Numerical analysis of a planar motion: GeoGebra as a tool of investigation

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Abstract: The paper is inspired by the method of the numerical approximation of the planetary motion that was used by Richard Feynman in the book "The Feynman Lectures on Physics." Using Newton's Second Law and replacing infinitesimally small changes in variables by their very small changes, he arrived at a respectable approximation of the elliptical motion of a planet around the sun. To organize all numerically computed data, which include the sequence of coordinates of the consecutive positions of the planet during its motion around the sun, Richard Feynman used a large table. Then, the plot of the resulting successive positions of the planet was in accordance with Kepler's laws. We show that GeoGebra, which enables a user to combine the spreadsheet with the numerical and graphical tools, is very suitable for grasping such complex tasks based on repetitive numerical computation and graphical planar representation.

Keywords: GeoGebra, science, applications, planetary motion

1. INTRODUCTION

The aim of the paper is to present GeoGebra [2] as a means to bring the possibility of solving various real-world problems into classrooms. Its native operation and the dynamic interconnection between representations of an object enable students to investigate, by themselves or with the guidance of their teacher, the mathematical background of various real-world phenomena and to produce reliable geometrical and mathematical models. Specifically, we will show, through an example of the numerical approximation of the motion of a planet around the sun that GeoGebra, thanks to its unique combination of `Spreadsheet View`, `Algebra View` and `Drawing Pad`, represents a powerful tool that enables us to simply and naturally perform a numerical iterative computation and to plot its results.

2. GEOGEBRA FEATURES

The numerical approximation of the planetary motion together with other problems that deal with the numerical approximation of a motion can be solved in a very comfortable and simple way in GeoGebra, mainly thanks to the use of the next two features of the program: the dynamic interconnection between 'Views' and the possibility of the iterative computation via the 'Spreadsheet'. First, let us look at these features in detail.

By the term 'Views' we understand the GeoGebra components 'Algebra View', 'Spreadsheet View' and 'Drawing Pad' in this paper. The usefulness of the dynamic interconnection of 'Views' is illustrated in the next simple example.

Example: A projectile motion. *Draw a trajectory of a projectile that moves under gravity in a vacuum after its shooting with the angle of elevation α and the initial velocity v_0.*

See Figure 1 to consider the utility of the dynamic interconnection of 'Views'. It works as follows: We define functions $h(t)$ and $v(t)$ for coordinates of the instantaneous position $p(t) = (h(t), v(t))$ of the projectile at the time t in the 'Algebra View', then use them in the 'Spreadsheet View' to generate a list of coordinates of the projectile's positions corresponding to the arithmetical progression of times t_i with the difference $\Delta t = 0.1\,s$ (see Figure 1, 'Spreadsheet', column 'A') and finally plot the list on the 'Drawing Pad'. Parameters α and v_0 of the functions are controlled by the sliders in such a way that the movement of a slider causes corresponding changes within all the 'Views'.

The GeoGebra 'Spreadsheet' enables us to realize the iterative computations that form the core of numerical approximation methods. Let us illustrate this technique with the next geometric example.

Example: Nested triangles. *Plot a sequence of triangles so that the vertices of the subsequent triangle are the midpoints of the sides of its predecessor.*

GeoGebra 'Spreadsheet' enables us to perform an iterative computation in such a way that the values computed

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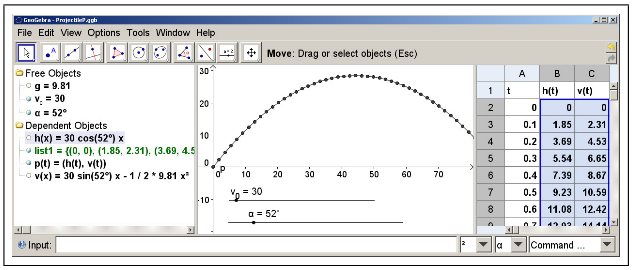

Fig 1: Projectile motion - solution in GeoGebra

on one line of a table can be used as input values for computation on the following line and so on; see the pairs of lines 1-2 and 2-3 in Table 1. Of course, the data from the table can be plotted on 'Drawing Pad' if it makes sense.

	A	B	C	D
1	`(-4,-2)`	`(5,-1)`	`(2,6)`	`Polygon[A1,B1,C1]`
2	`(A1+B1)/2`	`(B1+C1)/2`	`(C1+A1)/2`	`Polygon[A2,B2,C2]`
3	`(A2+B2)/2`	`(B2+C2)/2`	`(C2+A2)/2`	`Polygon[A3,B3,C3]`
4	...	...	...	...
5	...	...	...	...

