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of the 4th Annual Southern Connecticut GeoGebra Conference

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PART I

REFEREED PRESENTATIONS

GEOGEBRA TOOLS FOR THE POINCARÉ DISK

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Abstract: The Poincaré disk model played an important role in the acceptance and development of hyperbolic geometry. Although exceptionally useful, the pedagogical value of the model can be further enhanced via experimentation in a dynamic geometry environment. The focus of this article is on the creation of custom tools in GeoGebra for constructing hyperbolic lines and circles in the Poincaré disk. In an effort to make this material accessible to a wider audience, the necessary mathematics is also included.

Keywords: Poincaré disk, GeoGebra: custom tools

Introduction

Euclid's Elements was a comprehensive summary of the mathematics known to the ancient Greeks and remained the definitive textbook on geometry for more than 20 centuries. His postulates laid bare many of the assumptions made by the Greek geometers and the fifth postulate, called the parallel postulate, has been of particular interest. All attempts to prove that the fifth postulate was dependent on the first four ended in failure and most of these efforts contributed little to the evolution of geometry. Carl Friedrich Gauss may have been the first mathematician to recognize the independence of the fifth postulate, but he never published his results [7]. Models of hyperbolic geometry came 30 years later and were used to establish the relative consistency of Euclidean and hyperbolic geometry and to prove the independence of the fifth postulate. The Poincaré disk model was originally developed by the Italian mathematician Eugenio Beltrami in 1868 and was reintroduced and popularized by French mathematician Henri Poincaré in 1881. The model is conformal and provides a link to several other branches of mathematics including complex function theory and differential equations. The Poincaré disk model entered the popular culture via the work of the Dutch graphics artist M.C. Escher.

The Poincaré disk and other models made hyperbolic geometry easier to visualize and played a role in its gaining acceptance. Although the models are helpful, students can benefit from the use of dynamic geometry software which provides an environment suitable for experimentation. At present there are several software packages available for this purpose including Cinderella [3] and NonEuclid [2] which both provide tools for working with the Poincaré disk model. These tools are useful, but they are not as well known as GeoGebra [9] and it is natural to wonder whether or not the same tasks could be performed in GeoGebra. Currently, GeoGebra does not provide any built in tools but several custom tools are available on GeoGebraTube. Some of the best include those developed by Maline Christersson

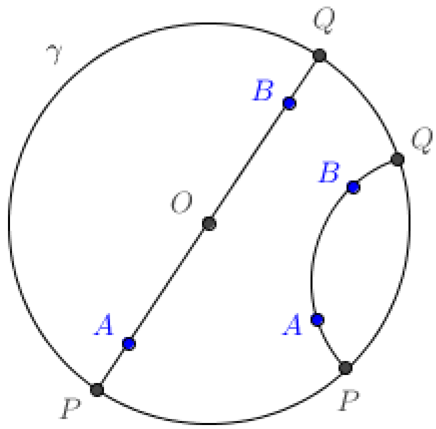


Figure 1.1: Lines in the Poincaré disk model are either open diameters or open circular arcs orthogonal to γ. Each line is shown passing through points A and B and with their ideal endpoints P and Q.

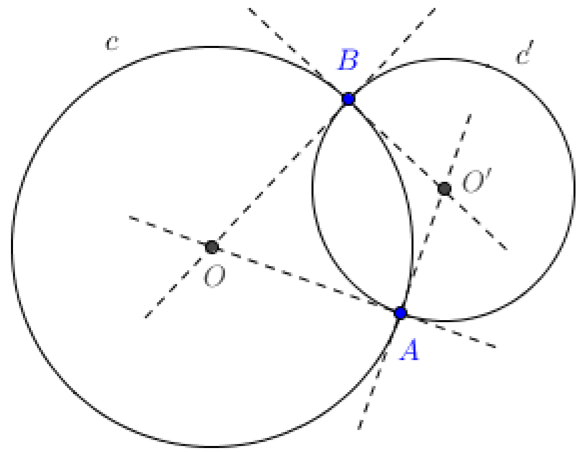
[4] and Stehpan Szydlik [10]. However, both of these tools have some defects. For example, their hyperbolic line tools do not work properly in all cases. The emphasis in this paper is on learning how to create robust custom tools in GeoGebra for constructing hyperbolic lines and circles in the Poincaré disk model. In an effort to make this material accessible to a wider audience, the mathematics needed to construct the tools will also be provided.

Poincaré disk model

The Poincaré disk model begins with a circle γ with center O in the Euclidean plane. Points in this model are points in the interior of γ. Points on γ are called ideal points. They are not included in the model and are considered to be infinitely far away from points in the interior. There are two types of lines in the Poincaré disk model. The first type are the open diameters of γ and the second type are the open circular arcs orthogonal to γ. Taken together, the two types of lines are called p-lines and they are geodesics. It is worth noting that a diameter may be regarded as the limiting case of a circle whose radius is approaching infinity.
Circles orthogonal to γ can be constructed in several different ways. One method is to use the following result from Euclidean geometry which makes use of tangents [11].

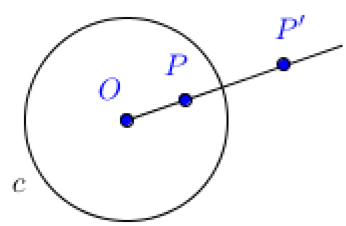
Definition 1.1 *Two circles are orthogonal if their tangent lines are perpendicular at the point of intersection.*

Theorem 1.1 *Let c be a circle with center O and c' be a circle with center O' and suppose the circles intersect at points A and B. Then the circles are orthogonal if and only if the tangent lines to c' at A and B intersect at O and the tangent lines to c at A and B intersect at O'.*

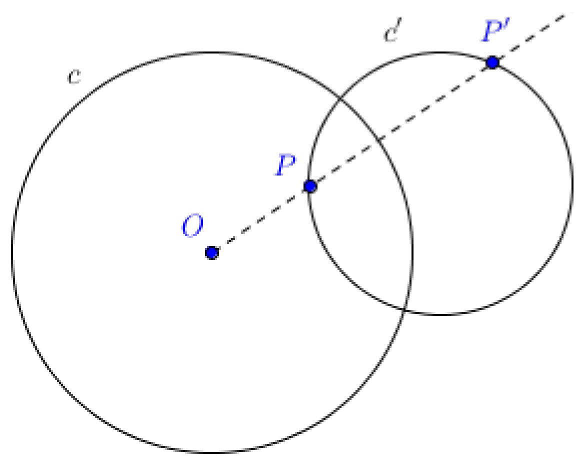


Another method involves using inversive geometry. In this approach, circles orthogonal to γ are constructed using inverse points [5, 6, 7].

Definition 1.2 *The inverse of a point P with respect to a circle c with center at point O and radius r is the unique point P' on the ray OP such that $OP \cdot OP' = r^2$.*



Theorem 1.2 *Let c be a circle with center O and radius r and P be a point that is not on c and that is not the center of c. Define P' as the inverse of P with respect to c. Suppose a circle c' passes through point P. Then the circles c and c' are orthogonal if and only if c' passes through P'.*



Angles in the Poincaré disk model are defined as the Euclidean angle between tangents. The fact that angles are preserved is considered to be an advantage of the disk model over other models of hyperbolic geometry.
Distance is more complicated in the Poincaré disk model and is significantly distorted. The ratio of distances is also distorted. However, the cross ratio defined as a ratio of ratios of distances is preserved [7].

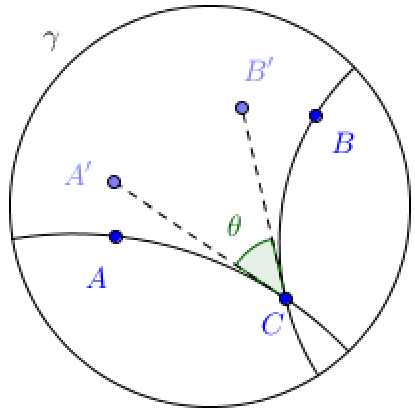


Figure 1.2: The angle between two hyperbolic lines is defined as the Euclidean angle between the tangents at their point of intersection. The angle ACB is defined as the Euclidean angle $A'CB'$.

Definition 1.3 *Suppose that A and B are any two points in the Poincaré disk and let P and Q be the ideal endpoints of the line that connects them. Then the cross ratio is defined as*

$$(AB, PQ) = \frac{AP}{AQ} \div \frac{BP}{BQ}$$

where the AP, AQ, BP and BQ are Euclidean lengths.

Theorem 1.3 *Let c be a circle with center O. If A , B, P, Q are four distinct points different from O and A', B', P', Q' are there inverses with respect to c, then their cross ratio is preserved, $(A'B', P'Q') = (AB, PQ)$.*

Definition 1.4 *The distance between points A and B in the Poincaré disk model is defined in terms of the cross ratio as $d(A, B) = |\ln(AB, PQ)|$.*

A hyperbolic circle with center A and radius r is defined as the set of all points X in the Poincaré disk such that $d(A, X) = r$. The relationship between hyperbolic and Euclidean circles in the Poincaré disk model is captured in the following theorem [1, 7].

Theorem 1.4 *Hyperbolic circles in the Poincaré disk model are also Euclidean circles but their centers are not the same unless the hyperbolic circle is centered at O.*

An isometry is a distance preserving transformation. There are four isometries in the hyperbolic plane including hyperbolic reflections, hyperbolic translations, hyperbolic rotations and the parabolic isometry which has no Euclidean counterpart. In this paper, we focus on the hyperbolic reflections [8] since they are required for the construction of the hyperbolic circle.

Definition 1.5 *A hyperbolic reflection is either a Euclidean reflection across a diameter or an inversion with respect to a circular arc orthogonal to γ.*

Theorem 1.5 *Hyperbolic reflections are isometries. If points A', B' are reflections of points A, B about a hyperbolic line, then $d(A', B') = d(A, B)$.*

Hyperbolic line tool

Although in theory the defining circle γ can be any circle, for the purpose of creating custom tools in GeoGebra for constructing hyperbolic lines and circles, we will take γ to be the unit circle. When constructing a hyperbolic line passing through two points, we must consider several possibilities depending on whether or not the points lie on a diameter of γ and whether the points are located in the interior or on the boundary of γ. These possibilities can be divided into three cases; diameters, arcs and ideal arcs.

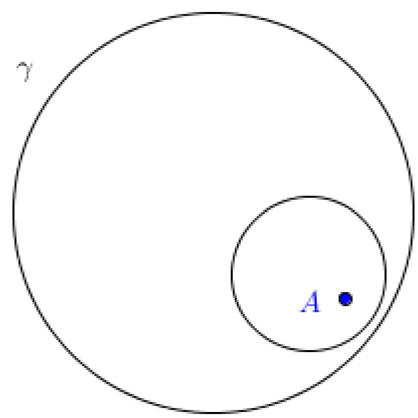


Figure 1.3: A hyperbolic circle in the Poincaré disk is also a Euclidean circle. The hyperbolic center is offset towards the boundary of the disk since distances near the boundary are larger.

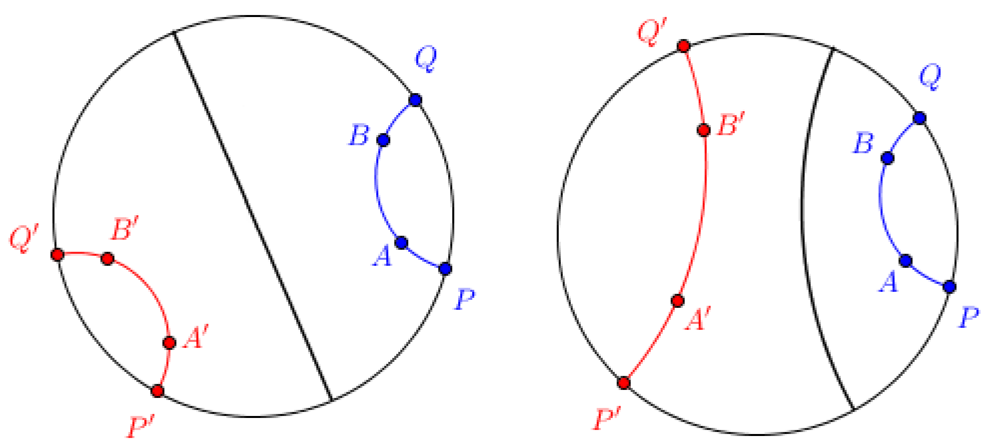


Figure 1.4: Hyperbolic reflections. (a) If the hyperbolic line is a diameter of γ, the hyperbolic reflection is simply the Euclidean reflection about the diameter. (b) If the hyperbolic line is a circular arc orthogonal to γ, then the hyperbolic reflection is an inversion about the hyperbolic line.

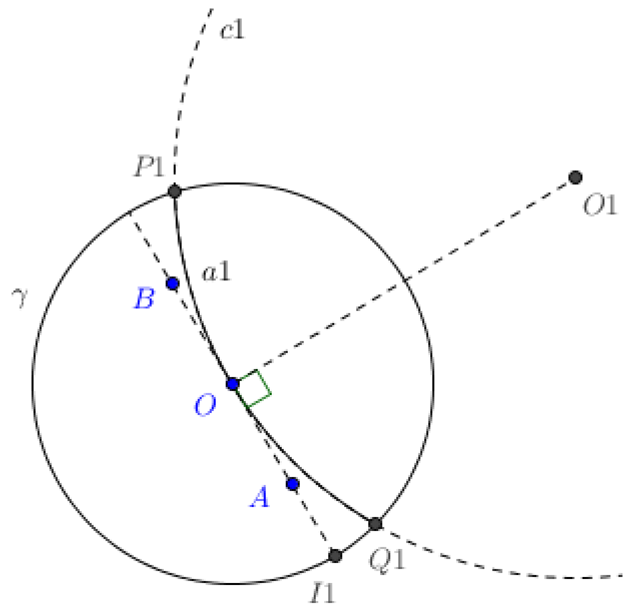


Figure 1.5: Use an arc of circle $c1$ to approximate the diameter passing through points A and B. The circle $c1$ is tangent to the diameter at O and its center $O1$ lies on the perpendicular bisector of the diameter. Define $I1$ as one of the intersections of the diameter and the defining circle γ. Locate $O1$ by rotating $I1$ about O counterclockwise by 90 degrees and then dilating by a factor equal to the radius of $c1$. Find the intersections $P1$ and $Q1$ of $c1$ and γ and use them as the endpoints of the circular arc $a1$. Note that the circle $c1$ is shown with a radius of 2 in order to make the details of the construction easier to see. When the radius is set equal to 1000, the arc $a1$ appears as a straight line.

Diameters

If A and B are collinear with O, then the hyperbolic line passing through A and B is a diameter of γ. Although diameters are simple in theory, there are some complications when implementing this case in GeoGebra since in the end all three cases will be combined into a single custom tool. In an ideal world, we could use the Segment [<Point >, <Point >] command to construct the diameter. Unfortunately, the If [<Condition >, <Then >, <Else >] command needed to combine the cases requires that the <Then >and <Else >objects be of the same type. Since segments and arcs do not meet this criterion, the diameter will be approximated using the small arc of a large circle as shown in Figure 1.5.

Begin the construction by creating γ using the Circle[<Point >, <Radius Number >] and two points inside γ using the PointIn[<Region >] command

```
// Poincarè disk --------------
// Construct the unit circle γ
// Create point A inside γ
// Create point B inside γ
// ----------------------------

γ: Circle[(0,0), 1]
A: PointIn[Circle[(0,0), 1]]
B: PointIn[Circle[(0, 0), 1]]
```

Then construct the approximate diameter. Note that the Intersect [ ] command has several forms and in the form used below, the third argument is the index of the intersection.

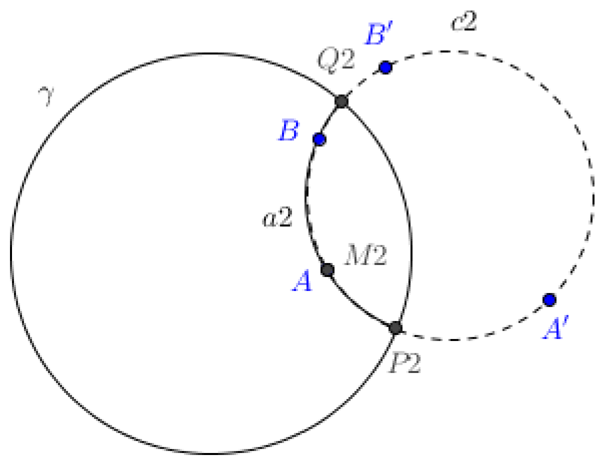


Figure 1.6: Construct the hyperbolic line for the case in which points A and B are not collinear with O and at least one of the points is in the interior of γ. Begin by determining the inverses of A and B with respect to γ. If A is in the interior of γ, then by Theorem 2 the circle $c2$ passing through points A, A' and B is orthogonal to γ. If A is on γ then A and A' coincide (points on the circle of inversion are invariant) and $c2$ can be constructed using points A, B and B'. Determine the intersections $P2$ and $Q2$ of $c2$ and γ and use them as the endpoints of circular arc $a2$. Locate an additional point $M2$ on arc $a2$. If A is on γ, choose $M2 = B$, otherwise choose $M2 = A$. Finally construct arc $a2$ using points $P2$, $M2$ and $Q2$.

```
// Diameters-----------------------------------
// Approximate the diameter using the small arc
// of a large circle
// Locate an endpoint I1 of the diameter
// Use I1 to locate the center O1 of circle c1
// Construct circle c1
// Locate endpoint P1 of arc a1
// Locate endpoint Q1 of arc a1
// Construct circular arc a1
//---------------------------------------------

I1: Intersect[Circle[(0,0), 1], Line[A, B],1]
O1: 1000 Rotate[I1, 90°]
c1: Circle[O1, 1000]
P1: Intersect[Circle[(0,0), 1], c1, 1]
Q1: Intersect[Circle[(0, 0), 1], c1, 2]
a1: CircumcircularArc[P1, (0, 0), Q1]
```

If you study the GeoGebra commands in this section, you will notice that γ is purely decorative and is included only to make the unit circle visible during the construction process. If the unit circle is required on the right hand side of any command, it is created using Circle [(0,0), 1] command. If the decorative γ were used instead, then the final tool would require the user to input two points and the circle. In other words, a little extra effort at this stage will result in a tool that requires two inputs instead of three.

Arcs

If A and B are not collinear with O, then the hyperbolic line passing through A and B is the open arc of a circle orthogonal to γ. If in addition, at least one of the points lies in the interior of γ, then the hyperbolic line can be constructed using inversive geometry as described in Figure 1.6.

Prepare for this construction by hiding all of the objects from the previous construction except for points A and B and the defining circle γ. Then construct the circular arc using the GeoGebra code provided below. Note the use of the Reflect[<Object >, <Circle >] command to determine the inverses of A and B with respect to γ and the Length[<Object >] command to determine whether or not A is on γ. The If[<Condition >, <Then >, <Else >] command is used to construct the circle passing through points A and B and orthogonal to γ. If A is on γ, then the circle is constructed using A, B and B', otherwise it is constructed using points A , A' and B.

```
// Arcs----------------------------------------------
// Construct a circular arc using Theorem 2
// Determine the inverse of point A with respect to γ
// Determine the inverse of point B with respect to γ
// Construct circle c2 passing through A and B and
// orthogonal to γ (if A is on γ, use points A, B, B',
// otherwise use points A, B, A')
// Locate endpoint P2 of arc a2
// Locate endpoint Q2 of arc a2
// Locate an additional point M2 on arc a2
// (if A is on γ, set M2 = B, otherwise set M2 = A)
// Construct circular arc a2 using points P2, M2 and Q2
//-------------------------------------------------------

A': Reflect[A, Circle[(0, 0), 1]]
B': Reflect[B, Circle[(0, 0), 1]]
c2: If[Length[A] == 1, Circle[A, B, B'], Circle[A, B, A']]
P2: Intersect[c2, Circle[(0,0), 1], 1]
Q2: Intersect[c2, Circle[(0, 0), 1], 2]
M2: If[Length[A] == 1, B, A]
a2: CircumcircularArc[P2, M2, Q2]
```

Ideal arcs

Consider the situation for which A and B are not collinear with O and both points lie on γ. In this case, the previous construction fails since both points are invariant under inversion about γ and inversion cannot be used to obtain the third point needed to construct the circle. The solution is to use a result from Euclidean geometry as illustrated in Figure 1.7.

