

EXPLORATION OF THE SIERPINSKI TRIANGLE WITH GEOGEBRA

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Abstract

In this paper, the authors explore using fractals in the classroom to teach more complex ideas. In GeoGebra, the Sierpinski triangle is created using a combination of the midpoint and polygon tools. Once created, the triangle can be used to explore the concept of limits by looking at the decreasing area of the figure. Learners can then engage in open inquiry to find a pattern that describes the decreasing area. After the pattern is found, learners can use the sequence function to summarize their results. The variety of functions in GeoGebra allow fractals to be used in a meaningful way in a high school classroom or any setting in which people are willing to learn and inquire.

Keywords: GeoGebra, fractals, Sierpinski Triangle, inquiry, limits

1 INTRODUCTION

The study of geometry in grades 6-12 is marked by exploration of common shapes - squares, circles, triangles - and formulas to describe them. In the secondary curriculum, geometry is largely a study of idealized shapes informed by the Platonic theory of Forms. However, nature is far more complex and many objects are better described by fractals. For example clouds, mountains, coastlines, snowflakes, and DNA. The more one looks, the more examples of fractals are found, yet fractals are not a central part of the 6-12 mathematics curriculum. In particular, a search for “fractal” in the *Common Core State Standards for Mathematics* yields no results.

This is unfortunate, for fractals are well-suited for exploration of various mathematical topics from geometric measurement and numerical patterns to algebraic formulas and limits. These ideas extend beyond geometry into algebra, precalculus, and calculus. The creation of fractals relies on repetition and transformation of basic geometric shapes, therefore they are also suited to younger grades or as an introduction to more advanced geometry. Technology expands the opportunities to create and explore fractals even further. In this paper, we explore some of these connections using GeoGebra to solve problems involving fractals using methods appropriate and accessible for grades 6-12.

1.1 What are fractals?

Fractals are geometric figures that are “self-similar,” meaning each small part looks the same as the whole. Zooming in or out on a fractal does not change the image that one sees. Famous examples of fractals include the Sierpinski Triangle, Koch’s Snowflake, the Hilbert Curve, the Cantor Set, the Apollonian Gasket, and many more (see Figure 1). Because there are algorithms to generate fractals, learners can easily create and explore.

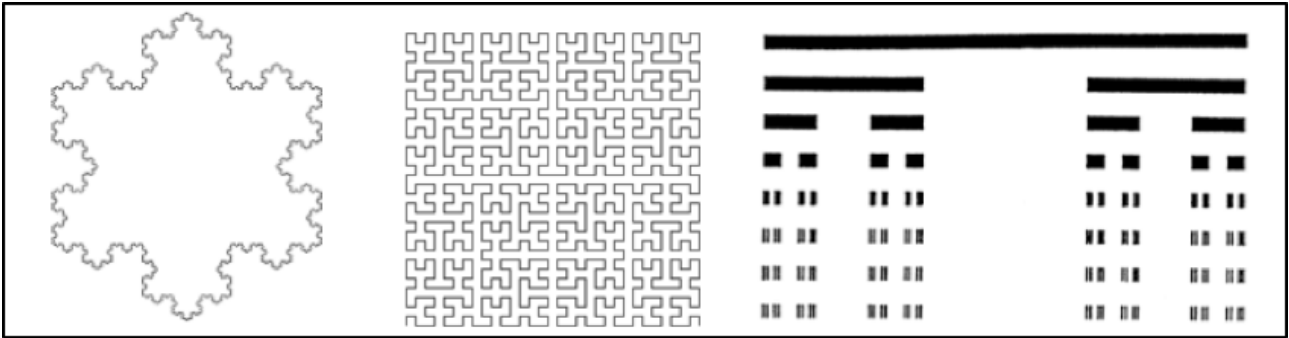


Figure 1. Examples of fractals include (left) Koch’s Snowflake (“[Koch Snowflake]”, n.d.), (middle) the Hilbert Curve (Jaffer, 2009), and (right) the Cantor Set (“[Cantor Set]”, n.d.).