Table 1: Computation of triangles' vertices through the 'Spreadsheet'

In the case of the given example we start with filling in the first two lines of the table in the way that is shown in Table 1, then we copy the second line into the next several lines, the number of which is given by the number of nested triangles that we want to draw.

Thanks to the relative addresses of the cells that are used in formulas for the computation of the midpoints of the sides of the preceding triangle and in the definition of the triangle (see Table 1) we will get successive iterations of the desired triangles, as shown in Figure 2.

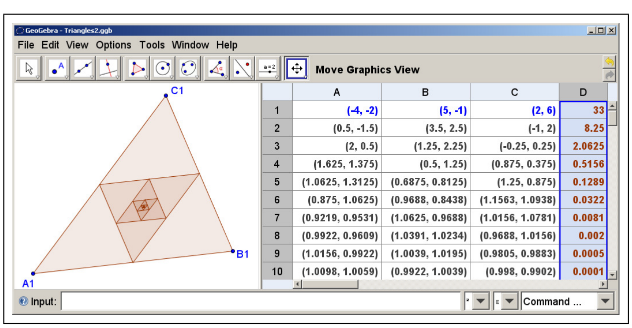

Fig 2: Nested triangles - solution in GeoGebra

3. PLANETARY MOTION

The effort to understand the planetary motion was an important moving force in the process of the formation and development of modern science. This history is well known and has been described in many publications, see for example [1, 3, 7].

Among the ground-breaking events in this story can be included the formulation of the laws of planetary motion [4] by Johannes Kepler, which describe the motion of planets around the sun, and the introduction of the laws of motion [5] and gravitation [6] by Isaac Newton, which give the cause of planetary motion. At the present time these laws are taught at secondary schools and each attentive student is able to express the laws and to explain their meaning. But what if that student asks a question about the possibility of deriving the Kepler's laws from the more general Newton's laws. The usual solution to this problem, especially in the case of the first two Kepler's laws, is based on advanced knowledge of calculus and so not comprehensible to a secondary school student [4].

Now we will show how we can take advantage of GeoGebra to solve the above mentioned problem numerically in such a way that is understandable and doable for secondary school students.

3.1. The problem assignment

The presented solution is inspired by the method of the numerical analysis of the motion of a planet around the sun that was presented by Richard Feynman in the well known book "The Feynman Lectures on Physics" (Volume 1, chapter "Newton's Laws of Dynamics", first published in 1964) [1]. Here on page 9-6 he assigned the next problem.

Problem: *"Can we analyze the motion of a planet around the sun? Let us see whether we can arrive at an approximation to an ellipse for the orbit. We shall suppose that the sun is infinitely heavy, in the sense that we shall not include its motion. Suppose a planet starts at a certain place and is moving with a certain velocity; it goes around the sun in some curve, and we shall try to analyze, by Newton's laws of motion and his law of gravitation, what the curve is."* [1]

The complete solution to this problem is given in [1], in an intelligible and gripping style, so characteristic of Richard Feynman. Here we will briefly mention only the main steps of Feynman's solution in this text.

The main idea of the numerical solution to the problem was to compute coordinates of the series of consecutive positions of a planet during its motion around the sun corresponding to times t_i, $i = 1, 2, \ldots$, that differ by the constant difference Δt, i.e. $t_{i+1} = t_i + \Delta t$.

First, he made a convenient choice of physical constants κ (gravitational constant) and M (mass of the sun) to simplify the computation. Specifically, he chose them so that to get $\kappa M = 1$ in the equation of the law of gravitation $F = \kappa \frac{Mm}{r^2}$ instead of $\kappa M = 1.372 \cdot 10^{20}$ for their real values $\kappa = 6.673 \cdot 10^{-11} m^3 kg^{-1} s^{-2}$ and $M = 1.989 \cdot 10^{30} kg$.