Prepare for this construction by hiding all of the objects from the previous constructions except for points A and B and the defining circle γ. Then construct arc $a3$ using the GeoGebra code provided below. Note that the most obvious method for constructing the tangents would be to use the Tangent[<Point >, <Conic >] command. However if this command is used, the final hyperbolic line tool will fail when imported into another GeoGebra file (most likely due to the fact that the Tangent command was designed to construct multiple tangents from a point to a conic). To work around this problem, unique tangents to γ at A and B are constructed using the fact that the tangent to a circle is perpendicular to the radius of the circle.

```
// Ideal arcs----------------------------------------
// Construct ideal arcs using Theorem 1
// Construct tangent t3a to γ at point A
// Construct tangent t3b to γ at point B
// Locate the center O3 of circle c3
// (intersection of the tangents)
// Construct circle c3 passing through A and B
// and orthogonal to γ
// Locate an additional point M3 on arc a3 by finding
// the intersection of circle c3 with the ray from O3
// to the midpoint of segment AB
// Construct circular arc a3 using points A, M3, B
// ---------------------------------------------------
```

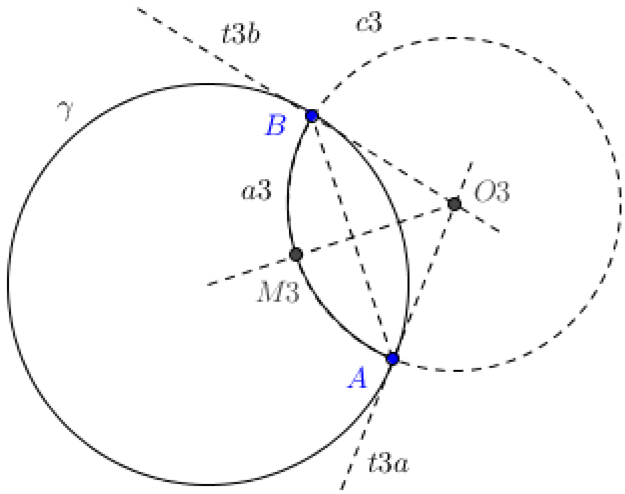


Figure 1.7: Construct the hyperbolic line passing through points A and B for the case in which A and B are not collinear with O and both points lie on γ. Begin by constructing tangents $t3a$ and $t3b$ to γ at points A and B respectively. By Theorem 1, the tangents intersect at the center $O3$ of a circle $c3$ which passes through A and B and is orthogonal to γ. Construct $c3$ using $O3$ as the center and either point A or B. Use A and B as the endpoints of arc $a3$. Determine an additional point $M3$ on arc $a3$ by constructing a ray from $O3$ to the midpoint of segment AB and locating the intersection of this ray with $c3$. Then use points A, $M3$, B to construct arc $a3$.

```
t3a: PerpendicularLine[A, Line[(0, 0), A]]
t3b: PerpendicularLine[B, Line[(0, 0), B]]
O3: Intersect[t3a, t3b]
c3: Circle[O3, A]
M3: Intersect[c3, Ray[O3, Midpoint[A, B]]]
a3: CircumcircularArc[A, M3, B]
```

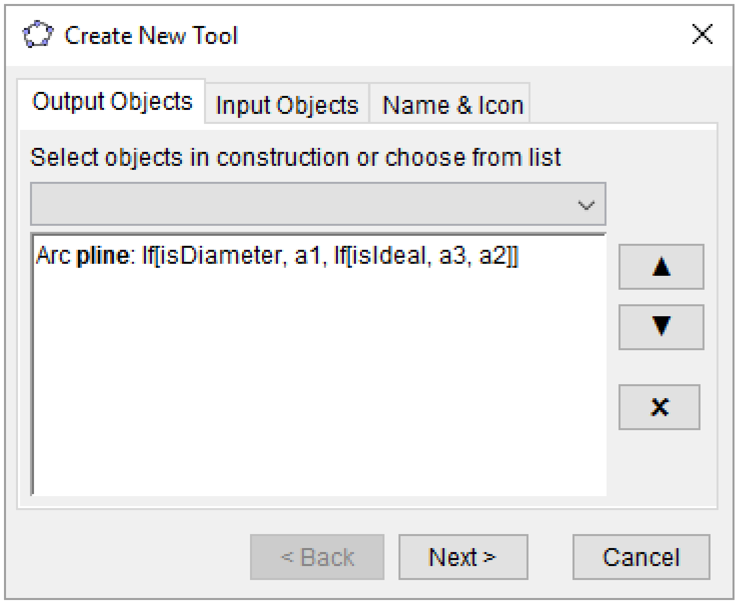
Create the hyperbolic line tool

Use the Boolean variables $isDiameter$ and $isIdeal$ to distinguish between the three cases. The $isDiameter$ variable tests whether or not points A and B are collinear with O and $isIdeal$ tests whether or not both A and B are on γ. Then use nested If[<Condition >, <Then >, <Else >] commands to combine the three cases. Note that in traditional programming languages, only the objects which are needed would be constructed. In this construction, however, the objects used in all three cases are always created and the nested If statement is used to display the valid object.

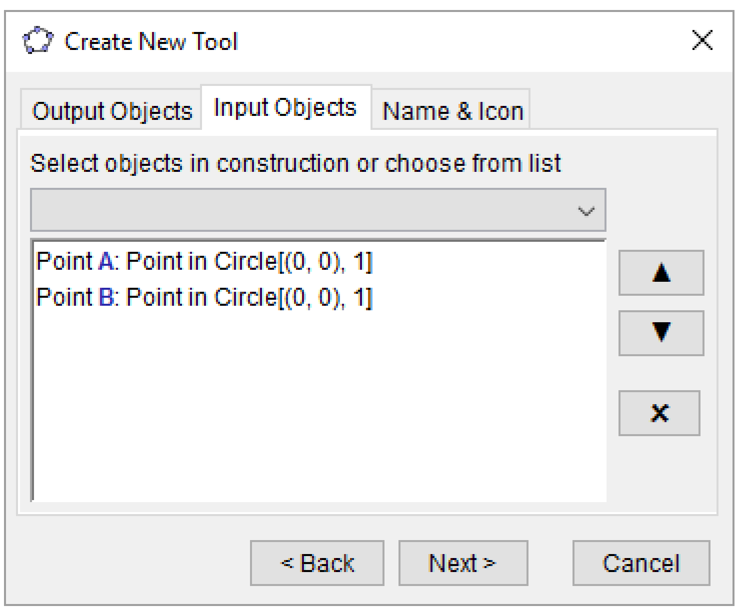
```
// Combine cases-----------------------------------------
// Boolean isDiameter is true if A, B and O are collinear,
// otherwise false
// Boolean isIdeal is true if A and B are on γ,
// otherwise false
// Combine diameters, arcs and ideal arcs into a single
// hyperbolic line command
//-----------------------------------------------------

isDiameter: AreCollinear[(0,0), A, B]
isIdeal: Length[A] == 1 & Length[B] == 1
pline: If[isDiameter, a1, If[isIdeal, a3, a2]]
```

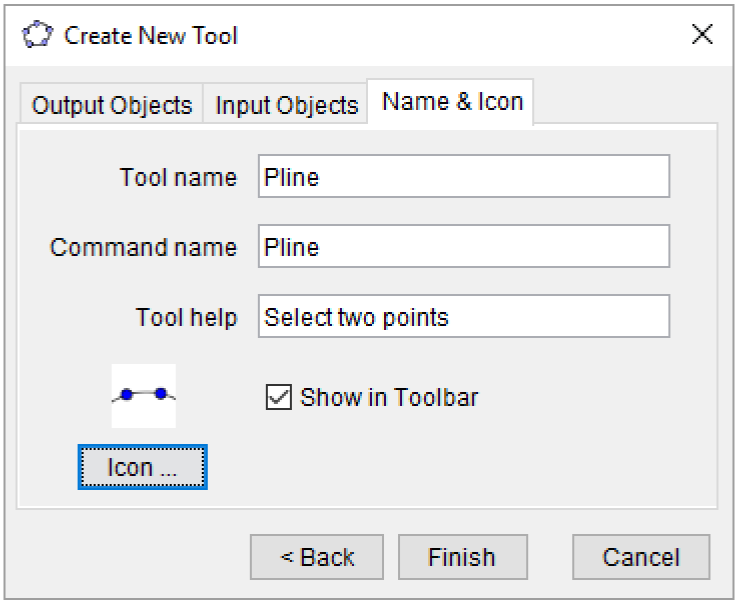
We are finally ready to create a custom tool for creating hyperbolic lines in the Poincaré model. Be sure to hide all of the objects except A, B and $pline$. For $pline$, it is best to turn off the label. Then go to the main menu in GeoGebra and select Tools/Create New Tool. In the Output Objects tab, use the drop-down menu to select $pline$.



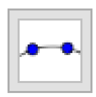
Use the Next button to move to the Input Objects tab and use the drop-down menu to select points A and B.



Use the Next button to move to the Name & Icon tab and provide a name for the tool and a command name. Provide a tool tip such as "Select two points" to provide users with information on how to use the tool.



Although it is not necessary, it is possible to create an icon. If you decide to do this, consider using a separate GeoGebra Worksheet to create a suitable image and then use screen capturing software such as Jing (free) or SnagIt (commercial) to create an image file. Then press the Icon button and navigate to the image file and press Open. Press the Finish button and you now have a tool for creating lines in the disk model.



This custom tool can be used both as a tool and as the command Pline[<Point >, <Point >] in the Input bar. In addition, the custom tool can be saved to your computer as GGT file and then imported into other GeoGebra files. To do this, select Tools/Manage Tools, then select Save As and press Save. Then open a new GeoGebra file and use File/Open to import the GGT file. Be sure to do this prior to beginning the hyperbolic circle construction, since this approach hides all of the details of construction of the hyperbolic line tool and allows you to focus on constructing the hyperbolic circle. This works since all of the details of the construction are contained in the GGT file and they are not displayed in the Algebra window.

Hyperbolic circle tool

Hyperbolic circles in the Poincaré disk model are also Euclidean circles but in general the hyperbolic center is not the same as the Euclidean center. The key to constructing a hyperbolic circle with center at points A and passing through point B is to locate its Euclidean center C and then construct a Euclidean circle with center C passing through B. The construction of the hyperbolic circle is broken into three cases depending on whether or not $A = O$ and whether or not the A and B are collinear with O.

Hyperbolic center at the origin

If $A = O$, then by Theorem 4 the Euclidean center of the circle is also at O and the hyperbolic circle can be constructed using a Euclidean circle centered at O and passing through B . Begin the construction by creating γ and two points inside γ.

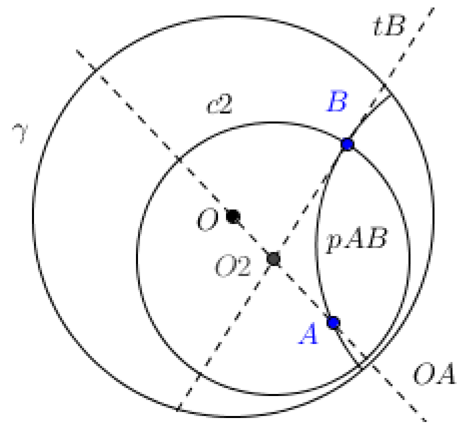


Figure 1.8: Construction of the hyperbolic circle $c2$ with hyperbolic center $A \neq O$ and passing through B for the case in which points A, B and O are not collinear. All hyperbolic lines passing through the center A are orthogonal to the circle. Since line OA is one such line, it is orthogonal to $c2$ and the center of $c2$ must lie on this line (if a line cuts a circle orthogonally, the line must be a diameter of the circle). In addition, $c2$ is orthogonal to the hyperbolic line pAB (circular arc) passing through points A and B . By Theorem 1, the Euclidean center $O2$ of $c2$ must lie on the tangent tB to the hyperbolic line pAB at point B . Thus the Euclidean center $O2$ of $c2$ is located at the intersection of line OA and tangent tB. The hyperbolic circle $c2$ can then be constructed using a Euclidean circle centered at $O2$ and passing through B.

```
// Poincarè disk --------------
// Construct the unit circle γ
// Create point A inside γ
// Create point B inside γ
// ----------------------------

γ: Circle[(0, 0), 1]
A: PointIn[Circle[(0, 0), 1]]
B: PointIn[Circle[(0, 0), 1]]
```

Then construct the circle.

```
// Center at the origin -------------------------
// Construct hyperbolic circle c1 using Theorem 4
// ---------------------------------------------

c1: Circle[A, B]
```

Points not on a diameter

If $A \neq O$ and points A and B are not collinear with O, then the hyperbolic circle centered at A and passing through B can be constructed using the fact that every hyperbolic line passing through A must be orthogonal to the circle as explained in Figure 1.8.

Begin by hiding $c1$ from the previous case and then implement the construction using the following GeoGebra code. Although the most obvious way to construct the tangent is to use the Tangent[<Point >, <Conic >] command, we forego this approach since the command fails for some locations of B on γ (returns undefined, most likely since the

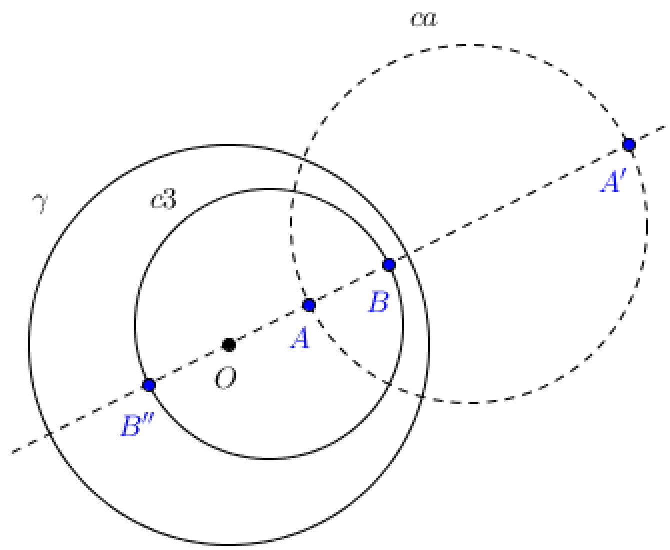


Figure 1.9: Construction of the hyperbolic circle $c3$ with hyperbolic center A and passing through B for the case in which $A \neq O$ and A and B are collinear with O. Begin by determining the inverse of A with respect to γ and then construct circle ca passing through A and A' and centered at the midpoint of segment AA'. By Theorem 2, ca is orthogonal to γ and is therefore a hyperbolic line in the Poincaré disk. Define A'' and B'' as the inverses of A and B with respect to ca (be careful since in all previous constructions, the inverses have been with respect to γ). To finish the construction, we use the distance metric. By Theorem 5, $d(A'', B'') = d(A, B)$. Furthermore since A is invariant after inversion with respect to ca , we obtain $d(A, B'') = d(A, B)$. Hence, the hyperbolic circle $c3$ can be constructed using a Euclidean circle centered at the midpoint of segment BB'' and passing through the point B.

numerical algorithms occasionally fail to find the tangent at the endpoint of an arc). The approach taken here is to use the fact that the tangent to a circle is perpendicular to its radius.

```
// Points not on a diameter ----------------------------
// Construct hyperbolic circle c2 using Theorem 1
// Construct line OA
// Construct hyperbolic line pAB through A and B
// Construct tangent tB to the hyperbolic line pAB at B
// Locate the center O2 of hyperbolic circle c2
// (intersection of line OA and tangent tB)
// Construct the hyperbolic circle c2
// -----------------------------------------------------

OA: Line[(0,0), A]
pAB: Pline[A, B]
tB: PerpendicularLine[B, Line[Center[pAB], B]]
O2: Intersect[OA, tB]
c2: Circle[O2, B]
```

Points on a diameter

If $A \neq O$ and points A, B and O are collinear, then the previous construction fails since the intersection of OA and tB is no longer defined. In this case the circle can be constructed using the distance metric as explained in Figure 1.9.

Begin by hiding all of the objects from the previous construction except points A and B and the defining circle γ and then move the A and B onto a diameter. Use the following GeoGebra code to implement the construction.

```
// Points on a diameter ------------------------------
// Construct hyperbolic circle c using Theorems 2 and 5
// Determine the inverse of point A with respect to γ
// Construct circle ca through points A and A′ centered
// at midpoint of segment AA′
// Determine the inverse of point B with respect to ca
// Construct the hyperbolic circle c3
// --------------------------------------------------

A’: Reflect[A, Circle[(0, 0), 1]]
ca: Circle[Midpoint[A, A’], A]
B’’: Reflect[B, ca]
c3: Circle[Midpoint[B, B’’], B]
```

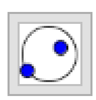
Create the hyperbolic circle tool

Combine the three cases by defining the Boolean values $Origin$ and $Collinear$ and using nested If[<Condition >, <Then >, <Else >] statements.

```
// Combine cases---------------------------------------
// Boolean isOrigin is true if A = O and false otherwise
// Boolean isCollinear is true if A, B and O are
// collinear and false otherwise
// Combine cases into a single hyperbolic circle command
//---------------------------------------------------

isOrigin: A == (0,0)
isCollinear = AreCollinear[(0, 0), A, B]
pcircle: If[isOrigin, c1, If[isCollinear, c3, c2]]
```

Hide all of the objects except A, B and $pcircle$ and turn off the label for $pcircle$. Then use Tools/Create New Tool to create a custom tool, $Pcircle$, which accepts points A and B as inputs and outputs $pcircle$. Provide a helpful tooltip such as “Select center, then point” and create an icon for the tool if you wish.



Conclusion

The primary focus of this paper has been on the construction of custom tools in GeoGebra for creating hyperbolic lines and hyperbolic circles in the Poincaré disk model. The hyperbolic line tool is robust and works well in all cases including when points A and B are in the interior of γ, on the boundary of γ and when they are on the same diameter. The hyperbolic circle is similarly robust and works for the full range of positions of the center point A and point B within and on γ. Tools for creating hyperbolic segments and rays can be constructed in a manner similar to that of hyperbolic lines with some modifications. If the reader wishes to create a more complete suite of tools for working with the Poincaré disk, then it is worth noting that the construction of hyperbolic perpendicular bisectors, hyperbolic midpoints, hyperbolic perpendicular lines and hyperbolic angle bisectors mimic their corresponding Euclidean compass and straightedge constructions.

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VISUALIZING FUNCTIONS OF COMPLEX NUMBERS USING GEOGEBRA

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Abstract: This paper explores the use of GeoGebra to enhance understanding of complex numbers and functions of complex variables for students in a course, such as College Algebra or Pre-calculus, where complex numbers are introduced as potential solutions to polynomial equations, or students starting out in an undergraduate Complex Variables course. The paper introduces methods to create interactive worksheets for students seeing complex numbers and functions for the first time and for those who have some experience with them, but struggle to visualize their meaning. Acknowledging limitations of GeoGebra concerning complex functions, we create new learning opportunities as we develop workarounds.

Keywords: complex numbers, GeoGebra

Introduction

Overview

At first glance, GeoGebra may not seem to be a very powerful tool for functions involving complex numbers. Indeed, if the goal is simply to perform very complicated calculations, there are many better tools available. Many of them, such as Sage, are also free. But for teaching the basics to College Algebra students or beginning students in a Complex Variables course, a strong case can be made that GeoGebra is the best tool for students to interact with visualizations involving complex numbers. At the time of this writing, GeoGebra lacks the ability to directly process functions involving complex variables, although it handles operations on complex numbers just fine. We will exploit those capabilities, and GeoGebra's unique ability to easily show two graphics windows interacting with each other. In some cases, GeoGebra's limitations can be used to an advantage, requiring students to break down functions into real-valued parts.

Entering and Displaying Complex Numbers

A complex number may be entered very easily in GeoGebra. Simply use the Complex Number tool or type

```
z1 = 2 + 2i
```

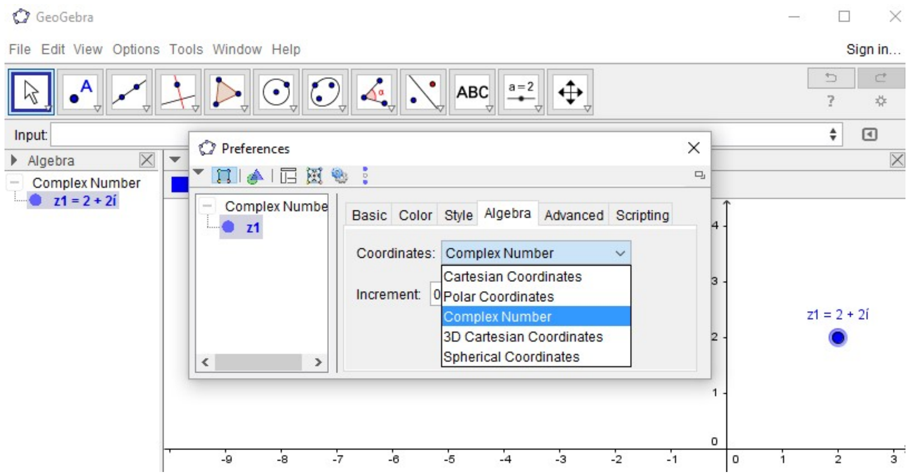


Figure 2.1: Graphics and Algebra view with $z1 = 2 + 2i$.

in the input bar and the complex number $z1$ will be saved and displayed as a point.[1] Except for the label, the complex number $2 + 2i$ is indistinguishable from the ordered pair $(2,\ 2)$ in the graphics window. The user can change the point to and from being treated algebraically as a complex number using the Algebra tab of the properties dialog. Referring to Figure 2.1, selecting *Complex Number* results in $z1$ being treated algebraically, and labeled (if *Value* is selected in the *Show Label* dialog) as a complex number. Only the *Complex Number* selection results in the point being treated algebraically as a complex number. In the Algebra view, points are either listed under the heading *Point* or *Complex Number*; this is based on how the point was initially entered and will not be affected by changing the setting in the Algebra tab.[2] Note that z cannot be used as the name because z is reserved as a variable in GeoGebra, but Z can be used. An existing point that had not been entered as a complex number can be changed to a complex number using the same tab. A complex number may be entered in polar form $(re^{i\theta})$, but will still be displayed as $a + bi$. For example, the same point $z1$ may have been entered as

```
z1 = 2 * sqrt(2) * e^(i * pi/4).
```

GeoGebra will accept the entry, but will automatically convert it and display $z1 = 2 + 2i$. [3] With a little extra work we can label the points with polar $\left(re^{i\theta}\right)$ form or any other form.

Complex Numbers as Vectors

To display a complex number as a vector, the user must enter a vector in GeoGebra, then use the Algebra tab in the object properties as in figure 2.1. The result will be a complex number displayed as a vector. For example, to have $z2 = 2 + i$ displayed as a vector, enter

```
z2 = Vector[2 + i].
```

This will result in the vector $z2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, but the Algebra setting defaults to *Cartesian Coordinates*, so the setting must be changed in the Properties dialog (see Figure 2.1). Figure 2.2 shows $z1$ and $z2$ as a point and a vector respectively.

Functions

Stored functions in Geogebra are always functions of real numbers. For example, an entry of

[1]There is nothing special about the choice of $z1$. entering `a = 2 + 2i` will result in the same complex number stored as a.
[2]Previous versions of Geogebra would change the heading of the point based on the Algebra setting.
[3]The *Polar Coordinates* selection in the pull-down menu in figure 2.1 will convert $z1$ to a point with polar coordinates (r, θ). GeoGebra will not process the point algebraically as a complex number.