1.2 Why study fractals?

Though fractals are not specifically included as content standards in the Common Core, their development and patterns created may be used to motivate many standards. For instance, teaching topics addressed in Common Core standards HSF.BF.A.2, 7.G.A.1, and 8.G.A.4 (2010) such as geometric sequences, scale drawings, and similarity can be viewed from this perspective. Working with fractals also encourages the study of series, limits, and infinite sums since visual representation helps make algebraic concepts more concrete for learners. Learners find fractals interesting because of their puzzle-like nature and the creativity they allow, providing teachers with a more engaging way to present this material (Reiter, Thornton, & Vennebush, 2014).

2 THE SIERPINSKI TRIANGLE

2.1 By-Hand Construction

The Sierpinski Triangle is one of the most well-known fractals. It is created by “infinite repetition” of the following steps: (1) for every filled equilateral triangle, locate the midpoints of each side, (2) connect these midpoints to form a smaller triangle, and (3) remove that triangle. Figure 2 shows the first four iterations of the algorithm.

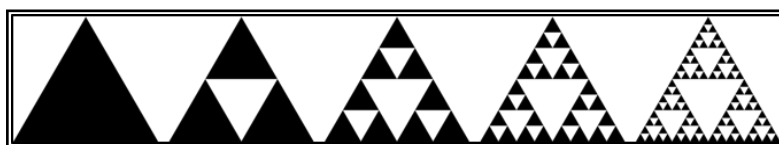


Figure 2. Iterations 0-4 of the Sierpinski Triangle (Knowlton, 2015).

Midpoints of line segments are the only mathematical concept used in the creation. The first few iterations can be done by hand, however, because the number of (filled) triangles grows exponentially with each iteration, constructing subsequent iterations becomes tedious and time consuming. GeoGebra offers an efficient, interactive way to construct stages of the triangle.

2.2 Construction with GeoGebra

2.2.1 Building the First 2 Stages

The initial equilateral triangle (poly 1) is made using the regular polygon tool. With the midpoint tool, select each side of this triangle and create a midpoint. Use the polygon tool again to construct another equilateral triangle from these midpoints named poly2. GeoGebra creates this new polygon on top of the initial. To help differentiate the new triangle, change its color by selecting it, going to object properties, and then the color tab. These steps are summarized in Figure 3.

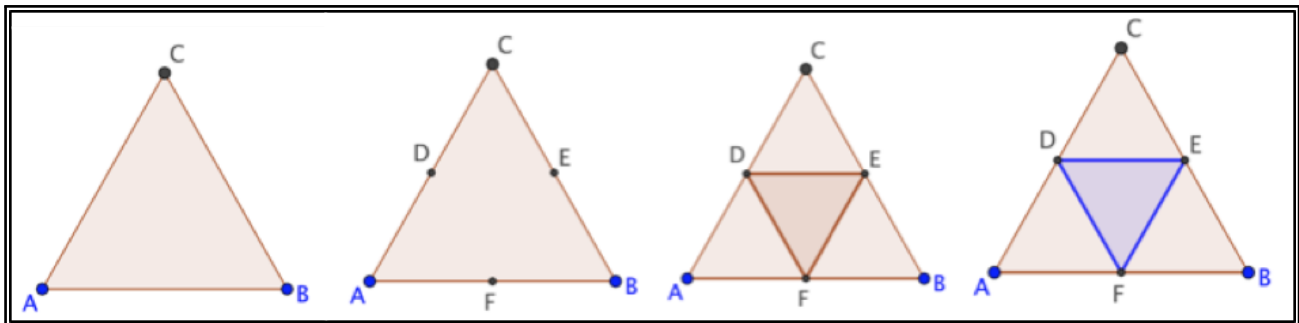


Figure 3. Iterations 0 and 1 of the Sierpinski Triangle are created in GeoGebra.

2.2.2 Creating an Automated Tool in GeoGebra

To automate the process, GeoGebra offers a Create new tool option found under the Tools tab. After selecting this option, a pop-up will appear (see Figure 4). Select poly2, Point D, Point E, and Point F as the output objects, and Point A and Point B as the input objects. This tool produces a midpoint triangle by selecting the two bottom points of a triangle. A screencast of this process can be found at the following link: <http://youtu.be/AuEySzFDJas?hd=1>.