Second, Feynman applied Newton's second law of motion and Newton's law of gravitation to derive formulas for the computation of the coordinates $a_x(t)$, $a_y(t)$ of the acceleration vector $\vec{a}(t)$ at time t_i:

$$a_x(t_i) = -\frac{x(t_i)}{r(t_i)^3}, \quad a_y(t_i) = -\frac{y(t_i)}{r(t_i)^3}. \tag{1}$$

These formulas allowed him to define relations between velocities in two consecutive times t_i and $t_{i+1} = t_i + \Delta t$:

$$v_x(t_{i+1}) = v_x(t_i) + \Delta t a_x(t_i)$$
$$v_y(t_{i+1}) = v_y(t_i) + \Delta t a_y(t_i). \tag{2}$$

After this he could express relations between two successive positions $X(t_i) = [x(t_i), y(t_i)]$ and $X(t_{i+1}) = [x(t_{i+1}), y(t_{i+1})]$ of a planet during its motion around the sun by the formulas:

$$x(t_{i+1}) = x(t_i) + \Delta t v_x(t_i), \quad y(t_{i+1}) = y(t_i) + \Delta t v_y(t_i). \tag{3}$$

Finally, after the determination of the initial position $X(0) = [x(0), y(0)]$ and velocity $\vec{v}(0) = (v_x(0), v_y(0))$ of the planet at time $t_1 = 0s$, he started the sequential computation of its consecutive positions according to the given formulas (1) - (3). With one subtlety - to improve the approximation accuracy he computed each vector $\vec{v}(t)$ of instantaneous velocity in the half-time between two positions $X(t_i)$ and $X(t_{i+1})$. To organize all numerically computed data Feynman used a large table (see [1], page 9-8, Table 9-2). Then, he plotted all the points with the coordinates $[x(t_i), y(t_i)]$ to get the picture of the elliptical trajectory during its motion around the sun (see [1], page 9-7, Fig. 9-6).

3.2. Solution in GeoGebra

The above presented Feynman's solution, based on the connection of the numerical computation with a table and a graph, calls for the use of GeoGebra. The numerically computed coordinate values, which Feynman stored in the large table, can be simply generated within the GeoGebra 'Spreadsheet' and drawn on the plane equipped with the Cartesian coordinate system with initial parameters controlled by the sliders, exactly in the way that was illustrated in section 2. of this paper. We apply the relations given by the equations (1) - (3) on cells of the two consecutive lines of the 'Spreadsheet', see the pairs of lines 2-3 and 3-4 in Table 2; with one distinction - instead of Δt we use a simpler symbol d. We start with filling in the first three lines in the way that is shown in Table 2, then we copy the third line into the next several lines, the number of which is given by the number of the planet's positions that we want to compute.

The resulting appearance of the 'Spreadsheet' is shown in Figure 3. Here, in columns B and C, we can find the desired coordinates of the individual positions of the planet

Table 2: Planetary motion spreadsheet

	A	B	C	D
1	`t`	`x`	`y`	`r`
2	0	x_0	y_0	$\mathtt{sqrt}(B2^2 + C2^2)$
3	A2 + d	B2 + d G2	C2 + d H2	$\mathtt{sqrt}(B3^2 + C3^2)$
4	A3 + d	B3 + d G3	C3 + d H3	$\mathtt{sqrt}(B4^2 + C4^2)$
5	...	...	...	...
6	...	...	...	...

	E	F	G	H
1	`ax`	`ay`	`vx`	`vy`
2	$-B2/D2^3$	$-C2/D2^3$	$vx_0 + d/2\,E2$	$vy_0 + d/2\,F2$
3	$-B3/D3^3$	$-C3/D3^3$	G2 + d E3	H2 + d F3
4	$-B4/D4^3$	$-C4/D4^3$	G3 + d E4	H3 + d F4
5	...	...	...	...
6	...	...	...	...

during its motion around the sun. To plot them, we simply select the content of those columns and invoke the *Create List of Points* command. To get the same result as in Figure 3 we must suppress the 'Show Label' options to each point and draw the sun, i.e. a coloured circle, at the origin of the Cartesian coordinate system.

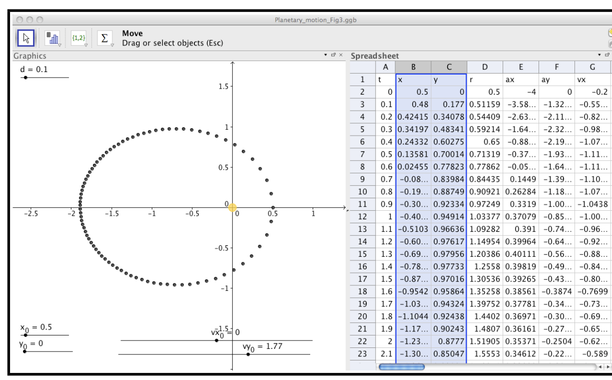

Fig 3: Planetary motion - solution in GeoGebra

3.3. Inquiry into the GeoGebra solution

The resulting trajectory of the planet, corresponding to the initial conditions: $X(0) = [0.5, 0]$, $\vec{v}(0) = (0, 1.77)$, which is indicated by the successive positions of the planet in Figure 3, appears consistent with Kepler's laws.