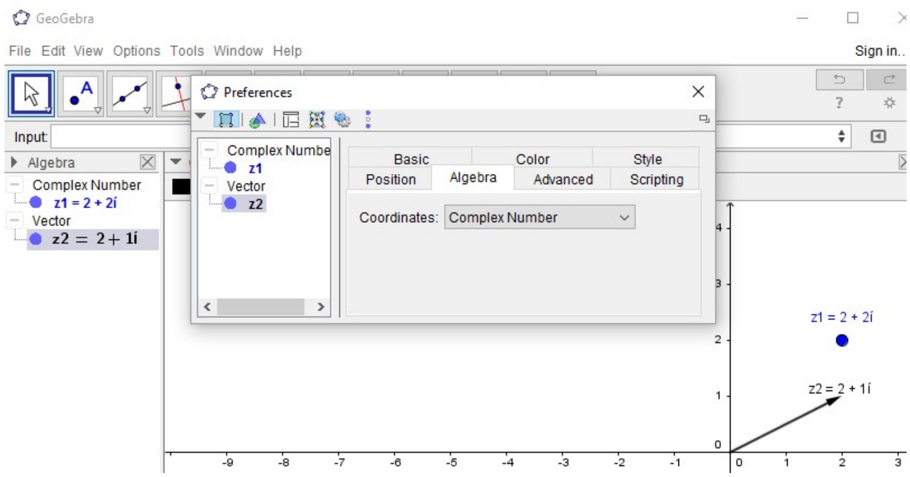


Figure 2.2: Complex numbers $z1$ as a point, and $z2$ as a vector.

```
f(z) = z^2,
```

might be expected to generate a function whose inputs may be complex numbers. However, if $f(z)$ is entered as above, followed by

```
w1 = f(z1),
```

where $z1 = 2 + 2i$, GeoGebra will process only the real part of $z1$, so the result is $w1 = 4$. Entering

```
f(2+2i)
```

results in an alert saying the argument $2 + 2i$ is illegal. The correct result can only be obtained by entering the function directly as a simple operation, not using function notation. For example, entering

```
w1 = (z1)^2,
```

yields the result $w1 = 0 + 8i$, as expected. The value of $z1$ can then be changed manually or by dragging $z1$ around the graphics view. As $z1$ updates, $w1$ will also update by the definition $w1 = (z1)^2$.

Operations With Complex Numbers

Addition and Subtraction

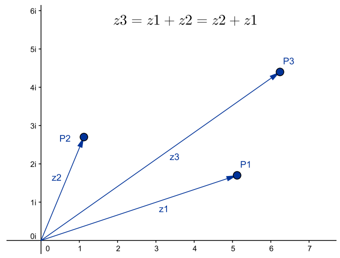
Clearly, the best way to visualize addition and subtraction is with vectors, but there are some obstacles to directly displaying vector addition with the desired parallelograms. The trick is to perform all algebra with points, then use the points to define vectors. For example, suppose we want to display the addition of 2 complex numbers $z1$ and $z2$ as $z3 = z1 + z2$. We would enter points $P1$ and $P2$ as complex numbers, then enter

```
P3 = P1 + P2.
```

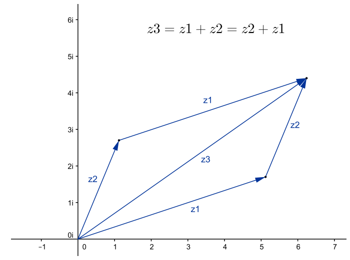
We then define our displayed complex numbers as vectors tied to each point by entering

```
z1 = Vector[P1]
z2 = Vector[P2]
z3 = Vector[P3].
```

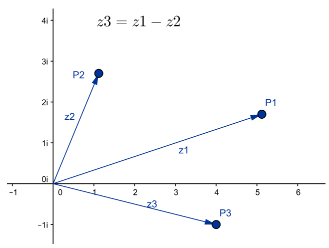
Figure 2.3a shows the result. The points defining the vectors are left labeled in Figure 2.3a for clarity, but the labels should be hidden and the points reduced to minimum size in the final version of the display. The points themselves may be hidden, but then they cannot be moved and the worksheet loses its interactive capability. Figure 2.3b shows $z3 = z1 + z2$ again, but $z1$ and $z2$ are duplicated from the tip of $z2$ and $z1$ respectively, forming a parallelogram, highlighting the commutative property. The duplicate vectors must also be defined by their corresponding points using



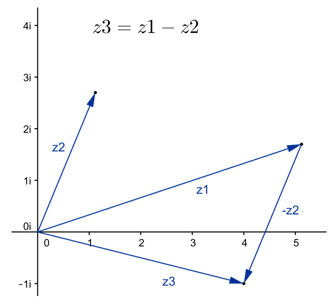
(a) Vectors $z1$, $z2$ and $z3 = z1 + z2$. Points defining vectors are shown.



(b) Same operation as Figure 2.3a, but point labels are off and point size is minimized.



(c) Subtraction of vectors, Points defining vectors labeled for clarity.



(d) Subtraction of $z2$ from $z1$.

Figure 2.3: Addition and Subtraction of Vectors.

the commands

```
z11 = Vector[P2, P3]
```
and
```
z22 = Vector[P1, P3].
```

The labels of $z11$ and $z22$ should be changed from *Name* to *Caption*, with captions of $z1$ and $z2$ respectively. Subtraction is essentially the same, except that $P3$ is defined as

```
P3 = P1 - P2
```

to subtract $z2$ from $z1$. Then, we label $z22$ (the vector from $P1$ to $P3$) as $-z2$ and display it, but do not display vector $z11$ (the vector from $P2$ to $P3$). The result is seen in Figures 2.3c and 2.3d.

Multiplication and Division

Multiplying two vectors whose coordinates are set to *Complex Number* will result in a point representing a complex number. If the Algebra setting of either of the vectors' coordinates are set to anything other than *Complex Number*, the result will be the scalar product of the two vectors [1]. Hence, the best way to display the multiplication of complex numbers as vectors is by using points to control the vectors, as described in Section 2. The goal is to demonstrate that the product, $z3 = z1 \cdot z2$, will have a modulus equal to the product of the modulus of $z1$ and the modulus of $z2$, and that the argument of $z3$ is the sum of the arguments of $z1$ and $z2$. Figure 2.4 shows $z3$ as the product of $z1$ and $z2$. As with addition, the algebra is done using unlabeled points $P1,\ P2$, and $P3$, where each vector is defined by its respective point. Division can be shown in a similar manner.

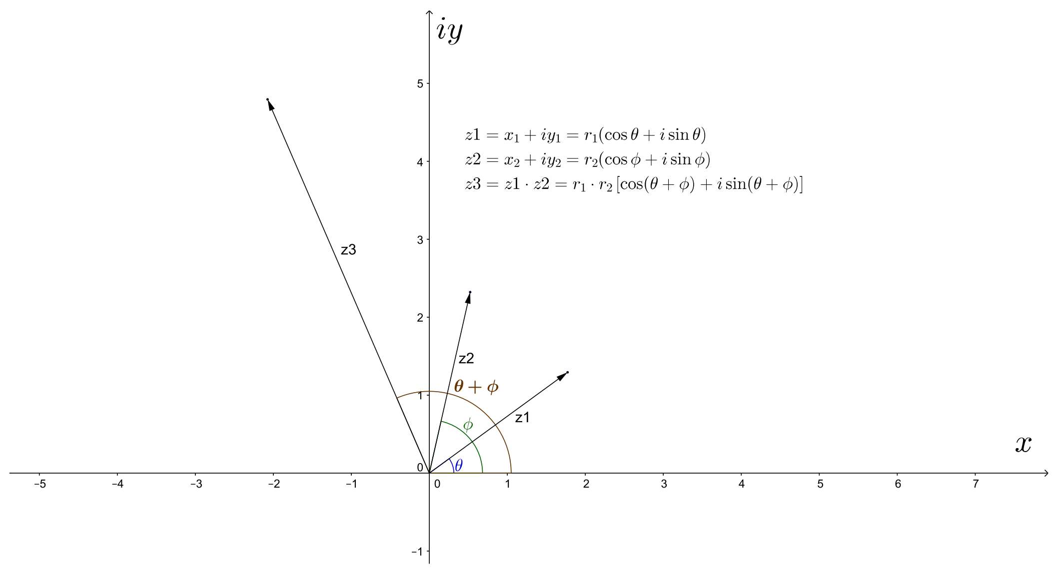


Figure 2.4: Multiplication of 2 complex numbers: $z3 = z2 \cdot z1$.

Functions as Transformations

Basic Setup

Since complex numbers are represented in 2 dimensions, we need a plane, usually the z-plane where $z = x + iy$, for the input of our function, and a second plane, usually the w-plane, where $w = u + iv$, to display the image of z [2]. We dedicate the *Graphics* and *Graphics 2* windows as complex planes and view them side by side. All function inputs, or pre-images, will be displayed in *Graphics*, which will be labeled as the z-plane. The outputs, or images, will be displayed in *Graphics 2*, which will be labeled as the w-plane. Starting with a simple function, we create a complex number (point), $z1$, visible only in the z-plane (*Graphics*). Next, define $w1$ by entering

```
w1 = (z1)^2
```

and making $w1$ visible only in the w-plane (*Graphics 2*). Figure 2.5 shows a trace of $z1$ and $w1 = (z1)^2$ in the z-plane and w-plane respectively.

The Locus Command

The trace in Figure 2.5 is simple and somewhat informative, but much better interaction and learning potential is possible using the `Locus[ <Point Creating Locus Line>, <Point> ]` command. First set up a geometric object in the z-plane and place a point with complex coordinates on the object. As an example, create 2 points, A and B, on the z-plane (Graphics window), then create a segment named $z2Segment$ (anticipating the use of point $z2$) between A and B. Next, create a point $z2$ on $z2Segment$ by using the *Point on Object* tool or by entering

```
z2 = Point[z2Segment].
```

The algebra of point $z2$ should then be set to *Complex Number* (see Figure 2.1). We can now move $z2Segment$ anywhere on the plane by dragging the endpoints A and B. The labels for Points A and B and $z2Segment$ should be off. Now create a point $w2$ in the w-plane that is defined by a function of $z2$. For now we will continue with $f(z) = z^2$. Then we define an object to be the locus curve of $w2$ as defined by $z2$; call it *w2Locus*. Enter the following commands in the input bar:

```
w2 = (z2)^2
w2Locus = Locus[w2, z2]
```

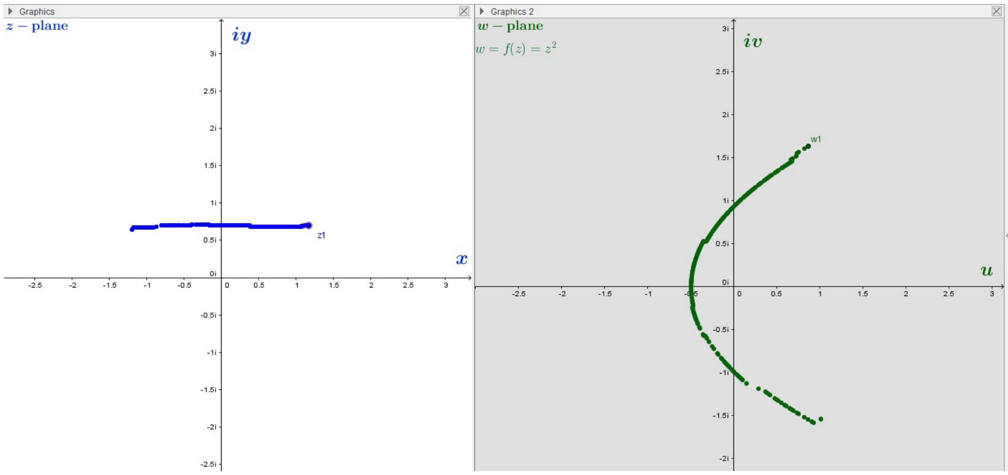


Figure 2.5: z and w planes side by side where $z = x + iy$, and $w = u + iv$.

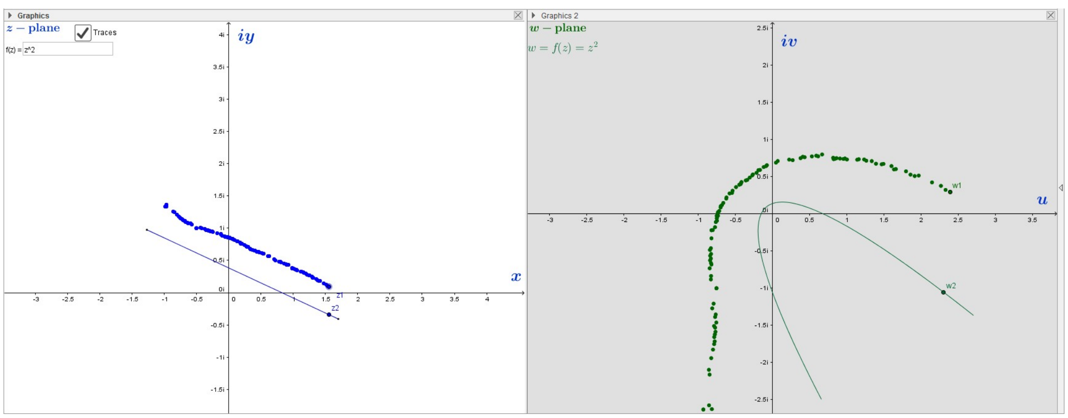


Figure 2.6: The locus of $w2$ as the image of the $z2$ along the segment.

Figure 2.6 shows the resulting locus of $w2$ compared with a similar trace. The locus is much simpler and cleaner in appearance than the trace and therefore more likely to grab the user's attention [3]. The locus is also more informative as we can see how the images of specific shapes look. In this case, the segment in the z-plane produces part of a parabola in the w-plane. Furthermore, the locus is permanent, whereas the trace will disappear as soon as an adjustment is made to the graphic. The segment $z2Segment$ can be moved arbitrarily as can the point $z2$ along $z2Segment$. $w2Locus$ and $w2$ automatically update as $z2Segment$ and/or $z2$ are updated, creating a nice interactive visualization. The process above is not limited to segments. The point $z2$ could have been attached to any geometric object. For example, Figure 2.7 explores the function $f(z) = z^5 - 1$. In the z-plane, a solid-lined fixed unit circle and a dashed-line movable circle have been created. Using the *Point on Object* tool, the point $z2$ is placed on the fixed circle and $z3$ is placed on the movable circle. In the w-plane, the corresponding image points, $w2$ and $w3$ have been created using the definitions $w2 = (z2)^5 - 1$ and $w3 = (z3)^5 - 1$. The loci were then created by entering

```
w2Locus = Locus[w2, z2]
w3Locus = Locus[w3, z3].
```

The line styles of the loci are set to match the styles of the corresponding geometric objects. The rest of the dots on the solid circle in the z-plane represent the complex roots of f. They are best entered as a list, so they are a single object in GeoGebra. By entering

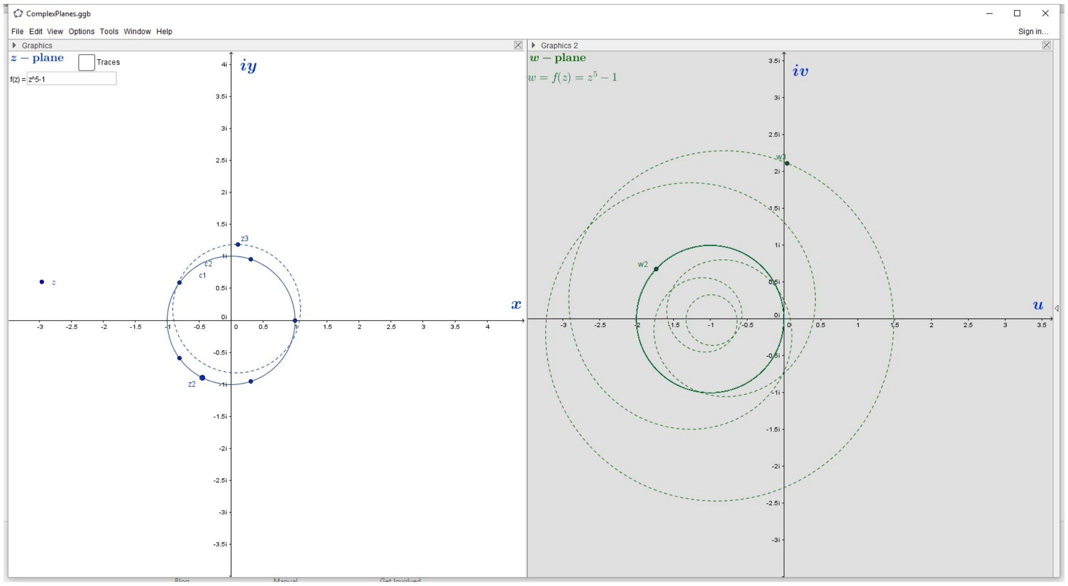


Figure 2.7: Image of $f(z) = z^5 - 1$.

```
roots = {ComplexRoot[f]},
```

the list *roots* is created whose elements are all the complex roots of $f(z) = z^5 - 1$.[4] With the roots displayed, the user should notice that when $z2$ is moved along the unit circle from one root to another, $w2$ travels all the way around the circle formed by its locus.

Returning to the function z^2, Figure 2.8 shows some more ideas for loci. The construction is created in a similar manner to that of Figure 2.7, with the addition of 2 polygons in the z-plane. For clarity, the line styles of the loci in the w-plane match the line styles of the objects in the z-plane used to create the loci. All the points in the z-plane are attached to the geometric objects as shown and all the objects are movable with the exception of the solid circle. As the objects are moved in the z-plane, the corresponding loci in the w-plane will be updated.

Advanced Interaction

Finally, we add some student interaction to the worksheet. This will require the student to calculate the real component, $u(x, y)$, and the imaginary component, $v(x, y)$, of w. The functions u and v are real-valued, multi-variable functions of x and y, where $z = x + iy$ and $w = u(x, y) + iv(x, y)$. As usual, z and w will have corresponding indices ($z1 \to w1$) on the GeoGebra worksheet. To accomplish this, we need two more functions to represent $u(z)$ and $v(z)$ in our worksheet. Since $z = x + iy$, this is easily accomplished with the two multi-variable function entries

```
u(x, y) = 1
```
and
```
v(x, y) = 1,
```

where the 1 on the right side of each equation is just a placeholder function; the student will change them later. The displays for both functions should be off. Once the functions u and v are in the worksheet, establish input boxes for each with u and v being the linked object respectively. Now the student can break a function $f(z)$ into its real and imaginary components and enter an image point, $w1$, of a complex number, $z1$, where the real coordinate is u and the imaginary coordinate is v. The student would enter the real-valued functions u and v into the input boxes and then simply create $w1$ by entering

[4]Note that the argument of the ComplexRoot[<Polynomial>] command must be a polynomial.

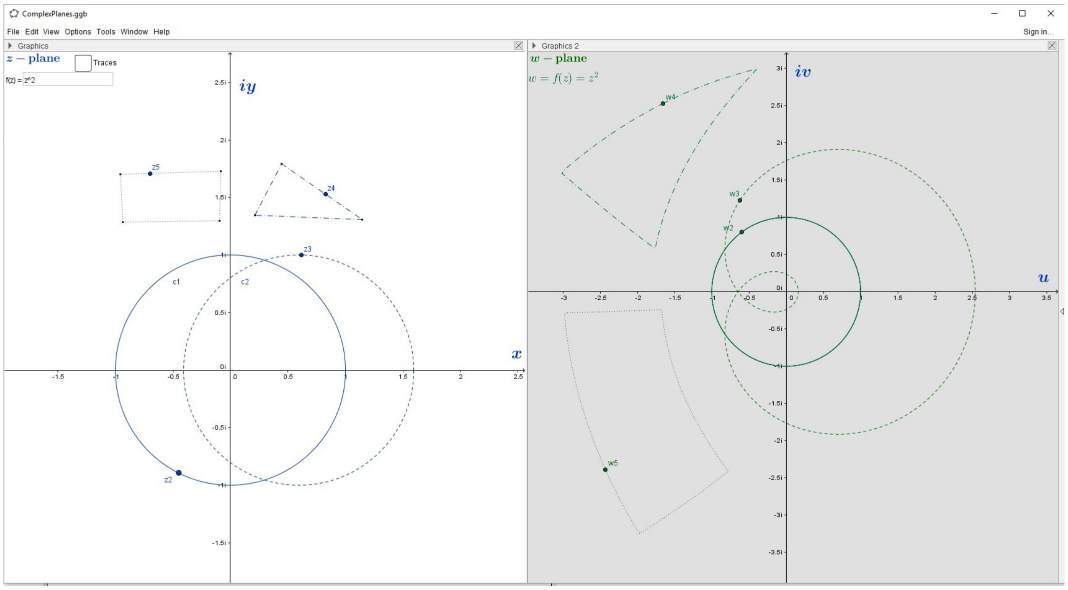


Figure 2.8: Image of $f(z) = z^2$ with multiple loci.

```
w1 = u(z1) + i * v(z1).
```

The functions $u(z1)$ and $v(z1)$ are multi-variable functions that take the coordinates of $z1$ as inputs. Alternatively, $w1$ could be defined by

```
w1 = u( x(z1), y(z1) ) + i * v( x(z1), y(z1) ).
```

Figure 2.9 shows another quadratic function $(2z^2 - 3z + 4)$. In this example, however, the functions u and v are used for the mappings $z1 \to w1$ and $z2 \to w2$ and the loci are created using techniques introduced earlier. The functions u and v are determined manually and entered into the input boxes. Once the functions u and v are entered, new mappings $zk \to wk$ can easily be created by defining a new complex number, zk, then entering `wk = u(zk) + i * v(zk).`

Conclusion

While GeoGebra is not the most powerful computational tool, it is perfectly suited for early studies of complex numbers and functions involving complex variables. GeoGebra provides a unique ability to work with two graphics windows that interact with each other, which provides an engaging experience. Since GeoGebra does not handle complex numbers with functions, the user must decide whether to use direct operations or construct functions of complex variables using only real-valued functions. Both methods work equally well and can be used to check each other. Students should be encouraged to explore these and limitless other possibilities.