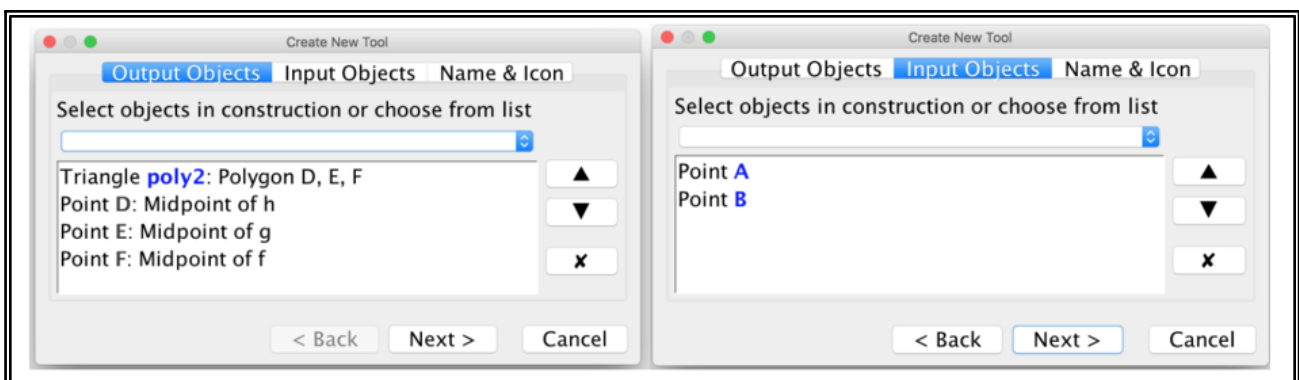


Figure 4. Input and output objects as shown will create a tool to quickly construct the fractal.

2.3 An Area Exploration

2.3.1 Posing the question

Now that the fractal is created, GeoGebra can also be used to explore area. Consider, for instance, the following question.

Using the Sierpinski Triangle, find a pattern that describes the area remaining after each iteration. What is the area for iteration 0? 1? 10? n ?

Viewing the original triangle as a whole, and each new triangle created inside it as a piece being removed, one can calculate the area remaining after each iteration. In Figure 3, the leftmost triangle (Step 0) represents the original area; the medial triangle within the rightmost triangles (Step 1) is removed. The Area tool (Figure 5) can be used to determine the area of the original equilateral triangle (i.e., Stage 0).

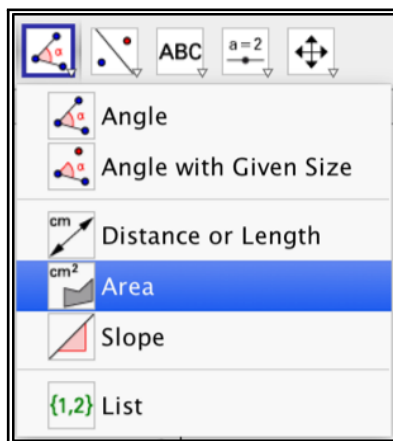


Figure 5. The Area tool can be found with this drop-down menu.

Find the area of iteration 1 by applying the Area tool to the first blue triangle (poly2), then subtracting this value from the area of the initial triangle (poly1). Figure 6 shows a table created by following this procedure for the first 5 stages of a specific Sierpinski Triangle with initial area 163.65 square units.

	A
1	163.65
2	122.74
3	92.05
4	69.04
5	51.78

Figure 6. Area remaining after each iteration of one Sierpinski Triangle.

2.3.2 Using the GeoGebra Sequence Tool

After calculating the area for a few iterations, look for patterns. We notice that the area decreases each time, but by how much? Is there a set number to subtract off each time or is it a proportion? It may be helpful to view the sample GeoGebra file <https://ggbm.at/kzWT27Hz>, which includes a dynamic Sierpinski Triangle with area for iterations 0-4.