GeoGebra tools provide us with the possibility of verifying this fact as well as performing other activities within the resulting approximation. First, we can try to verify the first two Kepler's laws with the help of the GeoGebra tools *Conic through Five Points* and *Polygon* (the option 'Show Label - Value' should be activated). The final results of both verifications are favourable, as you can see in Figure 4. Second, we can play with the values of the initial parameters $X(0)$ and $\vec{v}(0)$, which are controlled by sliders, and arrive at trajectories of different shapes, as shown in Figure 5.

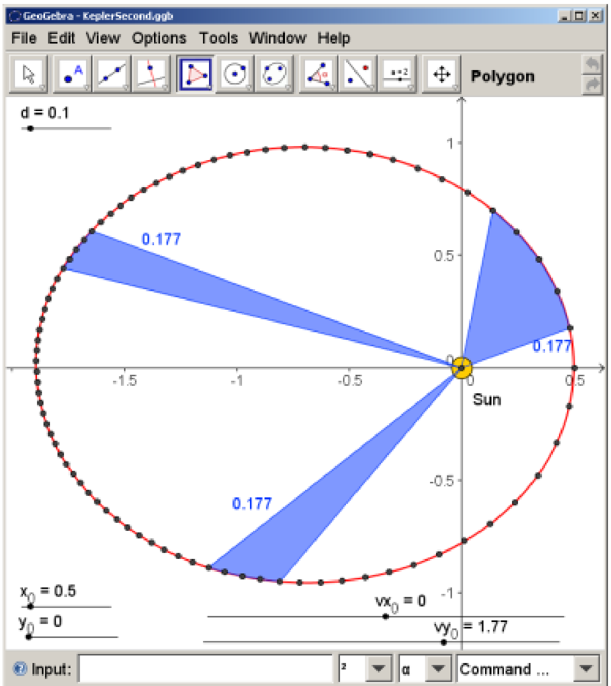

Fig 4: Inquiry into the solution by the GeoGebra tools: $X(0) = [0.7, 0]$, $\vec{v}(0) = (0.68, 1.85)$

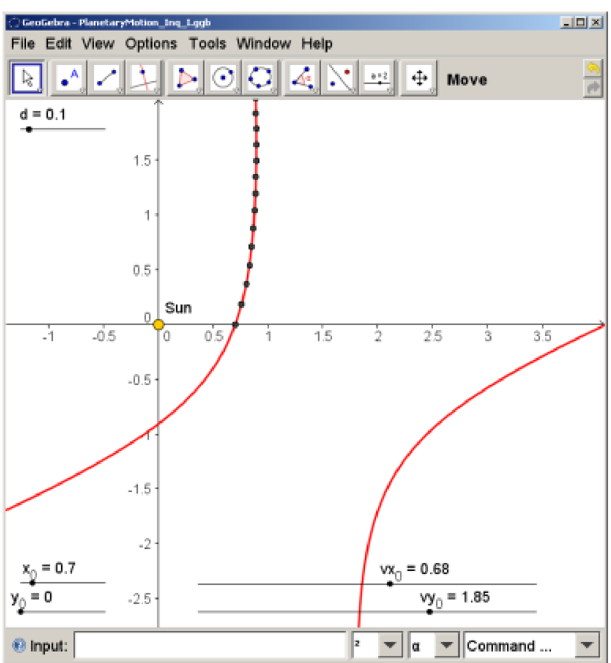

Fig 5: Inquiry into the solution by the GeoGebra tools: Kepler's laws of orbits and areas

4. CONCLUSIONS

The paper has introduced one particular real-world problem as a representative of the kind of problems dealing with the numerical approximation of a motion, the solutions of which, using the software GeoGebra, can be realized with students in a motivational and beneficial way. The process of solving such problems in the classroom introduces mathematics as a living and useful science, the application of which crosses boundaries between it and, what at first glance seem distinct disciplines, such as astronomy, biology, physics, geometry, computer science etc. It gives students a chance to experience the usefulness of the mathematical knowledge they have learnt at school. I would be delighted to inspire the reader of my paper to search for such problems and to solve them with his/her students.

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