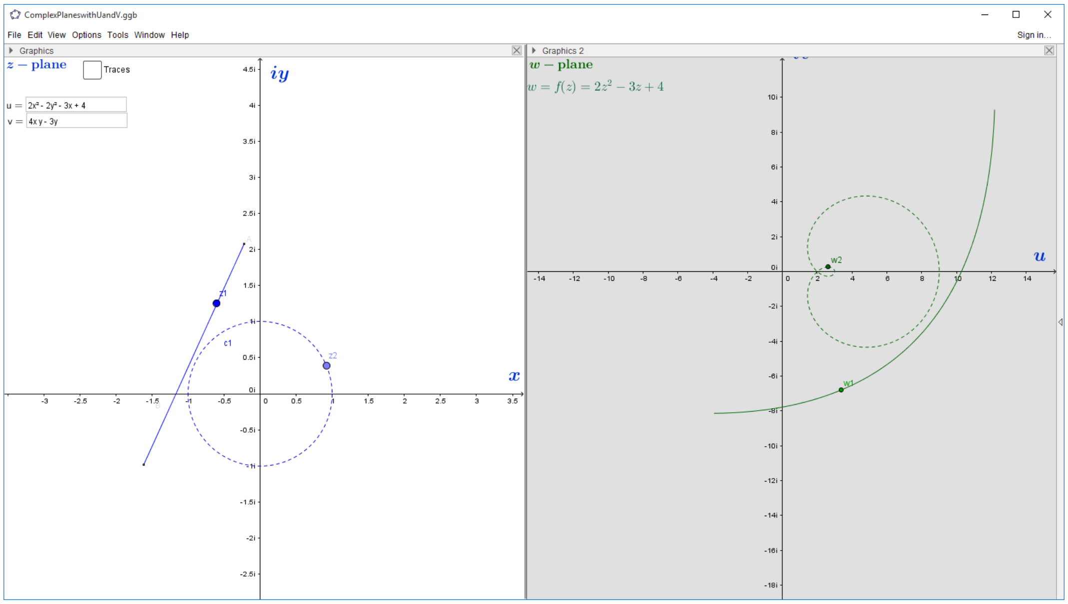


Figure 2.9: A function defined by real functions u and v entered in the input boxes shown on the z-plane. $w1 = f(z1) = u(z1) + iv(z1)$.

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A GEOMETRIC INTERPRETATION OF COMPLEX ZEROS OF QUADRATIC FUNCTIONS

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Abstract: Most high school mathematics students learn how to determine the zeros of quadratic functions such as $f(x) = ax^2 + bx + c$, where a, b, and c are real numbers. At some point, students encounter a quadratic function of this form whose zeros are imaginary or complex-valued. Since the graph of such functions do not intersect the x-axis in the xy-plane, students may be left with the impression that complex-valued zeros of quadratics cannot be visualized. The main purpose of this manuscript is to show that if the zeros of a quadratic function with real-valued coefficients are imaginary, the zeros can be seen if we use an appropriate coordinate system. For illustrative purposes, we have used the software program GeoGebra, which allows us to create a three-dimensional Cartesian coordinate system where imaginary zeros can be viewed simultaneously with the graph of the quadratic function they correspond to. To illustrate this, we will apply geometric transformations to the function given by $f(x) = x^2 - 6x + 13$ in order to visualize its zeros, which happen to be complex-valued. Then, we will identify a particular set of complex numbers that can be used as inputs for the function f. Using this set of complex numbers, we can construct the exact image that is produced by the geometric transformations. Then, we may deem the two methods as equivalent ways to ultimately construct the geometric images of complex-valued zeros of quadratic functions with real-valued coefficients.

Keywords: quadratic functions, complex roots, GeoGebra

Quadratic Functions and their Zeros

Students are taught in high school that one way to determine the zeros of quadratic functions with real-valued coefficients is to locate the function's x-intercept(s). For example, if we wish to find the zeros of the quadratic function $g(x) = x^2 - 5x + 6$, we graph it in $\mathbb{R}^2$ or the xy-plane (Figure 3.1), and we see that the function $g(x)$ has intercepts at $x = 2$ and $x = 3$, which correspond to the points $(2, 0)$ and $(3, 0)$ in $\mathbb{R}^2$. So, these are the zeros of the function $g(x)$ or, equivalently, the solutions of the quadratic equation $x^2 - 5x + 6 = 0$.

It is not long before students see the graph of a quadratic function such as $f(x) = x^2 - 6x + 134$ (Figure 3.2). Since the graph of f has no x-intercept(s), curious students might ask, "Where are the zeros located? How can we see them?" Indeed, the reason the graph of f has no x-intercepts is because the zeros are complex-valued, and the x-axis represents real numbers, not complex numbers. Some basic algebra allows us to verify, albeit without visualization in the xy-plane, that the zeros do exist, and $f(x) = 0$ precisely when $x = 3 \pm 2i$, where $i = \sqrt{-1}$ and $i^2 = -1$.

For ease of exposition, throughout the remainder of this manuscript we shall refer to quadratic functions with real-valued coefficients and complex-valued zeros as Type RC quadratic functions.

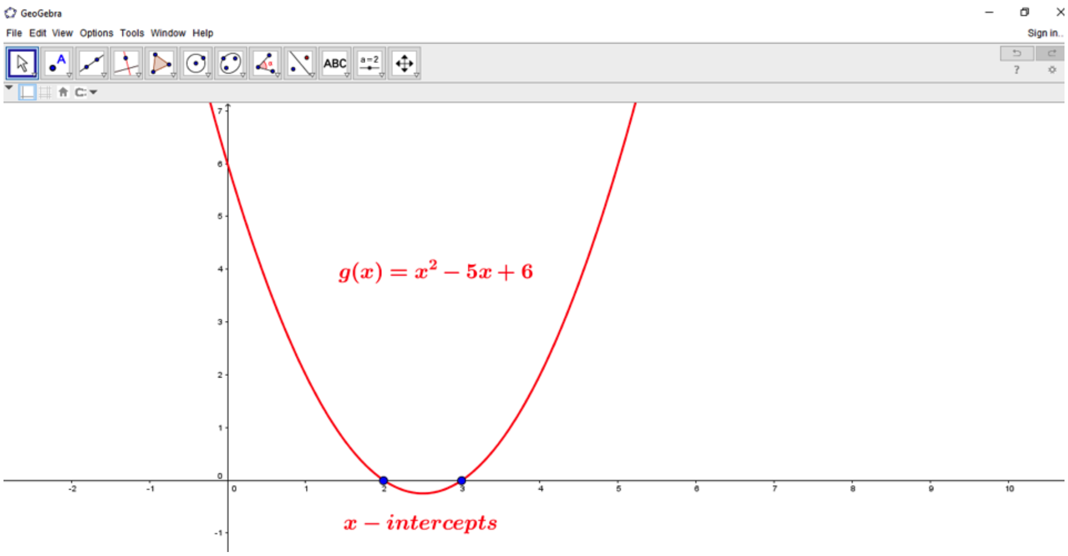

Figure 3.1: The graph of the quadratic function $g(x) = x^2 - 5x + 6$, which intersects the x-axis at the points $(2, 0)$ and $(3, 0)$. These points correspond to the real-valued zeros of $g(x)$, $x = 2$ and $x = 3$.

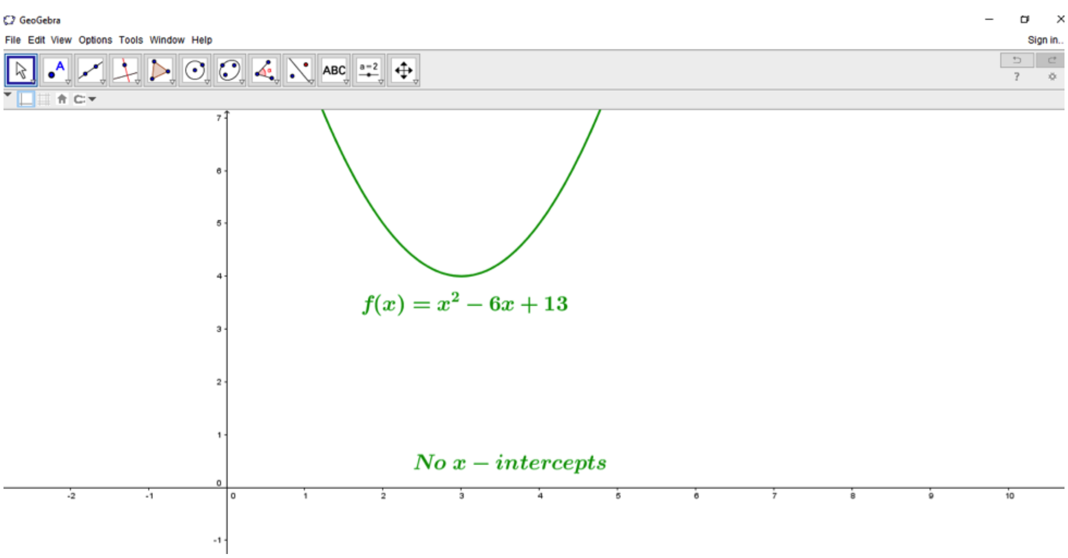

Figure 3.2: The graph of the quadratic function $f(x) = x^2 - 6x + 13$, which does not intersect the x-axis, indicating that the function f has imaginary or complex-valued zeros.

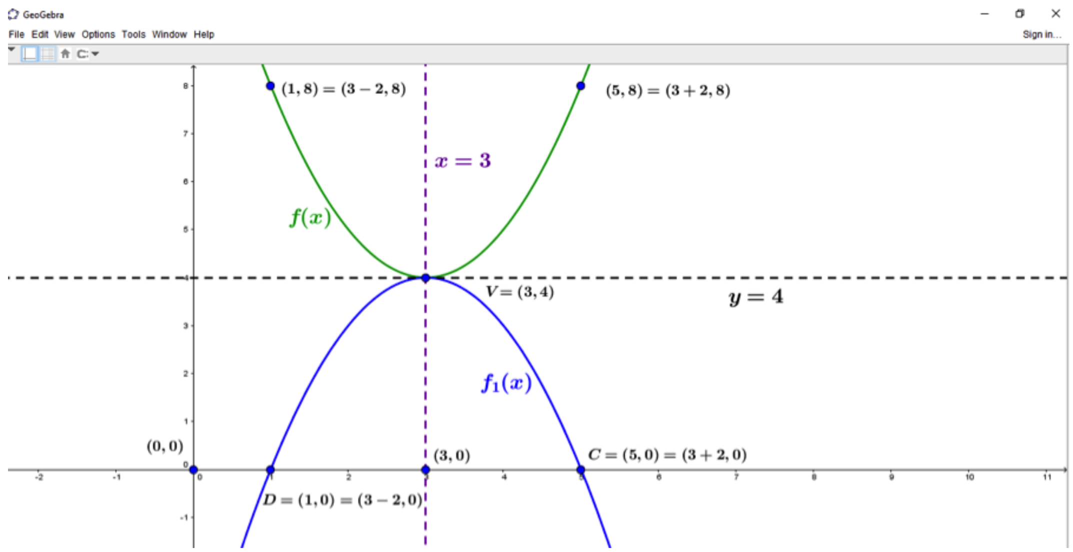

Figure 3.3: The graph of $f(x) = (x-3)^2 + 4$ is reflected about the line $y = 4$ in the xy-plane. This produces the graph of the quadratic function given by $f_1(x) = -(x-3)^2 + 4$, which has real zeros at the points $C = (5, 0)$ and $D = (1, 0)$.

A Process for Visualizing Complex Zeros of Type RC Quadratics

Using an appropriate coordinate system, we can apply simple geometric transformations to the graph of a Type RC quadratic function and ultimately obtain a visual image of its zeros. This process is outlined below in detail as it is applied to the function $f(x) = x^2 - 6x + 13$. The steps are accompanied by GeoGebra images that illustrate the process in a clear manner.

Step I: Write the Type RC quadratic function in vertex form $f(x) = a(x-h)^2 + k$, where the xy-plane vertex is the point (h, k). For our example, we write $f(x) = (x-3)^2 + 4$, which has xy-plane vertex $V = (3, 4)$. The graph of f is shown once again as the green parabola in Figure 3.3. We observe that the graph of f does not cross the x-axis since its zeros are $3 \pm 2i$.

Step II: In the xy-plane, reflect the graph of f about the line $y = 4$, which contains the vertex point V. The resulting image is an xy-plane parabola or the graph of the quadratic function $f_1(x) = -(x-3)^2 + 4$ [blue parabola in Figure 3.3]. In general, reflecting an xy-plane quadratic function $f(x) = a(x-h)^2 + k$ about the line $y = k$ will yield the graph of an xy-plane parabola or quadratic function $f_1(x) = -a(x-h)^2 + k$. In our example, the quadratic function f_1 has real zeros $x = 5$ and $x = 1$ or, equivalently, x-intercepts at the two points $C = (5, 0)$ and $D = (1, 0)$. These points may also be written as $C = (3+2, 0)$ and $D = (3-2, 0)$, indicating the manner in which the zeros of a quadratic function are distributed about the axis of symmetry. For f_1, each x-intercept is two units from the axis of symmetry, which is the line $x = 3$ in the xy-plane. Note that the graph of f contains the points $(5, 8) = (3+2, 0)$ and $(1, 8) = (3-2, 8)$, which are the preimage points of C and D, respectively, under the reflection.

Step III: Insert the zi-axis perpendicular to the xy-plane to represent the imaginary axis [blue axis in Figure 3.4], and define the resulting right-handed three-dimensional coordinate system as (x, y, zi)-space. We now denote our axis of symmetry as α since $x = 3$ represents an entire complex plane in (x, y, zi)-space. Each point (x, y) in the xy-plane may now be viewed as the point $(x, y, 0i)$ within (x, y, zi)-space. For example, our vertex point in (x, y, zi)-space is the point $V = (3, 4, 0i)$. The aforementioned points C and D are $(3+2, 0, 0i)$ and $(3-2, 0, 0i)$, respectively, and their respective preimage points are $A = (3+2, 8, 0i)$ and $B = (3-2, 8, 0i)$ [See Figure 3.5]. Although the insertion of the imaginary axis creates infinitely many complex planes within (x, y, zi)-space, the reader may want to consider that one of many ways to represent an arbitrary complex number $a + bi$ in (x, y, zi)-space is to write it as the point $(a, 0, bi)$. We choose this particular representation while reminding the reader that we want to find points that correspond to the zeros of our Type RC quadratic function f, and these points are expected to have a y-coordinate of 0.

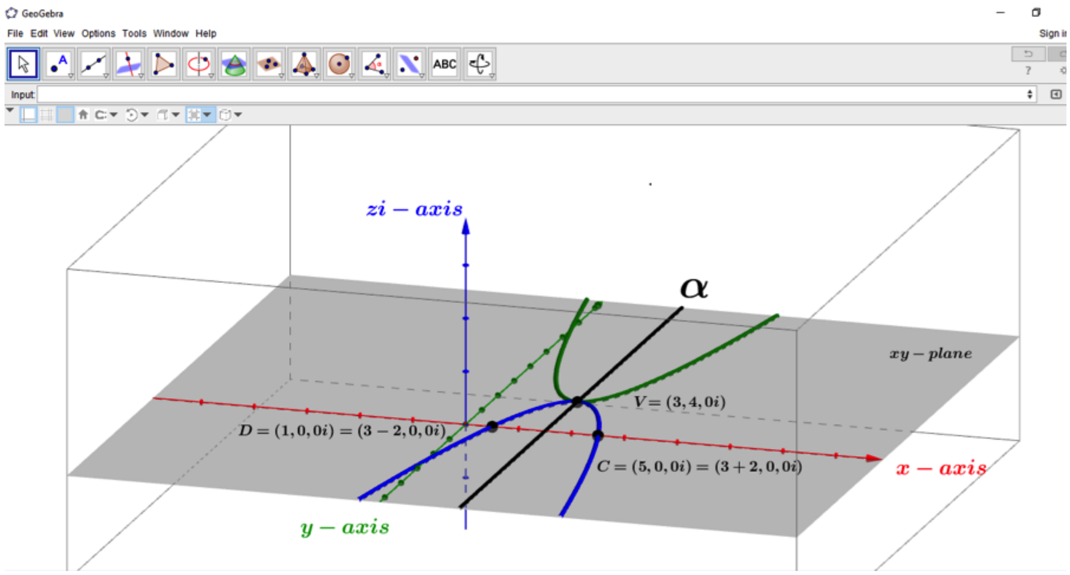

Figure 3.4: The (x, y, zi)-space image of the graphs of f, f_1, and α, all lying in the xy-plane.

Step IV: In (x, y, zi)-space, plot the points $(3, 0, 2i)$ and $(3, 0, -2i)$. Next, we rotate the graph of f_1 by $90°$ in the counterclockwise direction about α, the axis of symmetry[1]. We denote the resulting curve as the parabola φ, which lies entirely in the complex plane $x = 3$ [Figure 3.5]. The curves f_1 and φ must be parabolas since they are produced by applying rigid motions to f. Under this rotation, the points $C = (5, 0, 0i) = (3 + 2, 0, 0i)$ and $D = (1, 0, 0i) = (3 - 2, 0, 0i)$ are mapped to points $C' = (3, 0, 2i)$ and $D' = (3, 0, -2i)$, respectively, which are the points we plotted at the beginning of this step. Since the y-coordinate for each of C' and D' is 0, the parabola φ intersects the plane $y = 0$ precisely at two points that are equivalent to the complex zeros of f, namely $3 \pm 2i$ (see Figure 3.5). Recall that we set out to see the solutions to the equation $y = f(x) = 0$. Thus the points C' and D' appear to be the geometric images of the complex zeros of the quadratic function $f(x) = (x - 3)^2 + 4$.

To summarize what we have done thus far, we began with a Type RC quadratic function. By the procedure outlined above in Steps I – IV, the xy-plane graph of this quadratic function was reflected about the line $y = 4$, then rotated $90°$ counterclockwise about the axis of symmetry in (x, y, zi)-space, yielding the image of a parabolic curve that intersects the plane $y = 0$ at two points. The two points of intersection, C' and D', correspond directly with the complex zeros of our Type RC quadratic function $f(x) = (x - 3)^2 + 4$. Next, we show that this result was not a coincidence, meaning that the points C' and D' are indeed the geometric zeros of our function f.

The connection between the parabola φ and the quadratic function rule $f(x) = (x - 3)^2 + 4$

There is an interesting relationship between the quadratic function rule $f(x) = (x - 3)^2 + 4$ and points that lie on the parabolic curve φ. To uncover this relationship, we will first analyze the effects of the rigid motions as they are applied to points on the graphs of f and f_1 in succession, ultimately producing points on the curve φ. Since (x, y, zi)-space is a Cartesian coordinate system, we can identify the results of the rigid motions very easily. Then, we will verify the accuracy of these results using algebraic means.

First, we take full advantage of the vertex form of our function rule and write our input values in terms of the x-value of the vertex. Specifically, we write x-values in the form $3 + z'$, where z' is a real number or the signed-distance from the axis of symmetry, α. Writing x-values in this form will simplify any calculations involved. For any arbitrary

[1]Note that our choice of rotation above is somewhat arbitrary, as a $90°$ rotation in the clockwise direction would work as well. Under such a rotation, $(3 + 2, 0, 0i)$ would map to $(3, 0, -2i)$ instead of $(3, 0, 2i)$, and the point $(3 - 2, 0, 0i)$ would map to $(3, 0, 2i)$ instead of $(3, 0, -2i)$. This is not the correspondence we seek, thus the counterclockwise rotation was chosen instead.

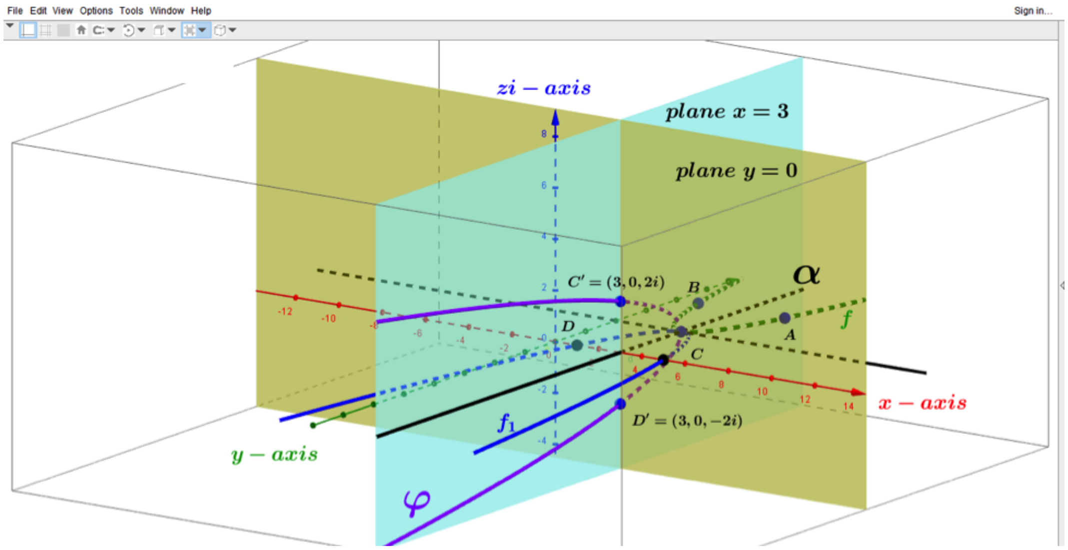

Figure 3.5: The (x, y, zi)-space image of the plane $x = 3$ (light blue), the plane $y = 0$ (yellow-grey), and the graphs of f (green), f_1 (blue), and the curve φ (purple). The complex planes $x = 3$ and $y = 0$ each contain the points C' and D', which correspond to the complex numbers $3 \pm 2i$, the zeros of the function f. Figure 3.6 provides an alternate image of this situation.

