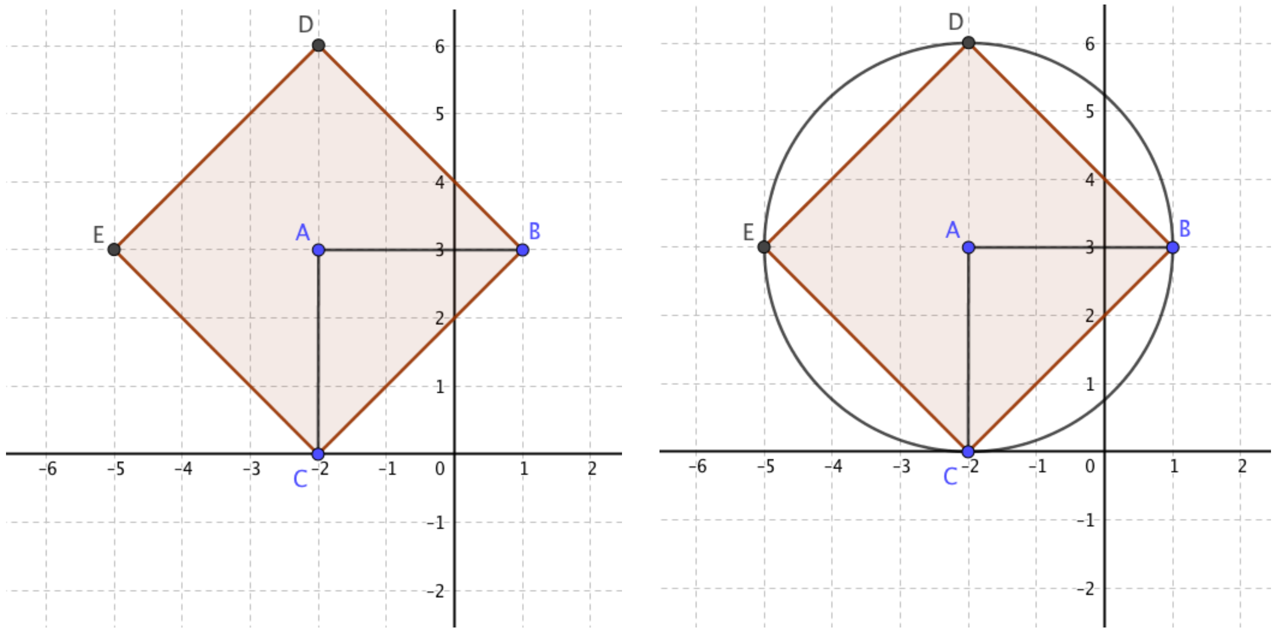


Figure 9. Perpendicular Segments $\overline{AB}$ and $\overline{AC}$

Use the tool "Regular Polygon" to construct a regular polygon with 4 sides by clicking on the points C and then B (see Figure 10a), then construct a Euclidean circle with center at $A = (-2, 3)$ and radius of 3 units using the tool "Circle with Center and Radius" (see Figure 10b).

What if you want to construct circles with radius of $1, 2, 3, 4, 5$, or 6 units dynamically using a slider? First create a slider (e.g., Slider `a` with `min=1`, `max=6`, and `increment= 1`) and then construct



(a) Taxicab Circle **(b)** Taxicab and Euclidean Circles

Figure 10

the segment with given length of "a" before you construct a 4-sided regular polygon and a circle with radius of "a" units.

Once students understand Taxicab and Euclidean distances and learn to construct Taxicab and Euclidean circles at a point, they should be asked to construct and explore possible intersections of two Taxicab or Euclidean circles. For instance, in the Euclidean geometry, two circles may have in common two points, one point, or no point. Whereas, in the Taxicab geometry, two circles can intersect to form a line segment in addition to having no point, one point, or two points in common (see Tasks 3, 4, and 5 in Student Tasks section). For the Tasks 3h, 3i, 4a, and 4b (see Student Tasks), you can construct two circles at $A = (-2, 3)$ and $B = (2, -1)$ with two different sliders and explore whether or not they intersect.