After exploration, it is seen that the area of the triangle decreases by one-fourth each time (in other words, the area of the shape at a particular stage is three-fourths the area of the previous stage). Once the pattern has been established, a formula can be created so the areas of multiple iterations can be calculated at once. For this, we will use the GeoGebra `sequence` function, which is generally used to compute lists of numbers that follow a set formula. By typing “sequence” in the input bar, a few options should appear; choose `Sequence[<expression>, <variable>, <start value>, <end value>]`.

As noted before, each stage of the fractal is three-fourths that of the previous stage. So, our formula will be modeled by $\frac{3}{4}^n \cdot \text{area}[\text{poly1}]$. At the input bar, enter this formula, variable n , starting value (0), and a “large” number for the end value (ideally infinity, but this is not allowed, so instead use 50 (see Figure 7)).

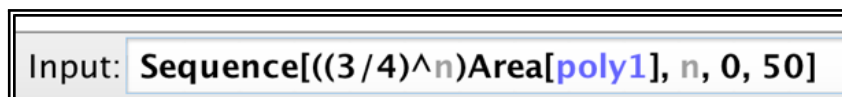


Figure 7. A sequence function to find area at each stage of the Sierpinski Triangle.

2.3.3 Using the GeoGebra Spreadsheet

After entering this expression, GeoGebra displays a list in the Algebra view (see Figure 8).

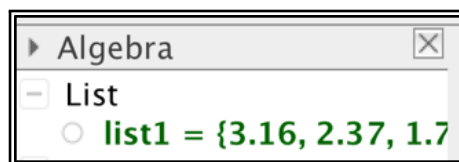


Figure 8. The list created by the Sequence command appears in the Algebra view.

In order to see all the values of a list, it is helpful to view them from the Spreadsheet view. Copy the list to the spreadsheet view by dragging the list over to the spreadsheet while holding down the shift key. A pop-up window will appear with an option to assign this list as a free object or dependent object. Select dependent to continue updating the list as the triangle changes. Check the transpose box so numbers are transferred from a horizontal list into a vertical column (see Figure 9 and <https://ggbm.at/H35BgtmU>).

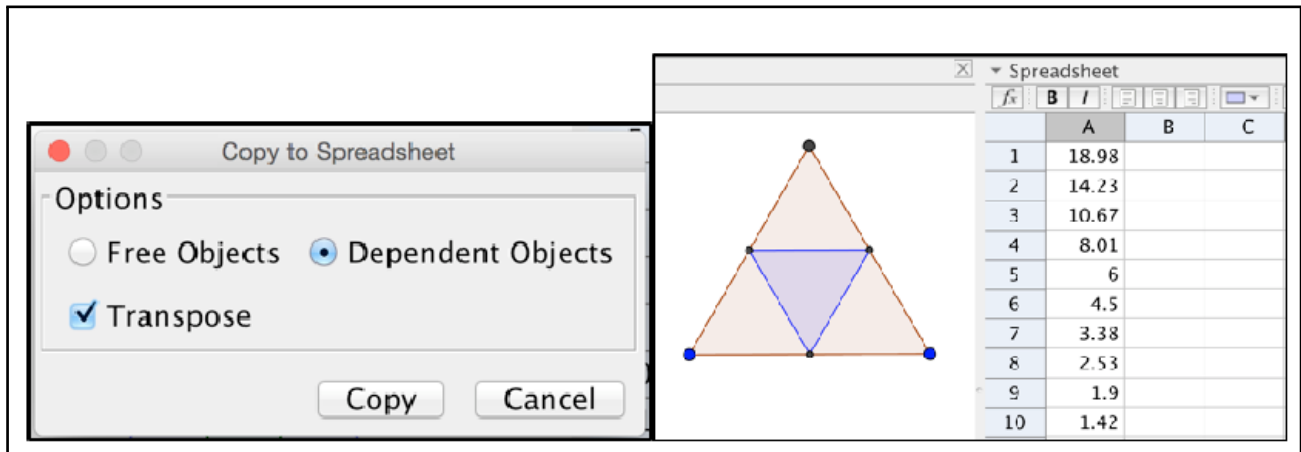


Figure 9. Transferring a list of numbers to the spreadsheet.