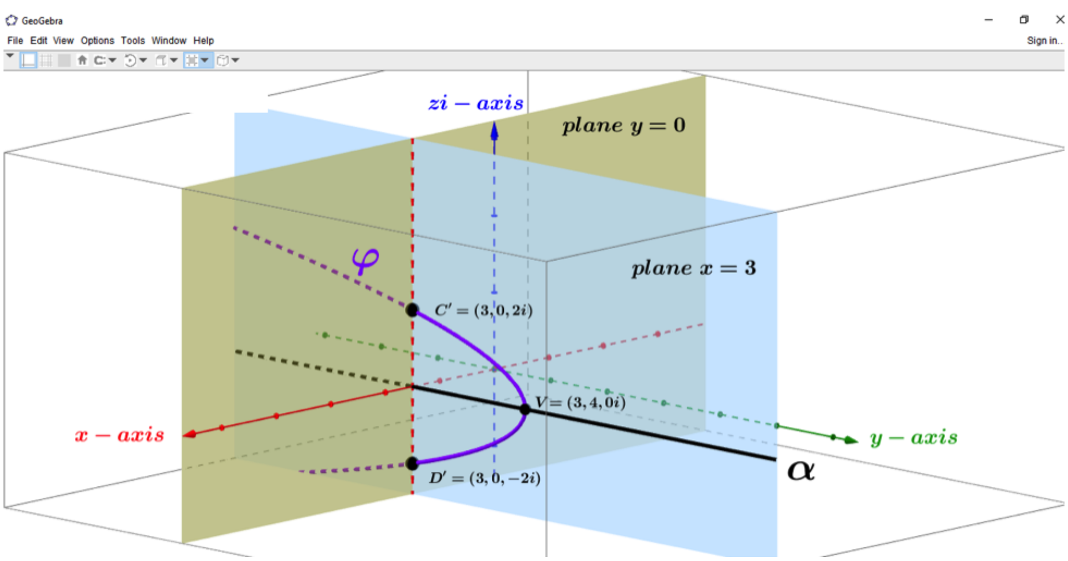

Figure 3.6: An alternate view of φ intersecting the plane $y = 0$. Note the significance of this since the complex zeros $3 \pm 2i$ correspond to the points C' and D', where $3 \pm 2i$ are solutions of equation $y = f(x) = 0$.

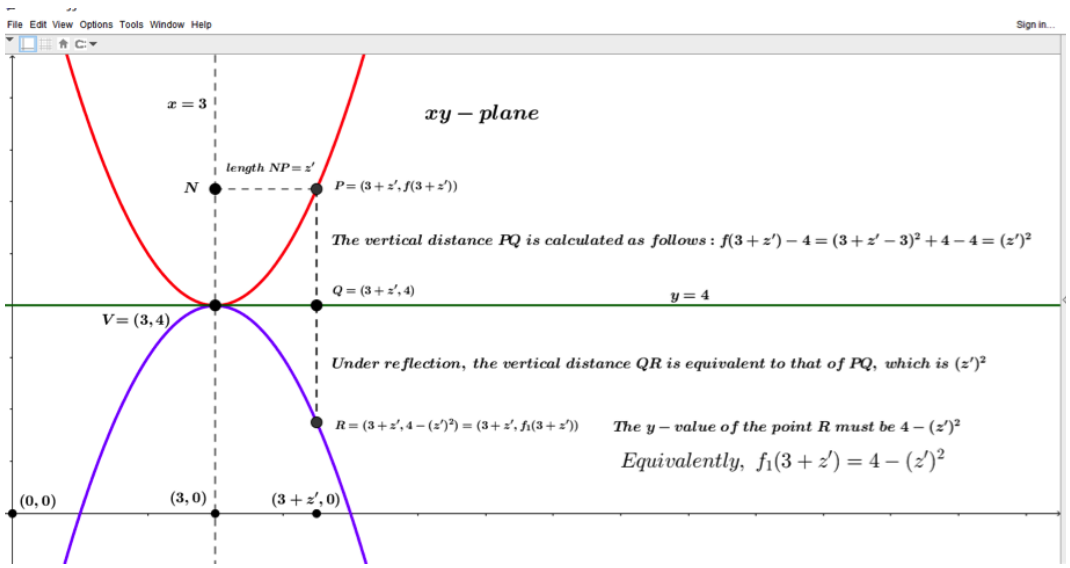

Figure 3.7: Calculation of the coordinates of an arbitrary point on the graph of f as it is mapped to a point on the graph of f_1 when f is reflected about the line $y = 4$ in the xy-plane.

$x = 3 + z'$, we have $f(x) = f(3+z') = (3+z'-3)^2 + 4 = (z')^2 + 4$. So, points on the graph of f in (x, y, zi)-space take the form $(3 + z', (z')^2 + 4, 0i)_f$. Note that any such point in the xy-plane lies at the vertical distance $(z')^2$ above the line $y = 4$ since $f(3 + z') - 4 = (z')^2 + 4 - 4 = (z')^2$.

Upon reflecting the graph of f about the line $y = 4$ in the xy-plane, we claim that all points on f are mapped to points $(3 + z', 4 - (z')^2, 0i)_{f_1}$ on f_1. Under the aforementioned reflection, it should be clear to the reader that the x and zi-coordinates remain fixed. With the exception of the vertex $V = (3, 4, 0i)$, all points on the graph f_1 in the xy-plane must lie below the line $y = 4$ at a vertical distance of $(z')^2$, the same vertical distance between points on f and the line $y = 4$. Consequently, points on the graph of f_1 must have a y-coordinate of $4 - (z')^2$. Equivalently, evaluating f_1 at $3 + z'$ yields $f_1(3 + z') = 4 - (z')^2$. Thus our claim is verified both geometrically and algebraically.

The results of the reflection described above are illustrated in clear detail in Figure 3.7. Since the reflection occurs in the xy-plane, for ease of exposition, points in Figure 3.7 are written in (x, y) form.

Finally, when the graph of f_1 is rotated in (x, y, zi)-space by $90°$ about α in the counterclockwise direction, we obtain the following results:

- The y-coordinate of points $(3 + z', 4 - (z')^2, 0i)_{f_1}$ must remain fixed under this rotation, so that points on φ must have y-coordinates of the form $4 - (z')^2$. For now, we seek points of the form $(x, 4 - (z')^2, zi)$.
- As a consequence of the $90°$ rotation, the parabolic curve α lies entirely in the complex plane $x = 3$. As a result, all points on φ must take the form $(3, 4 - (z')^2, zi)_\varphi$, for some real number z
- By the symmetry of the parabola f_1, under the $90°$ counterclockwise rotation, the value of the zi-coordinate must be precisely that of z' given by $3 + z'$. Recall that z' is the signed-distance from the axis of symmetry, α. As a result, all points of the form $(3, 4 - (z')^2, z'i)_\varphi$ lie on the graph of φ

To lend some additional perspective to the statements above, Figure 3.8 provides an overhead view of, in particular, the points C and D and their path of rotation about α (Step IV), which occurs entirely in the plane $y = 0$. The resulting image points C' and D' are shown as well.

Summarizing thus far, if we choose a real number z' and use our function rule to generate the point $(3 + z', (z')^2 + 4, 0i)_f$ on the graph of f, the reflection and subsequent rotation described in Steps I – IV naturally gives rise to the following mapping of points from f to f_1 to φ:

$$(3 + z', (z')^2 + 4, 0i)_f \rightarrow (3 + z', 4 - (z')^2, 0i)_{f_1} \rightarrow (3, 4 - (z')^2, z'i)_\varphi \tag{3.1}$$

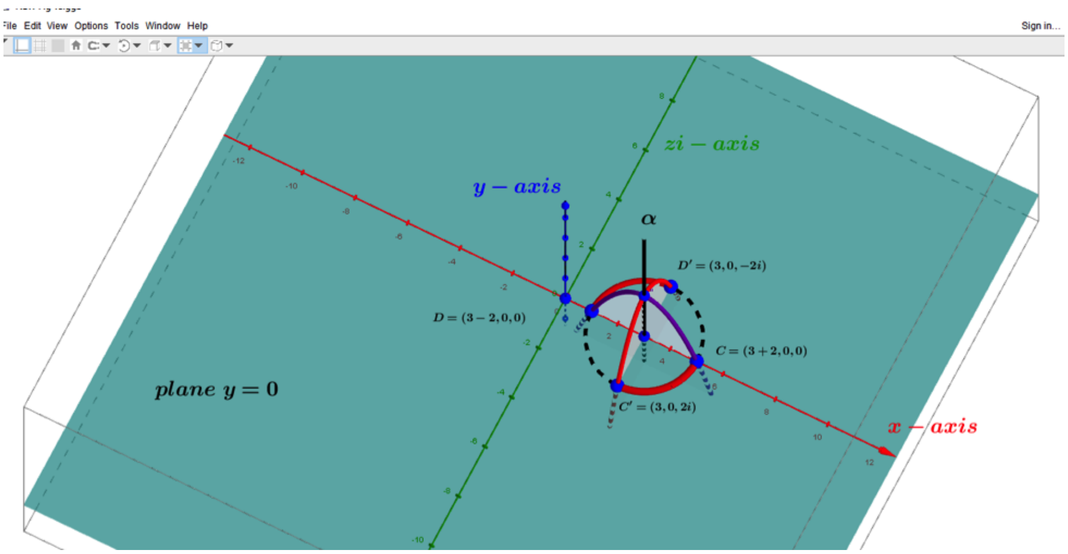

Figure 3.8: An overhead view of the plane $y = 0$, the graph of f_1, and the parabolic curve φ, which is produced by the rotation of f_1 about α in Step IV. The graph of the xy-plane circle $(x-3)^2 + y^2 = 4$ contains the path of rotation for the points C and D.

Taking $z' = 2$ gives us the mapping as it relates to the points C and D and their respective preimages and images in (x, y, zi)-space:

$$(3 \pm 2, 8, 0i)_f \to (3 \pm 2, 0, 0i)_{f_1} \to (3, 0, \pm 2i)_\varphi \tag{3.2}$$

In general, the mapping given by (3.1) suggests there are two identifiable properties of points lying on the parabolic curve φ. The first is that each has an x-coordinate of 3, which is our function's value for h in the vertex point. The second is that other than the vertex point, each point lying on φ contains a pure complex-valued zi-coordinate, meaning $z \neq 0$. Given these two properties regarding points lying on φ, perhaps pure complex numbers of the form $3 + zi$ play a role here. To establish the connection we seek between the function rule $f(x) = (x-3)^2 + 4$ and the parabolic curve φ, we shall modify the domain of our function f, allowing for complex number inputs of this form.

To see why this makes sense, let's analyze the transformation above more closely as it relates to the points of f_1 and φ. The x-coordinate $3 + z'$ for points on f_1 transformed to produce an x-coordinate of 3 and a zi-coordinate of $z'i$ for each point on φ. Suppressing the y-coordinate of a point $(3, 4-(z')^2, z'i)_\varphi$ lying on φ produces a point of the form $(3, z'i)$, which lies in the complex-plane $x = 3$. Equivalently, this point can be viewed as the complex number $3 + z'i$.

Following our suggestion above, if we take z' to be an arbitrary real number, evaluating f at $3 + z'i$ yields $f(3 + z'i) = (3 + z'i - 3)^2 + 4 = (z'i)^2 + 4 = 4 - (z')^2$, which is the precise value of the y-coordinate we had suppressed. Thus we can use our function rule to generate points $(3, 4-(z')^2, z'i)_\varphi$ on the graph of φ by simply picking an arbitrary signed distance from α, say z', and then compute $f(3 + z'i)$ to ultimately construct the points. All points produced in this manner are exactly the same points produced using the rigid motions described in Steps I – IV. Furthermore, the zeros of our function f, which happen to be the the mathematical objects we have sought to visualize, can now be constructed as geometric objects using either of two methods. We may construct the zeros of our function f by evaluating $f(3 \pm 2i)$ and then constructing the points $(3, 0, \pm 2i)$ in (x, y, zi)-space. Instead, we may apply the rigid motions of Steps I – IV to the xy-plane graph of our Type RC quadratic function f and determine the points in (x, y, zi)-space where the graph of φ intersects the plane $y = 0$.

Note that if we evaluate our Type RC function f using arbitrary complex number inputs of the form $x + zi$, where $x, z \in \mathbb{R}$ and $x \neq 3$, we will not obtain points associated with the curve φ. The following calculation illustrates why this is so:

$$f(x + zi) = (x + zi - 3)^2 + 4 = [(x-3)^2 + 4 - z^2] + 2z(x-3)i \tag{3.3}$$

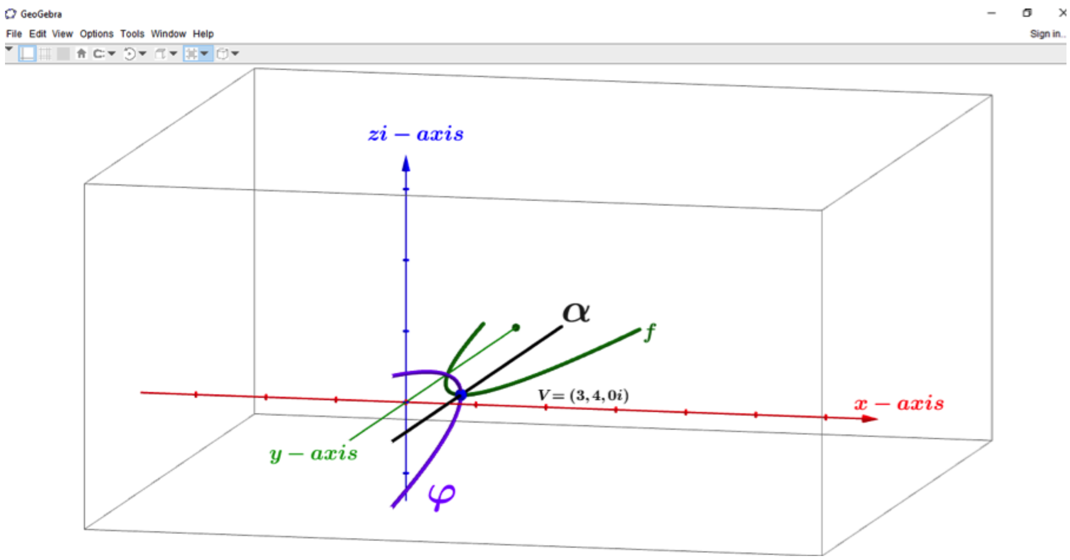

Figure 3.9: An image of the (x, y, zi)-space two-parabola system associated with $f(s) = (s-3)^2+4$.

Since $x \neq 3$, if $z \neq 0$, then $2z(x-3)i \neq 0$, hence $f(x+zi)$ is a pure complex number[2]. Yet all of the y-coordinates of points lying on φ are real-valued and take the form $4 - z^2$. Thus complex numbers of the form $3 + zi$ are the only complex number inputs that are associated with points on the graph of φ, as seen above by the mapping shown in (3.1).

A formal description about using the function rule for f to produce the parabolic curve φ

We now summarize in a formal yet concise manner how the function rule of our Type RC quadratic function f can be used to construct the parabolic curve denoted φ:

- Write the function rule for our Type RC quadratic function as $f(s) = (s-3)^2 + 4$, allowing s to take on all real number values as well as complex number values of the form $3 + zi$, where $z \in \mathbb{R}$.

- When s is a real number x, the output $f(x)$ is also a real number. These values of $f(x)$ produce the expected graph of a parabola in the xy-plane. Furthermore, this graph is unchanged when viewed in (x, y, zi)-space, where points on the graph of f are written as $(x, f(x), 0i)$.

- When s is any complex number of the form $3 + zi$, the output $f(3 + zi) = 4 - z^2$ is also a real number. These particular values of s and $f(s)$ can be used to construct all points lying on the graph of the parabolic curve φ, which lies entirely in the complex plane $x = 3$. Furthermore, all points lying on φ in (x, y, zi)-space take the form $(3, 4 - z^2, zi)$.

- As a result, we may construct points in (x, y, zi)-space that correspond to the zeros of f. These are the points $(3, f(3 \pm 2i), \pm 2i) = (3, 0, \pm 2i)$, which we now view as the geometric interpretation of the complex-valued zeros of the function f.

Thus we may combine the graphs of the xy-plane parabola f and the parabola φ to be one graph or a two-parabola system that is associated with a function rule of a Type RC quadratic function. Each individual parabola shares a common vertex and axis of symmetry. The two-parabola system for our function f is shown in Figure 3.9.

[2]Note that if $z = 0$, then $f(3 + zi) = f(3) = 4$, which corresponds to the vertex point $V = (3, 4, 0i)$.

The General Case for Type RC Quadratic Functions

Taking advantage of our hard work and computation above, addressing the general case is straightforward. We begin with a quadratic function $f(s) = a(s-h)^2 + k$, where $a, h, k \in \mathbb{R}$. If $ak > 0$, then f is a Type RC quadratic function and its zeros are the complex numbers $h \pm \sqrt{\frac{k}{a}}i$.

The rigid motions in Steps I – IV yield the following mapping of points in (x, y, zi)-space:

$$(h+z, k+az^2, 0i)_f \to (h+z, k-az^2, 0i)_{f_1} \to (h, k-az^2, zi)_\varphi \tag{3.4}$$

Taking $z = \pm\sqrt{\frac{k}{a}}$, the mapping that produces the complex-valued zeros of f is given as follows:

$$(h \pm \sqrt{\frac{k}{a}}, 2k, 0i)_f \to (h \pm \sqrt{\frac{k}{a}}, 0, 0i)_{f_1} \to (h, 0, \sqrt{\frac{k}{a}}i)_\varphi \tag{3.5}$$

Equivalently and alternatively, we can construct points on the graphs of f and φ by using the function rule and appropriate input values. If s is a real number $x = h + z$, where z is any real number and h is fixed, then $f(s) = f(h+z) = k + az^2$, and the points $(s, f(s), 0i) = (h+z, k+az^2, 0i)$ correspond to the usual xy-plane parabola.

On the other hand, if s is a complex number of the precise form $h + zi$, where h is the x-coordinate of the vertex and z is a real number, then $f(s) = f(h+zi) = k - az^2$. Hence $f(s) = f(h+zi)$ is real-valued for all such values of s and we write the points as $(h, f(h+zi), zi) = (h, k-az^2, zi)$, which correspond to points on the parabolic curve φ.

Closing comments

We would like to mention that some consideration was given to writing y as a function of x and z in an effort to approach the topic using the concept of function. Throughout this manuscript, the parabola φ was always referred to as a parabolic curve, not a function. Describing the two-parabola system of f and φ as a function is possible, and would first require us to construct what we define as the right-handed three-dimensional coordinate system (x, zi, y)-space. However, we ultimately decided not to go that route and perhaps address these ideas in future works.

As seen throughout this manuscript, GeoGebra is a great tool for illustrating the concepts under discussion. Much of the mathematical content in this article was both inspired by and deduced as a result of the authors analyzing images created by GeoGebra. We hope to inspire other practitioners of mathematics to explore mathematics using GeoGebra, and we encourage mathematics students to do the same.

There does not seem to be a lot of discussion in math-circles regarding the topic under discussion. We speculate this may be a consequence of the manner in which quadratic functions are presented at the secondary school level. Specifically, secondary school mathematics generally does not require students to evaluate Type RC quadratic functions using complex number inputs. Despite any unfamiliarity the reader may have with the content contained in this manuscript, we hope that the mathematics was presented in a clear and thorough manner, perhaps satisfying the curiosity of some mathematics instructors and students.

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2. *Complex Roots: A Graphical Solution*. URL: `https://education.ti.com/~/media/B2EBC742C06347CEBB2F748DF365ACBD`.

USING THE TAIL OF A SEQUENCE TO EXPLORE ITS LIMIT

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Abstract: Graphing sequences is a common approach to explore limits both conceptually and computationally. In the traditional approach, the first terms of the sequence are the primary focus, however, we present a GeoGebra applet that facilitates the determination and understanding of limits by illustrating an inverted perspective that emphasizes the tail of the sequence instead. Additionally, two robust and valuable GeoGebra topics/commands will be highlighted in the applet, in particular *Lists* and *Sequences*. Finally, we conclude with a discussion of several issues surrounding the limits and GeoGebra.

Keywords: limits, GeoGebra: `List, Sequence`

Introduction

The *limit* is a key concept in mathematics in general, and one of three fundamental ideas of calculus [5], the other two being the derivative and the integral, both of which are defined in terms of limits. As such, students are expected to know how to determine a limit, develop an intuitive understanding of the mathematical limit, estimate limits from graphs or tables, and find limits at or involving infinity [1]. So, it stands to reason that much instructional time is devoted to developing this concept. A recent NSF-funded study [4] explored the relative amount of instructional time spent teaching and learning various topics in first-semester calculus and found that, on average, 15% of instructional time involves limits. Although typically in first-semester calculus, only limits of functions are considered, perhaps working with limits of sequences is more natural and easier for students [5].

In this article, we present a GeoGebra [3] applet that utilizes the tail of a sequence instead of the traditional use of the first "few" terms. Prior to this development, we include prerequisite mathematical definitions and GeoGebra syntax and commands. We provide two examples that illustrate convergence and divergence of sequences. As a bonus, the method we use will provide visual insight into a related advanced topic. Lastly, we discuss issues discovered during the preparation and writing of this manuscript.