The challenge then becomes how can we explore possible intersections of two circles at two different points using only one slider? Since the shortest Taxicab distance between the points $A = (-2, 3)$ and $B = (2, -1)$ is 8 units, you need to construct a slider (e.g., Slider `a` with `min=0`, `max=8`, and `increment= 0.1`) and then construct two Taxicab circles at $A = (-2, 3)$ with radius of "a" units and at $B = (2, -1)$ with radius of "$8 - a$" units. As students explore these figures with dynamic GeoGebra woksheet, we suggest you encourage students to create real-life applications of Taxicab and Euclidean geometry. Additionally, comparing and contrasting the results of these explorations can be critical to their understanding of both systems and their use of Euclidean geometry. These explorations can be extended to include not only circles, but ellipses as students progress into upper level mathematics.

Student Tasks

Let $d_T(A, B)$ represents the Taxicab distance ("as a taxi would drive") from A to B and $d_E(A, B)$ represents the Euclidean distance ("as the crow flies") from A to B where $A = (x_1, y_1)$ and $B = (x_2, y_2)$. These distance formulas are given by:

$$d_T(A, B) = |x_2 - x_1| + |y_2 - y_1|$$
$$d_E(A, B) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

1. Plot each pair of points A and B and then find both $d_T(A, B)$ and $d_E(A, B)$.
   (a) $A = (1, 2), B = (5, 4)$
   (b) $A = (3, -1), B = (2, 4)$
   (c) $A = (-2, 3), B = (1, -4)$
   (d) $A = (-5, -1), B = (7, 6)$
2. Plot $A = (-2, 3)$.
   (a) Graph the set of all points P at taxi distance 3 from A; that is, graph $\{P \mid d_T(P, A) = 3\}$, the set of all points P such that the taxi distance from P to A is 3.
   (b) Graph the set of all points P at Euclidean distance 3 from A (i.e., $\{P \mid d_E(P, A) = 3\}$).
3. Given $A = (-2, 3)$ and $B = (2, -1)$.
   (a) Calculate $d_T(A, B)$ and $d_E(A, B)$.
   (b) Graph $\{P \mid d_T(P, A) = 2 \text{ and } d_T(P, B) = 6\}$
   (c) Graph $\{P \mid d_E(P, A) = 2 \text{ and } d_E(P, B) = 6\}$
   (d) Graph $\{P \mid d_T(P, A) = 4 \text{ and } d_T(P, B) = 4\}$
   (e) Graph $\{P \mid d_E(P, A) = 4 \text{ and } d_E(P, B) = 4\}$
   (f) Graph $\{P \mid d_T(P, A) = 5 \text{ and } d_T(P, B) = 3\}$
   (g) Graph $\{P \mid d_E(P, A) = 5 \text{ and } d_E(P, B) = 3\}$
   (h) Graph $\{P \mid d_T(P, A) + d_T(P, B) = d_T(A, B)\}$
   (i) Graph $\{P \mid d_E(P, A) + d_E(P, B) = d_E(A, B)\}$
4. Given $A = (0, 1)$ and $B = (0, 5)$.
   (a) Graph $\{P \mid d_T(P, A) + d_T(P, B) = d_T(A, B)\}$
   (b) Graph $\{P \mid d_E(P, A) + d_E(P, B) = d_E(A, B)\}$
5. Given $A = (0, 1)$ and $B = (3, 4)$.
   (a) Graph $\{P \mid d_T(P, A) = 3 \text{ and } d_T(P, B) = 3\}$
   (b) Graph $\{P \mid d_T(P, A) = 4 \text{ and } d_T(P, B) = 4\}$
   (c) Graph $\{P \mid d_T(P, A) = 6 \text{ and } d_T(P, B) = 6\}$
   (d) Graph $\{P \mid d_T(P, A) = d_T(P, B)\}$

3 DISCUSSION

This article illustrates how dynamic geometry worksheets can be used to model problems and situations to make visualizing solutions easier and clearer. More specifically, dynamic GeoGebra worksheets were designed to explore and reason about mathematical objects of Taxicab and Euclidean geometry.

As teachers, we should be aware that some of our students have difficulties visualizing mathematical objects, and that drawing mathematical objects and figures might not help them. On the other hand, in a dynamic environment, students can easily manipulate initial conditions by dragging elements of drawings or using sliders as they explore and analyze various situations. For instance, the Taxicab tasks were non-routine problems for our students; however, they quickly learned to construct dynamic worksheets and were able to explore how the Taxicab circles changed and what the overlapping area looked like. We believe that exploring mathematical objects dynamically is more powerful than simply thinking about the object drawn on paper or the computer screen, and that work with dynamic worksheets can improve students' ability to visualize and support their developing understanding of geometric concepts and relationships.

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