2.3.4 Conjectures with the Spreadsheet

Using the spreadsheet view, it is easy to identify the area of the original triangle (iteration 0), as well as iterations 1, 10, etc. So what about stage n ? As the number of iterations grow, we see the area approaches zero, however as long as GeoGebra is set to display decimals to multiple places, the area never equals zero. So as n approaches infinity, area approaches zero. While the idea of limits of area approaching zero may be abstract, GeoGebra provides a visual representation that makes it more concrete for students.

3 EXTENSIONS

Once a visual has been created, the image lends itself to a discussion of its related mathematical content. This content ranges from the complexity of calculus, with the use of sequences and limits of area, to the simplicity of basic geometry, with the application of points and line segments. While basic geometry is taught to children from a young age, higher-level thinking can be facilitated with questions that encourage learners to think abstractly. Examples of these questions are: “How many points will never be removed? How can we say that area is zero when an infinite number of points still remain?” (Naylor, 1999, p. 362). These questions require basic definitions of geometry while also synthesizing the information gained from this activity. Hopefully, the conclusion that the points on the corners of every triangle will never be removed will be reached. Additionally, learners should recognize that the area will never reach zero due to the fact that multiplying by $\frac{3}{4}$ repeatedly will never produce a product of zero. However, the area will decrease in size significantly with each multiplication, so the limit of the area as the number of iterations approaches infinity will be zero.

The Sierpinski Triangle can also be extended by using it in connection with Pascal’s Triangle or by creating a 3D version in GeoGebra. Of course, while this problem focused on the Sierpinski Triangle, there are many other fractals that can be explored. For example, Koch’s Snowflake is an example of how as a limit goes to infinity, the area will converge to a finite number. Lastly, because of the use in modeling real-world phenomena, fractals can be researched and explored in related scientific fields.

4 CONCLUSION

Due to their repetitive nature and many applications, fractals offer opportunities for learning that range from the very simple to the abstract. The use of software such as GeoGebra allows automation of the repetitive nature of the creation. In the context of the Sierpinski Triangle, GeoGebra can quickly calculate the remaining area of the triangle and display this information in a spreadsheet. Further inquiry into the changing area offers an opportunity to acknowledge and model patterns as functions. Additionally, exploring the pattern lends itself to a discussion of limits. As shown by this one example, fractals' applications offer an enriched way to teach mathematical content. Further exploration into other fractals will create additional opportunities for learning.

REFERENCES

- [Cantor Set]. (2015, May 9). Retrieved December 8, 2016, from <https://legiononomamoi.wordpress.com/2015/05/09/double-and-nothing-the-return-of-the-cantor-set/>
- Jaffer, A. (2009). [Hilbert Curve]. Retrieved December 8, 2016, from <http://people.csail.mit.edu/jaffer/CNS/PSFCDR>
- Knowlton, B. (2015, December 15). [Sierpinski Triangle]. Retrieved December 6, 2016, from <http://knowlton-group.com/kpis-and-the-sierpinski-triangle/>
- [Koch Snowflake]. (n.d.). Retrieved December 8, 2016, from http://www.math.ubc.ca/~cass/courses/m308-03b/projects-03b/skinner/ex-dimension-koch_snowflake.htm
- National Governors Association Center for Best Practices, Council of Chief State School Officers (2010). *Common Core State Standards - Geometry*. Washington D.C.: National Governors Association Center for Best Practices, Council of Chief State School Officers.
- Naylor, M. (1999). Exploring Fractals in the Classroom. *Mathematics Teacher*, 92(4), 360-366. Retrieved from <http://www.jstor.org/stable/27970992>
- Reiter, H. B., Thornton, J., & Vennebush, G. P. (2014). Using KenKen to Build Reasoning Skills. *Mathematics Teacher*, 107(5), 341-347. doi:10.5951/mathteacher.107.5.0341
- Shiffman, D. (n.d.). [Sierpinski Triangle]. Retrieved December 10, 2016, from <http://natureofcode.com/book/chapter-8-fractals/>



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