Mathematical Sequence and Limits

We start with the definition of a real sequence as an ordered list of numbers. A **sequence** of real numbers is a function $f : \mathbb{N} \to \mathbb{R}$ defined by $f_n = f(n)$ mapping the natural numbers to the real numbers. The subscript is referred to as the **index**. According to this definition, because f is a mathematical function, it is assumed that all domain values are used, therefore, all sequences are infinite.

Formally, the limit of a sequence f_n is the number L to which the sequence converges if for every real number ϵ, there exists a natural number N such that for every natural number $n > N$, the terms f_n satisfy $|f_n - L| < \epsilon$. Literature abounds documenting the difficulty in developing a firm understanding of this definition (e.g., [2], [6], [8], [9]).

GeoGebra Lists and the `Element` and `Sequence` Commands

Lists are important objects and are used in many contexts in GeoGebra. *Lists* can contain numbers, points, polygons, circles, and even other lists. For example, a *matrix* is defined in GeoGebra as a list of lists. *Lists* are created using bracket symbols, `{` and `}`. For example, a list, `L`, of even positive integers less than or equal to 20 can be created using the syntax at the input bar as follows:

```
L={2, 4, 6, 8, 10, 12, 14, 16, 18, 20}
```

Elements of a list can be obtained using the `Element` command. In particular, the nth element of a list `L` is obtained with the command: `Element[L,n]`. For example, the number 6 in the above example is obtained using `Element[L,3]`.

A *sequence* in GeoGebra is a (special) list and can be created using the `Sequence` command. The syntax to create a list using the command line is:

```
Sequence[<expression>, <variable>, <start>, <end>]
```

For example, the sequence of even numbers from 2 to 20 (`L` above) can be generated by

```
L=Sequence[2*k, k, 1, 10]
```

The `Element` and `Sequence` commands are essential when working with *lists* in GeoGebra.

The Tail Emphasis

When considering the limit of a sequence, the "**tail**" provides all the necessary information about the convergence or divergence of the sequence as well as other details of interest. The tail is considered to be the terms of the sequence greater than or equal a certain index value, i.e., f_n for all $n \geq m$ for some m. This is sometimes called an m**-tail**. Therefore, instead of graphing the sequence $f_1, f_2, f_3, \ldots$ using the points $(1, f_1), (2, f_2), \ldots$ and attempting to view this graph "at infinity", an alternative method presented by Olson [7] suggests plotting the points

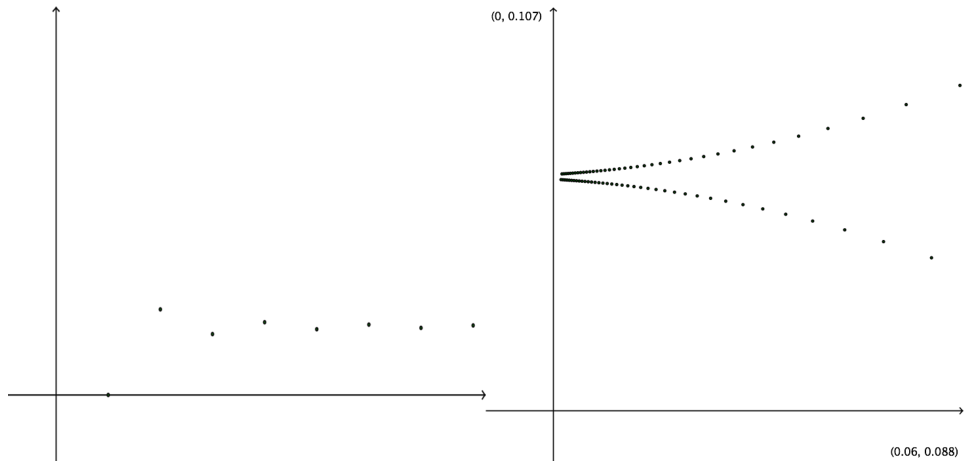
$$\left(\frac{1}{m}, f_m\right), \left(\frac{1}{m+1}, f_{m+1}\right), \left(\frac{1}{m+2}, f_{m+2}\right), \ldots$$

which represents the tail of the sequence. The successive terms are plotted from right to left approaching the $y-$axis and represents the "tail". In effect, the graph displays the ultimate behavior of the sequence. If the sequence converges, it will be visually obvious that the terms are approaching a particular value on the $y-$axis. Additionally, it is easy to add the ϵ tolerance to the limit value. Divergence is equally obvious in that there will not be one value toward which the terms are converging.

Modifications in the GeoGebra applet to increase n or produce (graphically) terms beyond a specified index are also easy to obtain. The general construction protocols for this method are provided below followed by two examples.

General Construction Protocols

- Define a value for `N` $\in \mathbb{N}$.
- Define a value for `m` $\in \mathbb{N}$, for the m tail.



(a) Traditional Approach (b) Alternative Approach

Figure 4.1: Plots of the sequence $f(n) = 1 + (-1)^n/n^2$

- Define a function in `n` to use in the sequence command: `f(n)` (only need to edit this function to produce other sequences)
- Define a sequence `f\_n = Sequence[f(k), k, m, N]`
- Define the tail sequence (of points) using the `Sequence` and `Element` commands: `Sequence[(1/k, Element[f_n,k]),k, m, N]`

Example 4.1 We first illustrate the tail method using the sequence defined by $f(n) = 1 + (-1)^n/n^2$ and produce the points $(1/1, f_1), (1/2, f_2), (1/3, f_3), \ldots$ and we show a side-by-side comparison with the traditional "early terms" depiction $(1, f_1), (2, f_2), (3, f_3), \ldots$. We enter at the command line the following:

```
N=50

m=1

f(n)=1+(-1)^n/n^2

f_n = Sequence[f(k), k, m, N]

Tail = Sequence[(1/k, Element[f_n,k]),k, m, N]
```

The result is shown in Figure 4.1.

Example 4.2 We illustrate divergence using the tail end of the sequence $f(n) = 2/5 + \sin(3n)(n+10)/n$, redefining only $f(n)$ in Example 4.1, `f(n) = 2/5+sin(3n)(n+10)/n`. The results is shown in Figure 4.2. Observe that in this case, the tail makes it possible to view the $\limsup$ and $\liminf$ of the sequence, two important concepts in mathematical analysis.

Discussion

Interactive dynamic software is one way to foster understanding of mathematical limits. In fact, there already exist hundreds of deftly designed GeoGebra applets available on GeoGebra Tube (tube.geogebra.org) dedicated to developing limits, including one-sided limits, limits at infinity, the ϵ-δ definition, the squeeze theorem, limit approximations and calculations, etc. This immediately begs the question concerning the effectiveness of the current applets used to develop

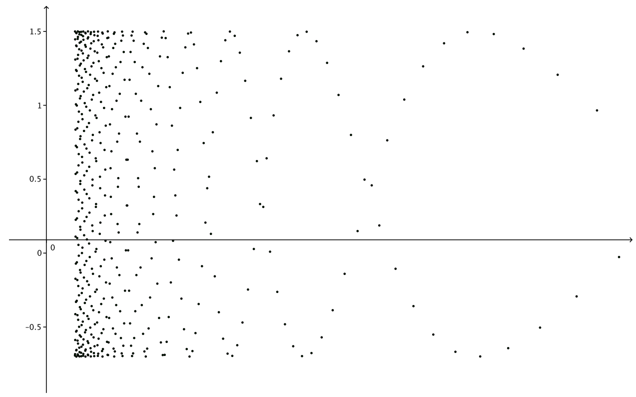


Figure 4.2: Tail of the divergent sequence $f_n = 2/5 + \sin(3n)(n + 10)/n$. In this case, however, the tail makes it possible to view the $\limsup$ and $\liminf$, two important concepts in mathematical analysis.

the limit concept. With all of these applets, are students still having difficulty with understanding limits? Furthermore, there seems to be a lack of literature reporting the use and effectiveness of these apps.

In preparation for this article, the task of wading through and exploring hundreds of available applets required a considerable amount of time and was fairly overwhelming. Clearly, this would be a daunting task for any teacher searching through materials while preparing a lesson on this topic. One suggestion in this regard would be to implement some sort of user rating to aid in identifying highly effective (highly used) applets. Additionally, more detailed descriptions and better/longer list of keywords could be helpful during the selection process.

Further, there needs to be more material devoted to truly creating an intuitive understanding of the limit, not just dynamically presenting examples found in standard textbooks. One suggestion is an applet that illustrate the limit as a process of successive approximations for which materials are scarce. Examples include: approximating geometrical figures (e.g., circle) with an increasing number of sides of a polygon, approximating arc length of a curve, approximating roots of an equation, the area under a curve, or the slope of a tangent line. Although several applets exist that demonstrate the definition of derivative as the limiting process of slopes of secant lines or the integral as the limit of Riemann sums as the number of rectangles increases, these applets are not tagged with “limit" or “concept of limit" as keyword(s).

Conclusion

We developed a GeoGebra applet based on [7] for presenting the concept of the limit. Although there are many GeoGebra materials covering limits, currently none exist that emphasize the tail of a sequence. This method is robust and can also provide an introduction into the related (but more advanced) notions of limit superior and limit inferior.

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“MESSING AROUND” IN GEOGEBRA: ONLINE INQUIRY THROUGH APPS AND GAMES

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Abstract: This article explores a problem solving strategy called “messing around” that is particularly apt for online course work in GeoGebra. While traditional math problems may require students to search deliberately for a specific solution, “messing around” employs a more fortuitous approach. We dissect this approach and examine multiple GeoGebra apps that support “messing around” by providing students with a sandbox for mathematical experimentation.

Keywords: problem solving, online inquiry, hanging out, messing around, geeking out, GeoGebra

Introduction

There is already a wealth of research exploring students’ problem-solving strategies in mathematics. Pólya [14] proposed students use a series of heuristics such as “work backwards,” “consider special cases,” and “solve a simpler problem.” These heuristics are employed through four steps: (1) understand the problem, (2) make a plan by selecting one of the heuristics, (3) carry out the plan, and (4) review your work. Most subsequent problem-solving research has simply expanded on Pólya’s heuristics, scaffolding the process in similar ways [4, 10, 13]. Schoenfeld [17] found that amateur problem-solvers tended to select one technique and stubbornly stick with it even when their approach was not working. On the other hand, advanced problem-solvers spent more time testing different strategies until they found the right one. In all of this work, the prevailing idea is to ask students to think metacognitively about the different strategies available to them so they can work more deliberately towards a clear end-goal.

Most of this research was conducted before the advent of so called new media technologies: interactive, digital media such as blogs, wikis, video games, social media, and dynamic graphing software. As such, this research does not consider how new media technologies might influence the problem-solving process. In a recent study [16], Santos-Trigo and Moreno-Armella examined how teachers and learners use new media technologies like GeoGebra to communicate about mathematics during the problem-solving process. They observed that new media can provide online resources for students to extend discussions beyond the classroom, and these technologies also offer interactive models of mathematical objects that students can use to test their understandings as they work. However, research in this direction is still very limited, and most new media research focuses on users as “digital natives” or “millennials,” attributing shifting forms of engagement to the demographics of users rather than to features of the medium itself [11, 15].

Do new media technologies like GeoGebra enable non-traditional problem-solving strategies? We demonstrate one particular problem solving approach, "messing around," which GeoGebra facilitates quite well. We'll describe several GeoGebra apps that encourage this approach, and explain how they were used in a hybrid, inquiry-based, community college Precalculus course. This class was "inquiry-based" in the sense that students used class time to work through scaffolded activities and make sense of mathematics on their own; there was very little teacher-led instruction in the form of lectures. The class was "hybrid" in the sense that students met face-to-face half as often as students in other sections of Precalculus, and this reduced class time was supplemented with online work. Students were asked to maintain a Wordpress blog about math. They selected two prompts from a menu of assignments each week and responded to them on their blogs. Many of these prompts used GeoGebra apps. We'll discuss these apps, and look at a few examples of student work.

The Challenges of Learning Online

Online learning presents new challenges and limitations not encountered in face-to-face classrooms. Consider some of the challenges a student might face while working on the following problem.

For what input x does the function $f(x) = -x^2 + 5$ produce the output 4?

1. The student may not understand key terminology like "function" or "input." In a face-to-face classroom, this student might ask their instructor or a peer for clarification, but these resources are not immediately available to the online student. At best, they might look these terms up in a book or online, or they might send their instructor an email and wait for a reply. Even if the student knows where to look, searching for these terms requires an additional investment of time and energy that's unnecessary for the face-to-face learner.

2. A student may not know how to evaluate the function. Again, the student would have to look this procedure up, or email an instructor or a peer and wait for a reply.

3. Even if the student understands the terminology and the process of evaluating a function, they may not know where to start. How does one find the correct input? A student could guess-and-check, plugging different x's into the function and hoping for the right y. That process is tedious, and it's difficult to measure one's progress. Suppose the student tries 34 different inputs and none of them produce the right output? Have they made any more progress than the student who tried only two inputs? At what point should this student give up and try something else? If the student does not already know to set up an equation and solve for x then nothing in this problem will suggest that approach.

4. In the best case scenario, the student understands the terminology, knows how to evaluate the function, and has the wise idea to set up an equation, but they still may not know how to solve the equation.

This list is not exhaustive, but the point is that none of these challenges can be overcome without looking the answer up or asking another person for help. As a result, many online classes follow an acquisition model of learning. Students watch a video and take a multiple choice quiz to ensure they've acquired the "correct" understanding of the material. They may learn to mimic a rote procedure, but how much are they engaged in authentic mathematical problem solving? This type of online instruction is akin to face-to-face lectures; it does not enable the type of active learning that inquiry-based classrooms aspire to engender.

Indeed, Marra and Jonassen [9] examined popular learning management systems (LMSs) like Blackboard and WebCT and concluded that because these systems "do not support the use of alternative forms of knowledge representation by learners, authentic forms of assessment, and the use of distributed tools to scaffold different forms of reasoning, the range of student learning outcomes is restricted to reproductive learning." Blackboard acquired WebCT in late 2005 [1], and remains the most popular LMS today with roughly 32% of the market share [8]. In a 2014 blog post [7], George Kroner, a former development manager at Blackboard, observed that "while LMSs have evolved over time, they generally have the same capabilities that they had back in the late 1990s." The world has changed a lot since 2001, but the limitations identified in Marra and Jonassen's study have not gone away.

Jonassen et al. [6] encourage instructors to look beyond these limitations and focus on technology not as a tool to teach with, but rather as "an excellent tool to learn with." Thinking about technology as a "tool to learn with" requires an understanding of how people learn online. Mizuko Ito and fourteen other researchers [5] conducted an extensive, three-year ethnographic study of the way young people communicate and learn online, categorizing this engagement into three basic "genres of participation": "hanging out," "messing around," and "geeking out."

- **"Hanging out"** refers to friendship-driven forms of participation like sharing photos from a party on Facebook or exchanging texts with a friend.
- **"Messing around"** describes casual, interest driven experimentation with new media technologies. This category includes what Ito et al refer to as "fortuitous searching": "moving from link to link, looking around for what many teenagers describe as 'random' information" [5, p. 54]. This type of search is open-ended rather than goal-driven.
- **"Geeking out"** refers to more intense interest-driven activity with more sharply focused goals. This might include activities like maintaining a blog on a niche interest or editing a video.

It's important to note that these are "genres of participation," not genres of people. Rather than casting people into rigid categories like "digital natives" or "millennials," Ito et al acknowledge that an individual can be a "luddite" in one setting and a "geek" in another.

These researchers found that certain activities can bridge the divide from one form of participation to another. They note that "'hanging out' with friends while gaming can transition to more interest-driven genres of what we call recreational gaming" [5, p. 17]. Perhaps, "messing around" with recreational mathematics can then transition to "geeking out" about algebra?

With this in mind, we created several GeoGebra apps that serve as "distributed tools to scaffold different forms of reasoning." Each of these apps enables students to "mess around" with new mathematical ideas before transitioning to more traditional work. In the next section, we present one such app, and then explain how it enables problem-solving strategies not available in a traditional algebra problem.

Function Machines in GeoGebra

The *Function Machines* app[1] presents users with four machines and asks them to make the outputs of all four machines the same. Students "mess around," testing different inputs in each machine to see what type of output it will produce. A label on each machine also provides a formula for the output. When the student provides an input that is not in the domain of the function, the output will display a question mark. See figure 5.1 below.

This task is open-ended; there are several different correct answers, but students can begin the task with a "fortuitous search" and discover the properties of functions along the way. The app will also confirm a correct solution, presenting the student with a congratulatory message upon their success.

In many ways, this is a richer, more complex exercise than the algebra problem described above, and yet most of the challenges attributed to that problem do not occur here.

- The student does not need to know any special mathematics terminology to start working on this problem. The app presents students with a readily identifiable metaphor about machines in the real-world, and any other properties of these machines can be discovered by experimenting with different inputs.
- The student does not need to know how to evaluate a function. In fact, they can experiment with different inputs to form their own hypothesis about how the formulas for each machine relate to the output.
- The GeoGebra app automates a lot of the tedious calculations that made guessing-and-checking a poor strategy in the algebra problem above. The app also provides four functions rather than one. If a student cannot produce

[1] `http://tube.geogebra.org/m/2870887`

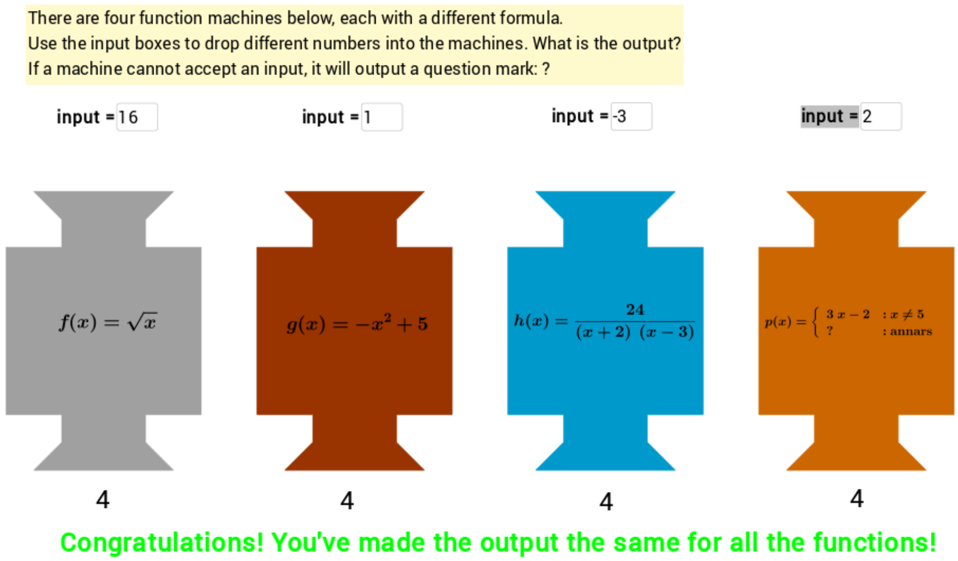

Figure 5.1: A student has successfully made all four outputs the same in the *Function Machines* app.

the desired output on one machine, they can switch to another and continue working. Thus, it's easier to get a sense of progress if the student can produce the desired output on two or three of the four machines.

This GeoGebra app is not a panacea. On its own, it will not teach students how to set up and solve the requisite equations, but it does allow students to engage in authentic online inquiry before learning more traditional techniques. By "messing around" in the app, students can begin to make meaning on their own, and this meaning will later support their understanding of the algebra. The *Function Machines* app enables "messing around" whereas the algebra problem requires some heavy "geeking out." Traditional algebra problems are still important, but "messing around" may be a necessary precursor to "geeking out."

Anatomy of "Messing Around"

By our own observations, the following five criteria seem to be necessary (if not sufficient) components of "messing around."

1. "Messing around" requires **very little domain-specific knowledge**. Students did not need any special terminology or algebraic skills to start working with the GeoGebra app.

2. "Messing around" allows for **"fortuitous searches."**[2] A student can play around with different inputs until they stumble across an answer that works. This often produces the perception for students that they are "just about to get it."

3. "Messing around" entails a **built-in "control of error"** in the sense that Montessori used this term [2, p. 80–81]. In other words, some element of the problem or app will verify the student's answers. The *Function Machines* app calculates the output of each function so the student can confirm whether or not they understand how to evaluate the functions. Likewise, when a student makes all four outputs the same, the app presents a congratulatory message, validating that they did it correctly.

[2] In the field of artificial intelligence, problem-solving is often conceived of in terms of search-algorithms [12]. In some sense, any problem-solving technique is a search (fortuitous or otherwise) for a solution. As mentioned in the introduction, Schoenfeld [17] observed that amateur problem-solvers tend to get stuck on one strategy even when that strategy is not working. Advanced problem-solvers take a more deliberate approach, switching between strategies until they find the right one. In some sense, the distinction between "messing around" and "geeking out" is as simple as distinguishing between fortuitous and deliberate searches.

4. "Messing around" is **portable**. The mobile nature of GeoGebra means that students can work on this app anywhere, thereby maintaining the casual nature of "messing around."
5. "Messing around" **does not require one's full attention**. A student can play around with the GeoGebra app while hanging out with a friend. It's a casual and therefore engaging activity that invites the transition to deeper forms of engagement when ready.

As an example of these principles at play, consider the following excerpt from a student's blog.

> I use the function app and challenged myself to see if I could make the outputs of all 4 functions the same. After 10 minutes of "Guess and Check", I figured it out. The first function needs an input of 16 to equal 4. The second function already equals 4 because the default of the input was 1. The third one was the trickiest one for me but eventually realized that the input needed to be 4. Last but not least, the fourth one needed an input of 2 to equal 4.

This student identifies their strategy as "Guess and Check," a form a "fortuitous search." However, they appear to be doing more than just randomly plugging in numbers. They note that "the third one was the trickiest one for me but eventually realized that the input needed to be 4." The student found a solution that worked for three of the four functions. They felt they were were "just about to get it" and thus felt confident enough to persist until they found an answer.

Sail Away

With these distinctions in mind, let's look at another example. The *Sail Away* app[3] prepares students to understand trigonometric functions and their connection to the unit circle. "Messing around" with this app will help students later "geek out" and solve problems such as the following.

`How are the graphs of` $\sin(x)$ `and` $\sin(4x)$ `related?`
`What is the maximum value of the function` $\sin(x)$`? For what` x `does it achieve this maximum?`

The app (in figure 5.2 below) shows a sailboat floating next to a dock. Beneath the dock is a circular dial with the instructions "rotate counter-clockwise." Rotating the dial raises and lowers the tide. The app's instructions suggest rotating the dial counter-clockwise, but the app does not forbid students from rotating the dial in the opposite (negative) direction. Students can also click on the "Animate Sail Boat On/Off" button, and the boat will begin to sail away, tracing its path as it moves. A "seconds" counter beneath the dock keeps track of time throughout the boat's animation. A wave is drawn along the water as a dashed line. Students are asked to trace the path of this wave using the sailboat and the dial. This task requires precise eye-hand-coordination and some trial-and-error.

The prompt also asks students how many times the dial goes around the circle in 8 seconds? This app is embedded in a GeoGebra book with subsequent examples, each showing waves of varying period. By focusing on the speed of the dial, students begin to realize that turning the dial faster produces a smaller period and turning it slower produces a larger period. As demontrated in figure 5.3 below, students were also able to connect the peaks and troughs of the wave with the top and bottom of the circle.

Again, consider how this app demonstrates all five criteria for "messing around."

1. **Little Domain-Specific Knowledge:** Students do not need to know anything about sine functions to begin working with the app. The context of a boat floating next to a dock is familiar to everyone and thus students can draw on their real-world experiences to reason about mathematics.
2. **Fortuitous Searches:** Through trial-and-error, students can experiment with navigating the boat. They can see how rotating the dial affects the graph and thus make sense of the scenario by playing with the app.
3. **Built-In Control of Error:** The app shows the wave as a dashed line and traces the boat's path as it moves. Students can visually compare the boat's trace with the sine function. They do not need to consult an instructor or a book to verify their answers.

[3] `http://tube.geogebra.org/m/JENexAZC`

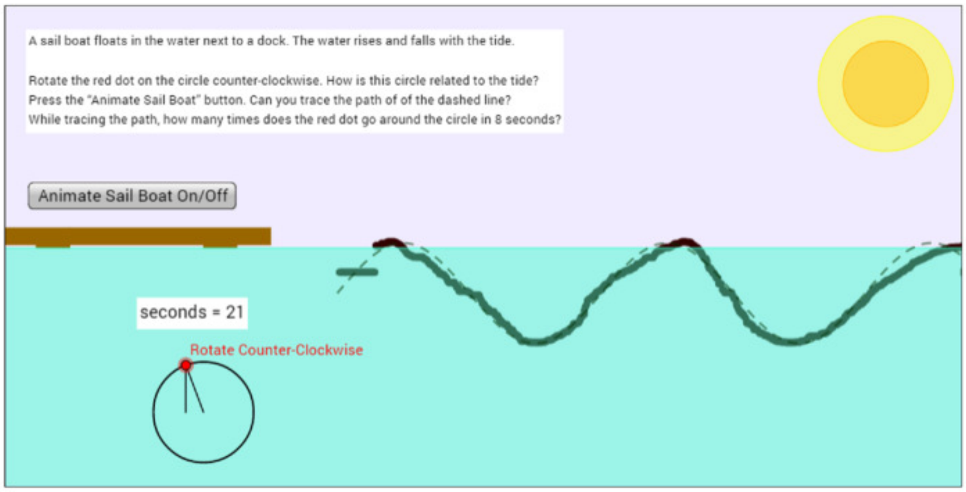

Figure 5.2: A student traces the wave by rotating the dial counter-clockwise.

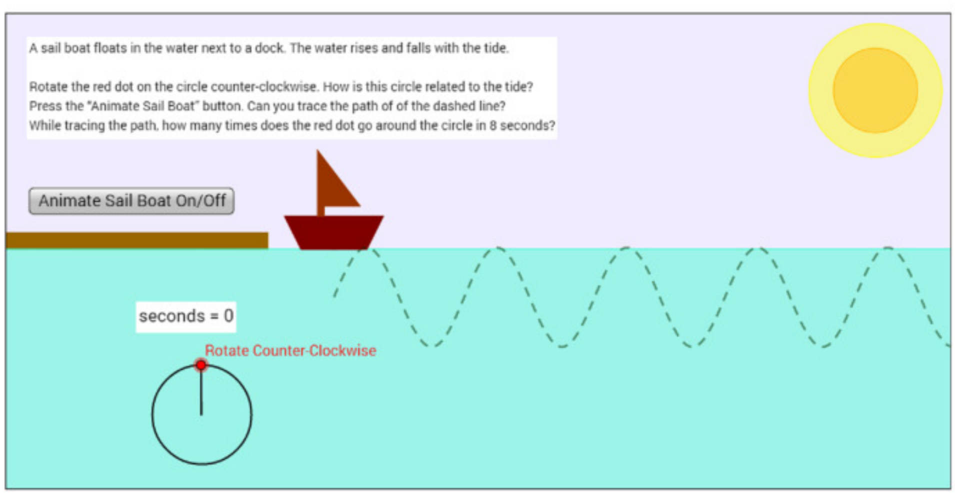

Figure 5.3: A student demonstrates their understanding of the curve's maxima and minima by observing that "at high tide the dot is located at the top of the clock (12)."

4. **Portable:** The app is portable by virtue of GeoGebra being portable.

5. **Does Not Require Full Attention:** Students can play with the dial and intermittently attempt to trace the wave while hanging out with a friend or working on other tasks.

As an example of how persistent students can be when "messing around," consider the following student blog entry titled "Annoying little Boat."

> This assignment is harder than it looks. I restarted many times because the tracing wasn't cute. In order to make the dot move you move it clock wise, it moves down until it fits 6 then it moves up until it hits 12 and then it goes down again. The highest tide is at 12 and the lowest is 6. You need to move the dot slow for you to be able to trace the ones with big gaps like the ones in sail boat 1 and 2. For sail boat 3 and 4 you need to move the dot faster. If you move it slow then you will create big waves and if you move it fast it will create tight waves.

This student is clearly irritated with the "annoying little boat," and yet they continue to work on the problem precisely because it enables them to "mess around" until they find a suitable solution. Likewise, the student is aware of the problem's built-in control of error as they "restarted many times because the tracing wasn't cute."

Note that this student overlooked the app's instructions and chose to move the dial clockwise. This also demonstrates some of the limitations of "messing around." The student is able to develop a surprising level of understanding about the trigonometry of the unit circle, but without explicit instruction, they will not be aware of the convention of tracing positive angles in the counter-clockwise direction. On it's own, this app will not teach the student formal trigonometry, but it does allow the student to start making sense of the mathematics on their own without a deliberate strategy in mind and without immediate access to their instructor.

Additional examples of apps that support "messing around" include the *Composition of Functions* app[4], the *Exponents Game*[5], and the *Radians Game*[6]. We suggest the reader "mess around" with each of these apps, and consider for themselves the extent to which they do or do not align with each of the five criteria identified above.

Why GeoGebra?

We've provided several apps designed in GeoGebra, each of which enables students to "mess around" with new mathematical ideas before engaging in more traditional classroom work. Each of these apps could have been developed without GeoGebra. They could have been programmed by hand in Javascript or constructed using a platform like GDevelop[7]. Programming apps by hand is time-intensive and requires specialized knowledge that most teachers do not have. GDevelop requires a smaller time investment and virtually no coding, but it does not have the same built-in mathematical functionality as GeoGebra.

The advantage of GeoGebra is that it provides a quick and easy way to build one-off apps for specific mathematical problem-solving situations. Educators can eschew monolithic tools like Blackboard in favor of their own DIY creations much along the lines of what Jim Groom has termed "edupunk" teaching [3]. With a critical mass of creators, this could be a powerful pedagogy, enabling new problem-solving strategies for a new media landscape. As we've demonstrated, new media provide opportunities for students to engage in mathematical thinking without developing a deliberate strategy or a clear end-goal. By "messing around," students can engage in authentic mathematical problem-solving without appealing to their instructor every time they veer off the beaten path. This more casual form of engagement can form a gateway to more serious "geeking out" such as traditional homework problems and face-to-face classroom instruction.

[4] `http://tube.geogebra.org/m/xKCGjZxj`
[5] `http://tube.geogebra.org/m/3143617`
[6] `https://www.geogebra.org/m/VWcWKR9E`
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SURVIVING ON MARS WITH GEOGEBRA

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Abstract: In this paper, the authors describe an interdisciplinary lesson focused on determining how long an astronaut can survive on Mars. The lesson utilizes resources provided by NASA within an inquiry-based context and is aligned to the Common Core modeling standard. The authors detail the use of a GeoGebra applet that encourages students to explore their own questions.[1]

Keywords: problem solving, inquiry-based learning, modeling, GeoGebra

Introduction

With the increased popularity of technology in school classrooms, there has been a corresponding call for inquiry-based learning (IBL). Edelson (1999) and others define IBL as an instructional approach in which students are presented with problems that are challenging and unfamiliar. Within an IBL classroom, students are encouraged to ask questions, discuss ideas and possible solution strategies with peers, and apply newfound knowledge while solidifying their answers. IBL aligns well with Standards for Mathematical Practices advocated within Common Core State Standards[2].

When attempting to create a problem that is centered around IBL, Common Core State Standards, and Standards for Mathematical Practices, teachers must consider how much information, time and resources to provide their students. These concerns will be addressed through the exploration of the following problem:

How long can an astronaut survive on Mars with a stockpile of 8,000 potatoes?

The task is inspired by the popular film, *The Martian*. Lead actor, Matt Damon, is stranded on Mars until his team is able to rescue him. Damon's character keeps a video diary documenting the passing days. Students will be shown a short clip of *The Martian* depicting the first video diary entry once the astronaut is aware of his situation. After this clip has been shown, students will be asked how long they think the astronaut can survive on Mars.

The original problem is ill-defined by design. To successfully answer the problem, students must formulate and explore many secondary questions such as the following:

[1]Originally developed as part of a NASA Space Consortium Grant.

[2]HSF.LE.A.1.A: Prove that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals; HSF.LE.B.5: Interpret the parameters in a linear or exponential function in terms of a context.

1. How many calories does the average male/female need in a day to survive?
2. How many calories does a potato provide?
3. What is the distance between Earth and Mars?
4. How long does it take to get to Mars?
5. Are there factors in daily life that would reduce calorie intake (for example, exercising)?

Note that students are not required to answer all of these questions. As part of the IBL process, students are encouraged to work in groups to formulate their own researchable questions and determine solutions to the original problem. Such an approach supports a view of inquiry promoted by the National Academy of Sciences—one in which students (a) question, (b) investigate, (c) use evidence to describe, explain, and predict, (d) connect evidence to knowledge, and (e) share findings (Shavelson and Towne, 2002). These five steps are addressed in the lesson to ensure that students are working on critical thinking skills and developing a deeper understanding.

Students work in small groups to explore questions and visualize the problem scenario with the applet. Once students have generated estimates of the maximum survival time, the teacher plays a clip from the popular movie, *The Martian*, that reveals an answer to the initial problem scenario. This viewing leads to yet more student questions including queries regarding accuracy and precision, model parameters, and the plausibility of the movie's findings. Such questions reveal that quantitative findings, particularly those generated with mathematical models, are subject to argument, change, and interpretation.

Student Exploration

Surviving on Mars is envisioned as part of a three-day interdisciplinary lesson developed to expose students to NASA materials and linear relationships in an inquiry-based manner using a GeoGebra sketch. The three-day lesson is structured in the following manner.

Day 1 A clip from *The Martian* will serve as an engaging hook, resulting in curious and intrigued students. Then students will move between four different "space" stations in small groups. The stations are set up to foster student engagement and deeper student understanding of the problem by targeting four different styles.

1. Visual learning: Students at this station will watch a short clip from NASA discussing the importance of space food to astronauts. After viewing the video, all members of the group will work together to complete a comprehensive worksheet.
2. Auditory learning: Students at this station will take verbal directions to complete a collaborative task. Students will calculate the distance between Earth and Mars on a specific day and time given by the teacher, and then use the solution to determine how long it takes a space station to reach Mars.
3. Reading/Writing learning: Students will be presented with a NASA article explaining the concept of growing plants and vegetables. After reading the article, the group will work together to complete a reflective assignment showcasing the new information. Students will write a paragraph summary and come up with a short oral presentation or a multimedia project.
4. Kinesthetic learning. Group members will weigh an actual potato and estimate its caloric content.

Day 2 The second day of the lesson will consist of applied mathematical concepts and the collaborative journey toward a solution to *Surviving on Mars*. Students will rejoin their group members to discuss the data they collected and to discover relationships between calories needed and the time a human can survive. A Geogebra applet (see Appendix) will be provided to help students model the newly uncovered relationship. Students will be given this applet after completing several fundamental calculations to determine that the relationship between potatoes eaten and number of calories is, in fact, linear. The applet will allow students to formulate observations and questions regarding *Surviving on Mars*.

Target observations for students

1. When the variable ‘workout’ is factored into the problem the relationship remains linear.
2. There is a range of calories that the astronaut must stay within to survive. This range is the same range that students discovered during station work with NASA materials on day 1 of the lesson.
3. The point P represents the linear relationship between the number of calories in a potato and the number of potatoes eaten.
4. When workout time is considered, the point P moves with respect to workout time as well.

Target questions for students

1. Are there other factors that may affect the astronaut remaining in the optimal range on the graph?
2. Are there any additional factors that could be added to the graph to create a nonlinear relationship?
3. Is there any way to incorporate the concept of days survived into the applet?

Students will continue working with their groups to determine concrete answers to more direct questions. Specifically, students are asked to examine the following questions.

4. As the number of potatoes increases, what happens to the number of calories?
5. As the amount of workout time changes, what happens to the point P?
6. Upon examination of the point P, what relationships are causing the point to move?
7. What is the minimum number of calories the astronaut can consume in a day to survive?

Classroom time will be dedicated to applet exploration so students can better understand mathematical concepts behind the linear relationship. The main purpose of the applet is to provide students with a visual way to calculate the number of potatoes that can be eaten in a given day. Once students determine that number, they can use the result along with information provided in *Surviving on Mars* to find a solution. Students are then challenged to modify the applet in a way that produces a solution to *Surviving on Mars*. This task introduces various new elements of consideration. First, students must discover how to effectively manipulate the existing applet to ensure that it depicts the desired result. Second, the students will have to present a new condition to be considered in the existing relationship (number of calories burned while sleeping or number of calories burned when not exercising). By adding additional obstacles to the homework, students are required to utilize the problem solving techniques discovered during class to work through the homework alone.

Day 3 At the beginning of class, a final clip from *The Martian* will show students how long filmmakers expected the astronaut to survive. This information will allow students to compare their answers to the film while exploring the plausibility of various results. In addition, students will be able to contemplate what could have skewed their results or the results in the movie. Further analysis of results and supporting mathematics solidifies the applicability of linear relationships in everyday life.

Purpose of Technology

The implementation of technology within this lesson offers multiple benefits that allow more comprehensive understanding to students. The sketch allows students to work at their own pace as they uncover linear relationships in an interactive manner. Sliders in the sketch afford students the opportunity to interact with variables in a way that makes them come alive. As students drag sliders within the sketch, they discover that workout time effects the caloric intake but does not affect the number of potatoes eaten per day. Moreover, the applet provides students with visual representations of the problem scenario while automating many of the arithmetic calculations that otherwise distract students from conceptual understanding—specifically the linear relationship between potatoes consumed and calorie intake. By automating the repeated calculation of the y-value for point P, the students will be able to focus on the path that point P takes—a line with positive slope. The applet effectively eliminates discrepancies in results caused by student error. Given that there is a range of acceptable potato consumption, not all students will arrive at the same answer. However, all students should reach the understanding that an increase in potato consumption causes a direct increase in calories. In contrast, an increase in time spent exercising causes a decrease in calories and no change in potato consumption. This will result in students having a variety of answers, all of which are plausible. An additional benefit is the speed with which the students will be able to manipulate variables. Sliders allow students to quickly see how changes in variables affect the linear relationship in various instances. Sliders are not the only dynamic features of GeoGebra aiding students. Tracing the point on the graph to clearly show students the linear relationship between potatoes and calories per potato is another dynamic feature. The ability to factor in, or not factor in, calories burnt while working out by the use of a checkbox also adds to the benefits of the applet. Overall, students' use of technology will allow a fuller comprehension of linear relationships.

Conclusion

To implement this lesson successfully, students need to be familiar with the use of technology and inquiry in the mathematics classroom. When students are accustomed to both of these areas, the focus of the lesson can remain on the mathematics. As a result, throughout this Mars-themed lesson, students are encouraged to gain a deeper understanding of mathematical concepts, specifically linear relationships. Additionally, students reflect on their results by providing evidence for their conclusions. By the end of the lesson, students utilize their knowledge of modelling, prove linear functions grow by equal differences over equal intervals, and interpret the parameters in a linear or exponential function in terms of a context. This lesson challenges students, but, through the use of scaffolding and engaging activities, the goals of the lesson can be achieved. The addition of the GeoGebra applet allows students to visualize the relationship among variables. By asking students to produce a modified applet as a homework assignment, they are also given the opportunity to improve their critical thinking and problem solving skills.

REFERENCES

1. D. C. Edelson, D. N. Gordin, and R. D. Pea, *Addressing the challenges of inquiry-based learning through technology and curriculum design*, Journal of the learning sciences, 8 (1999), no. 3-4, pp. 391-450.
2. R. J. Shavelson, and L. Towne, *Scientific research in education*, National Academy Press, 2002.

Appendix—Steps for Creating the *Surviving on Mars* Applet

We begin by specifying parameters of the graphics window. The axes of the graph need to be modified to take into account caloric values of the problem scenario. Because there can not be a negative number of potatoes consumed or minutes working out, the axes need to be set to only show positive values. Furthermore, we decided to place the maximum number of potatoes consumed to be 30 and the maximum calorie amount to be 3500. These settings are illustrated in Figure 6.2. Figure 6.3 shows the completed graph with axes scaled. Additionally, we added text boxes to label the x-axis as potatoes and y-axis as kcals.

In keeping with realistic expectations of survival rates, we determined what level of kcals consumed per day for an average male would correspond to obese and malnourished levels. We determined that 2500 kcal and 1500 kcal, respectively, would fit this requirement. We then inserted a line a, corresponding to $y = 2500$, and line b, corresponding to $y = 1500$, by entering these equations into the input box. From there, we decided to alter the properties of the line for aesthetic purposes. In Figures 6.4 and 6.5, we illustrate how to modify object properties. Figure 6.6 shows both lines red and dotted

Next, we began to add the manipulated elements. First, we created a checkbox with the caption 'Workout' and corresponding Boolean value C. A slider was created with the same caption and values ranging from 0-15, incremented by one unit. A similar process was done to create the checkbox and slider associated with 'Potatoes' and Boolean value D. Figure 6.7 shows how to create the 'Workout' check box.

The final step included adding the point P, representing the relationship between potatoes consumed and calories available. The equation for this point shows that the x-value will be the number of potatoes consumed. The y-value

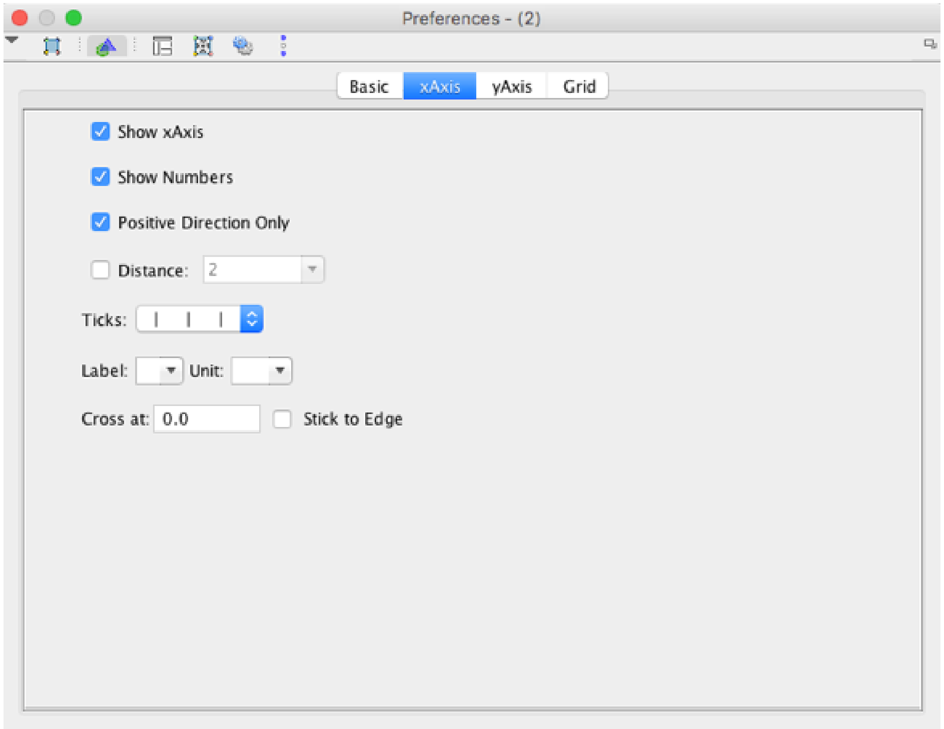

Figure 6.1: Window settings in GeoGebra.

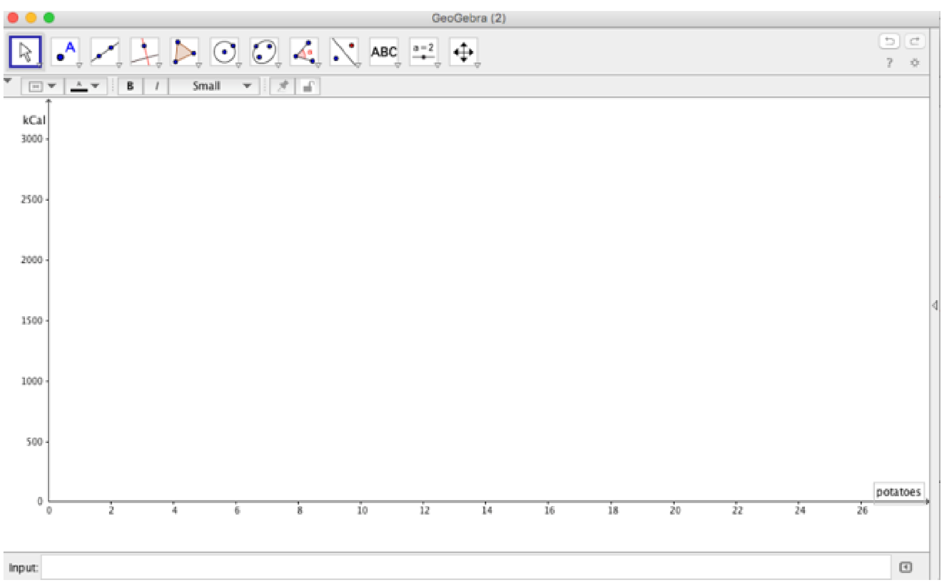

Figure 6.2: New window.

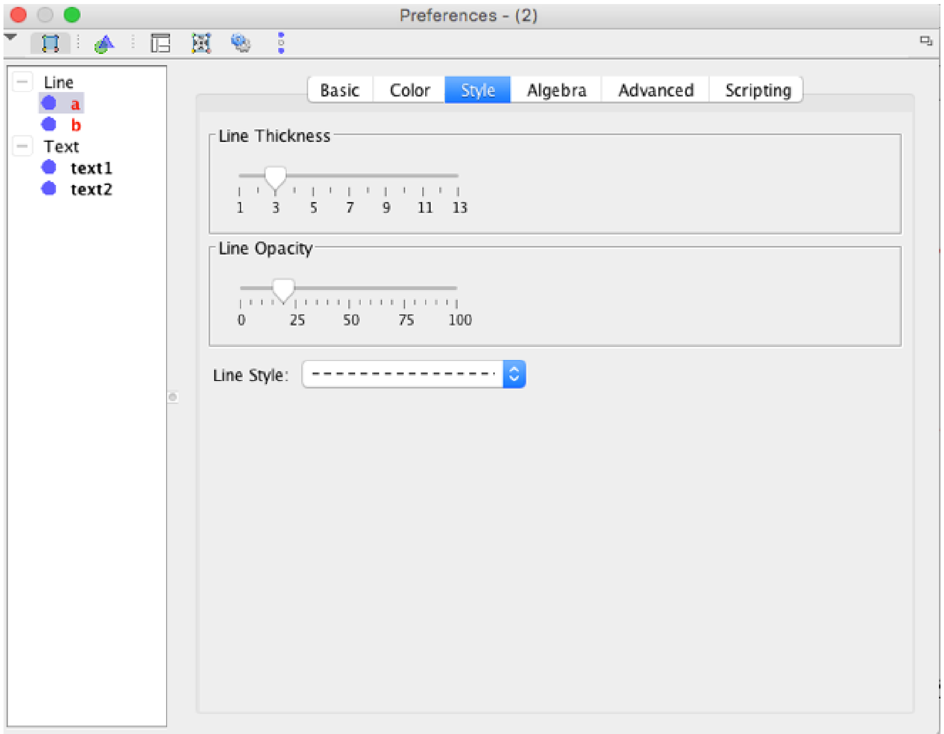

Figure 6.3: Line Style.

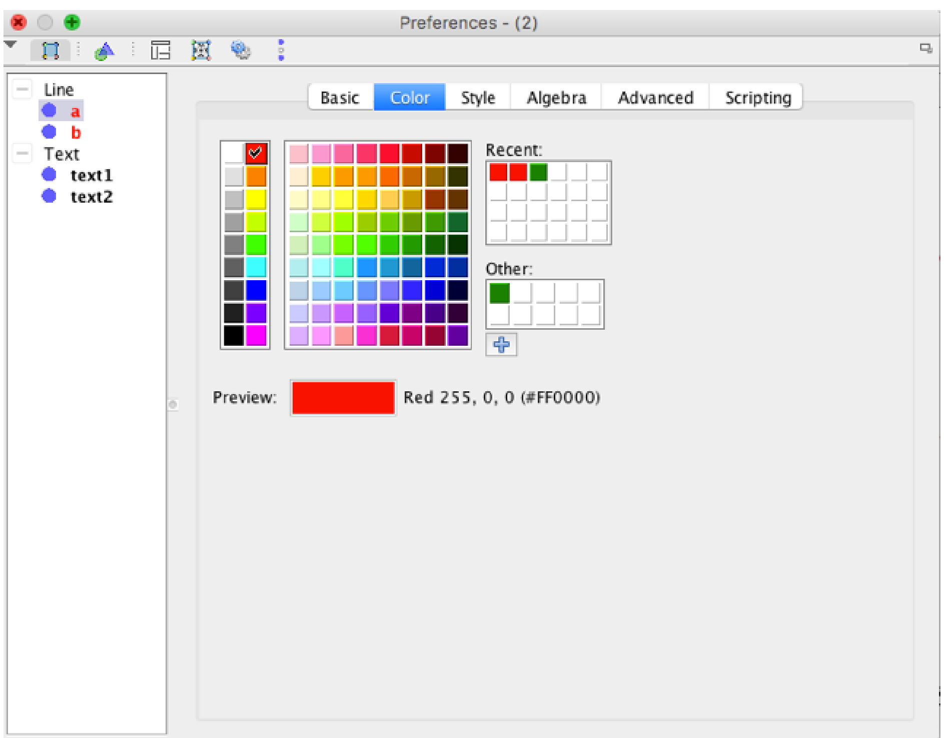

Figure 6.4: Line color.

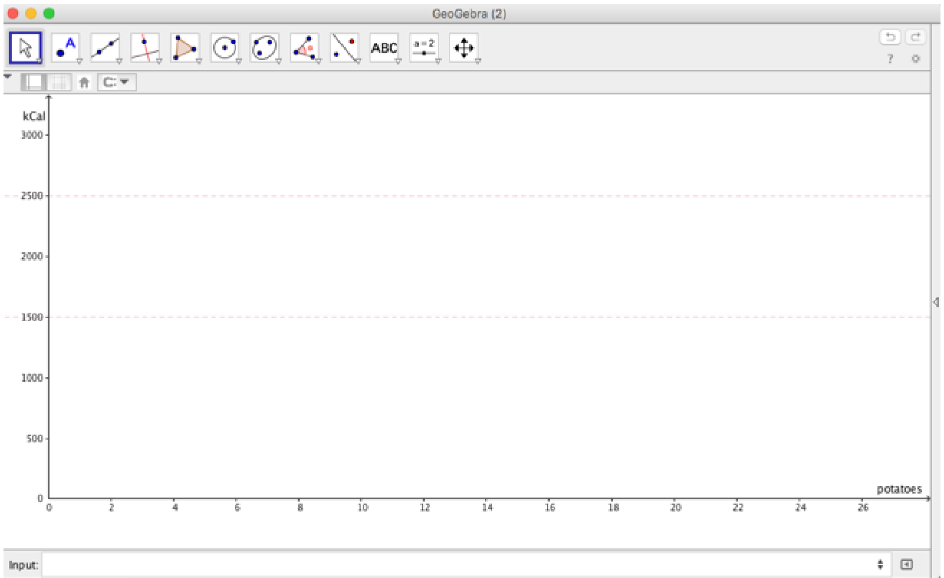

Figure 6.5: Completed lines.

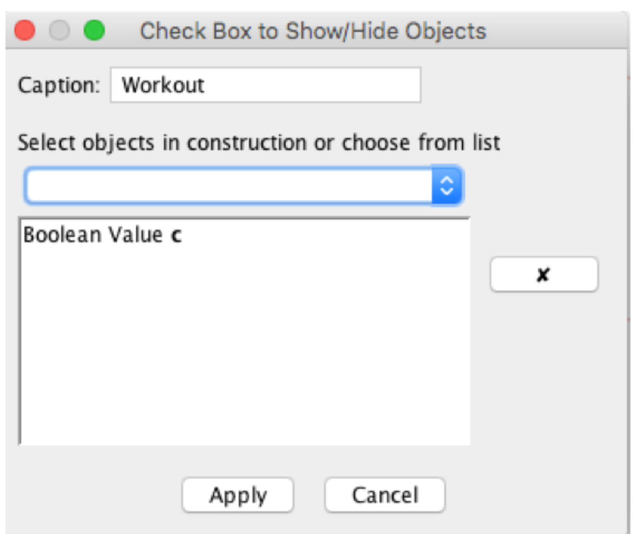

Figure 6.6: Workout checkbox.

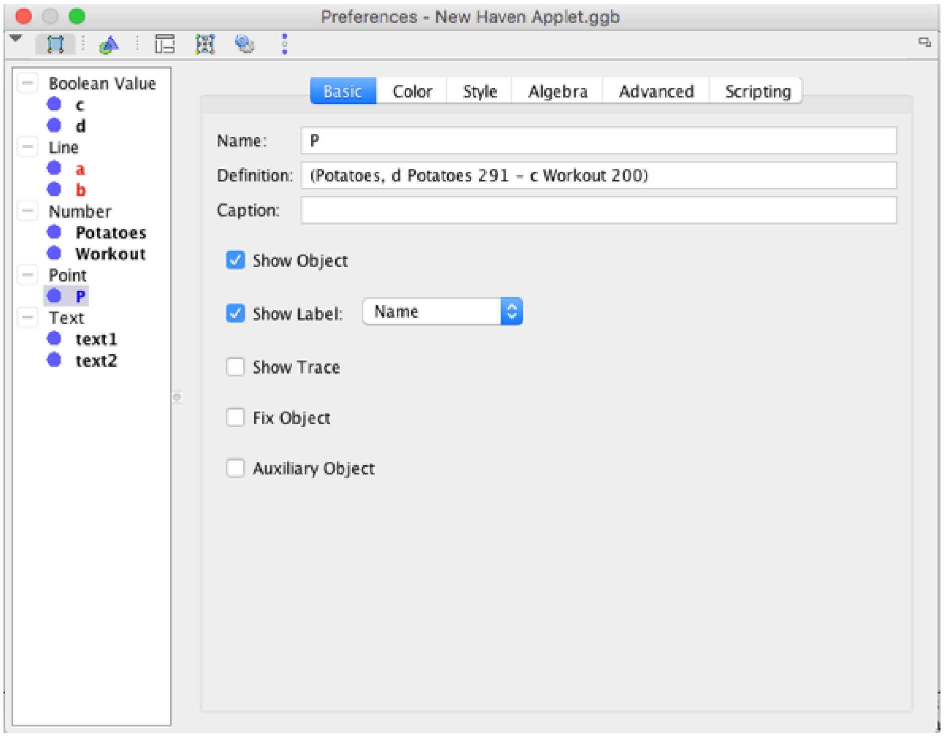

Figure 6.7: Workout checkbox.

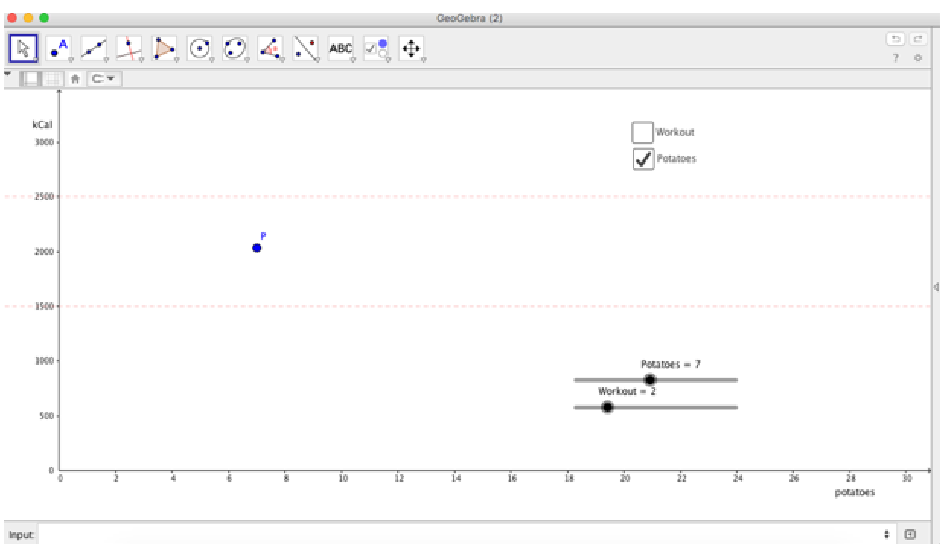

Figure 6.8: Completed initial sketch.

reflects the idea that calories available will be calories consumed. For calories consumed, we determined that an average potato has 291 calories. For calories burned, the teacher determined that an average male will burn 200 calories per hour of working out. Therefore, the calories available is the number of hours spent working out multiplied by 200 subtracted from the number of potatoes consumed multiplied by 291. Figure 6.8 shows where the equation should be entered. Figure 6.8 shows the final product.

PART III

SHORT PRESENTATIONS AND POSTERS

SHORT PRESENTATION ABSTRACTS

8.1 Volume of Solids with Known Cross Sections - Rasha Tarek, Staples High School, CT

Students often struggle with visualizing solids in 3D. This presentation will walk you through the process of creating 3D solids with known cross sections to help your students visualize them. We will start by creating the base of our solid using any two continuous functions, then we will construct infinitely many square or triangular cross sections that are perpendicular to the x-axis to form the solid.

8.2 Dynamic Illustrations and Validations of Geometric Theorems - Tim Brzezinski, Berlin High School, CT

In this session, participants will have the opportunity to interact with several GeoGebra applets Âthat dynamically illustrate definitions, concepts, theorems without words, segment lengths, and angle measures. Each applet contains a figure that the user can modify at any time. The main tool that controls all dynamics in each applet is the slider tool. These illustrationsÂprovide teachers with a powerful tool to foster student discovery and meaningful reinforcement of concepts.

8.3 Real Problems with Real Pictures - Ali Heery and Rob Belevich, Southern Connecticut State University, CT

In this presentation, we will showcase how we used real pictures to illustrate real-life problems. The activities that we propose allow students to interact with the problem and try different strategies. Other problems can be created using the same strategies, making the context of the problem more real and meaningful to students.

8.4 Geometric Constructions with Automatic Feedback - Jason Wofsey, Professional Childrenâ^s School, NY

I will present several applets that ask students to perform Euclidean constructions. The applets incorporate JavaScript with added listeners to check for the correct object and then provide affirmative feedback to students. I will explain how teachers can create their own such exercises and how to best make use of them in the classroom. I will also show how they can be used with the Moodle GeoGebra Quiz plugin to make automatically graded quizzes.

8.5 Pythagoras in Converse - Hunter Smith, Engineering & Science University Magnet School, CT

In this presentation, we will look at multiple applets to explore the converse of the Pythagorean Theorem in a high school geometry class. The applets will offer hands-on demonstrations and ways to visualize the inequality.

8.6 Graphing Surfaces with Polygon Cross Sections in GeoGebra - Doug Hoffmann, Northwestern Community College, CT

The purpose of this presentation is to provide a method of graphing surfaces in GeoGebra with parameterizable cross sections. Specifically, we will focus on surfaces that do not have circular cross sections. For a relatively small number of sides, the coordinate functions are easy to code, but when the number of sides gets bigger, the piecewise functions get troublesome to code by hand. One way around coding the coordinate functions by hand is to use a Sequence command. For each coordinate function, a Sequence command creates a list of all the functions needed to define a coordinate function.

8.7 A Dynamic Definition of $\tan(t)$ - Doug Kuhlman, former teacher at Andover Philips Academy, MA

The short talk would be showing how to teach the tangent function using GeoGebra: attaching a tangent line, $x = 1$, to the unit circle and then drawing a line though $(0, 0)$ and $W(t)$, where W is the wrapping function. $\tan(t)$ is the second coordinate of the point of intersection of that line and the line $x = 1$.

8.8 Constructing Meaning in Geometry with GeoGebra - Sandra Ollerhead, Mansfield High School, MA

GeoGebra can serve as a powerful tool to developing a deeper understanding of geometric concepts behind straightedge and compass constructions. This presentation will show the benefits of using GeoGebra versus pencil and paper to teach students constructions.

INTERACTIVE POSTER ABSTRACTS

9.1 Triangle Similarity Shortcuts - Rasha Tarek, Staples High School, CT

This is an interactive activity in whichÂstudents utilize their knowledge of Similar Polygons to verify the validity of several triangle similarity shortcuts. Students will be presented with 5 different scenarios that they will investigate. They also have the opportunity to check their understanding at the end of the activity.

9.2 Slope Field Generator - Rasha Tarek, Staples High School, CT

Generate any slope field and graph particular solutions with ease using this slope fields generator. Students can use this worksheet to visualize solutions to certain differential equations or simply to check their work.

9.3 Graphing Polynomials Using Zeros and their Multiplicities - Janet Zupkus, Naugatuck Valley Community College, CT

This activity uses an existing geogebra applet to direct the studentâ^s exploration of polynomial functions and graphs. The activity explores polynomial end behavior and x-intercept behavior related to the multiplicity of factors. The activity lends itself to a flipped classroom environment as an independent exploration prior to a classroom lecture.

9.4 Characteristics of Quadratic Graphs - Rayigam Thevaraja Mathiyalagan, Tamil Vidyalayam, Sri Lanka

The applet is designed to help students investigate the characteristics of the graphs of quadratic functions.

9.5 Pythagorean Theorem Game - Matthew Krebs, Boston Public Schools, MA

This applet gives students the opportunity to (1) create a right triangle with the red side as the hypotenuse and (2) click on the sides and use the Pythagorean Theorem to arrive at the length of the side that is not given. Students can opt to click on the square root symbol to begin their work or after they have written everything else.

9.6 Distance Game - Matthew Krebs, Boston Public Schools, MA

This applet gives students the opportunity to become comfortable with unbalanced axes. With each problem, the x and y axes change. As a result, students must be careful to create either a vertical line or a horizontal line to measure the appropriate length of a segment, or they must identify the length of a given segment according to the axes.

9.7 Proportions Game - Matthew Krebs, Boston Public Schools, MA

This applet gives students the opportunity to interpret problems using proportions through the creation of right triangles. Students make a right triangle based on the first situation, then make a similar right triangle that coincides with the second situation.

9.8 Jeopardy - Matthew Krebs, Boston Public Schools, MA

Every time a student plays this game, the applet will select a random five categories for Jeopardy, another five for Double Jeopardy, and another one for Final Jeopardy. The applet updates every time it is being played, so the problems will not be the same. Users get a new experience every time.

9.9 Interior Angles of Polygons - Ali Heery, Bridgeport Public Shools, CT

The applet is designed to help students make conjectures about the sum of the interior angles of polygons based on the number of triangles that can be built inside the polygon.

9.10 The Ambiguous Case - Ella Sayin, Southern Connecticut State University, CT

The applet is designed to help students learn about the ambiguous case of the law of sines.

9.11 Non-coplanar “Quad” Midpoints - Tim Brzezinski, Berlin High School, CT

Many geometry teachers and students are familiar with the theorem that states that the (four coplanar) midpoints of the segments of any quadrilateral form the vertices of a parallelogram. However, did you know this statement also holds true for any four non-coplanar points? That is, if segments are consecutively connected among 4 non-coplanar points, consecutive midpoints of these segments will always be vertices of a parallelogram!

9.12 9-point Circle Action (Part 1) - Tim Brzezinski, Berlin High School, CT

For any triangle, there are 9 special points that all lie on a circle (The midpoints of the triangleâ^s 3 sides, the points at which the triangleâ^s 3 altitudes meet the lines containing the triangleâ^s 3 sides, and the midpoints of the segments that connect the triangleâ^s orthocenter to each of its 3 vertices). This applet dynamically illustrates without words, segment lengths, or angle measures that the center of this 9-Point Circle is the midpoint of the segment that connects the triangleâ^s circumcenter and orthocenter.

9.13 Hexagonal Napolean Theorem? - Tim Brzezinski, Berlin High School, CT

If equilateral triangles are constructed off the 6 sides of any hexagon (convex or concave), then the midpoints of the segments that connect the centroids of an â^opposite pairâ^ of equilateral triangles will always form vertices of yet another equilateral triangle. (There are 3 such segments, seeing that there are 6 equilateral triangles.) Would you consider this to be a version of Napoleanâ^s Theorem for a hexagon? You be the judge!

9.14 Numerical Integration - Albert Navetta, University of New Haven, CT

This applet shows a visualization of 3 numerical integration techniques: Midpoint Rule, Trapezoid Rule, and Simpson's Rule. You can change the function, the number of divisions, and the limits of integration. To get the results for Simpson's Rule, the box must be checked. Simpson's rule takes a lot of processing, so be patient after checking the Simpson's Rule box. Your browser may even indicate that the script has stopped, but it is working, just wait. If you have a relatively new computer, it should not be a problem. Once Simpson's rule is displayed, you can cycle through the parabolas that make up the estimate.

9.15 Exploring Rotations - Sandra Ollerhead, Mansfield High School, MA

This applet and accompanying worksheet provide students the opportunity to develop a formal definition of rotations as well as to explore rotations on the coordinate plane.

9.16 Exploring reflections - Sandra Ollerhead, Mansfield High School, MA

This applet and accompanying worksheet provide students the opportunity to develop a formal definition of reflections as well as to explore reflections on the coordinate plane.