VISUALIZING FUNCTIONS OF COMPLEX NUMBERS USING GEOGEBRA

ALBERT NAVETTA

University of New Haven, West Haven, CT 06516

Abstract: This paper explores the use of GeoGebra to enhance understanding of complex numbers and functions of complex variables for students in a course, such as College Algebra or Pre-calculus, where complex numbers are introduced as potential solutions to polynomial equations, or students starting out in an undergraduate Complex Variables course. The paper introduces methods to create interactive worksheets for students seeing complex numbers and functions for the first time and for those who have some experience with them, but struggle to visualize their meaning. Acknowledging limitations of GeoGebra concerning complex functions, we create new learning opportunities as we develop workarounds.

Keywords: complex numbers, GeoGebra

Introduction

Overview

At first glance, GeoGebra may not seem to be a very powerful tool for functions involving complex numbers. Indeed, if the goal is simply to perform very complicated calculations, there are many better tools available. Many of them, such as Sage, are also free. But for teaching the basics to College Algebra students or beginning students in a Complex Variables course, a strong case can be made that GeoGebra is the best tool for students to interact with visualizations involving complex numbers. At the time of this writing, GeoGebra lacks the ability to directly process functions involving complex variables, although it handles operations on complex numbers just fine. We will exploit those capabilities, and GeoGebra's unique ability to easily show two graphics windows interacting with each other. In some cases, GeoGebra's limitations can be used to an advantage, requiring students to break down functions into real-valued parts.

Entering and Displaying Complex Numbers

A complex number may be entered very easily in GeoGebra. Simply use the Complex Number tool or type

```
z1 = 2 + 2i
```

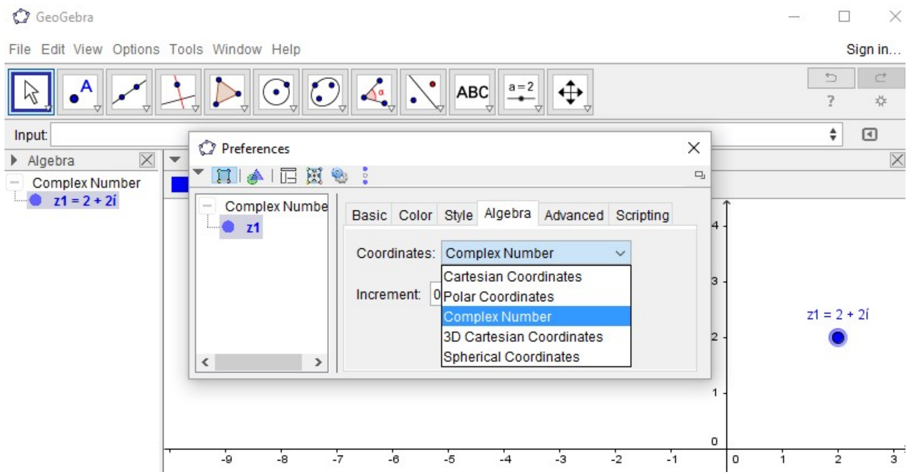


Figure 2.1: Graphics and Algebra view with $z1 = 2 + 2i$.

in the input bar and the complex number $z1$ will be saved and displayed as a point.[1] Except for the label, the complex number $2 + 2i$ is indistinguishable from the ordered pair $(2,\ 2)$ in the graphics window. The user can change the point to and from being treated algebraically as a complex number using the Algebra tab of the properties dialog. Referring to Figure 2.1, selecting *Complex Number* results in $z1$ being treated algebraically, and labeled (if *Value* is selected in the *Show Label* dialog) as a complex number. Only the *Complex Number* selection results in the point being treated algebraically as a complex number. In the Algebra view, points are either listed under the heading *Point* or *Complex Number*; this is based on how the point was initially entered and will not be affected by changing the setting in the Algebra tab.[2] Note that z cannot be used as the name because z is reserved as a variable in GeoGebra, but Z can be used. An existing point that had not been entered as a complex number can be changed to a complex number using the same tab. A complex number may be entered in polar form $(re^{i\theta})$, but will still be displayed as $a + bi$. For example, the same point $z1$ may have been entered as

```
z1 = 2 * sqrt(2) * e^(i * pi/4).
```

GeoGebra will accept the entry, but will automatically convert it and display $z1 = 2 + 2i$. [3] With a little extra work we can label the points with polar $\left(re^{i\theta}\right)$ form or any other form.

Complex Numbers as Vectors

To display a complex number as a vector, the user must enter a vector in GeoGebra, then use the Algebra tab in the object properties as in figure 2.1. The result will be a complex number displayed as a vector. For example, to have $z2 = 2 + i$ displayed as a vector, enter

```
z2 = Vector[2 + i].
```

This will result in the vector $z2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, but the Algebra setting defaults to *Cartesian Coordinates*, so the setting must be changed in the Properties dialog (see Figure 2.1). Figure 2.2 shows $z1$ and $z2$ as a point and a vector respectively.

Functions

Stored functions in Geogebra are always functions of real numbers. For example, an entry of

[1] There is nothing special about the choice of $z1$. entering `a = 2 + 2i` will result in the same complex number stored as a.
[2] Previous versions of Geogebra would change the heading of the point based on the Algebra setting.
[3] The *Polar Coordinates* selection in the pull-down menu in figure 2.1 will convert $z1$ to a point with polar coordinates (r, θ). GeoGebra will not process the point algebraically as a complex number.

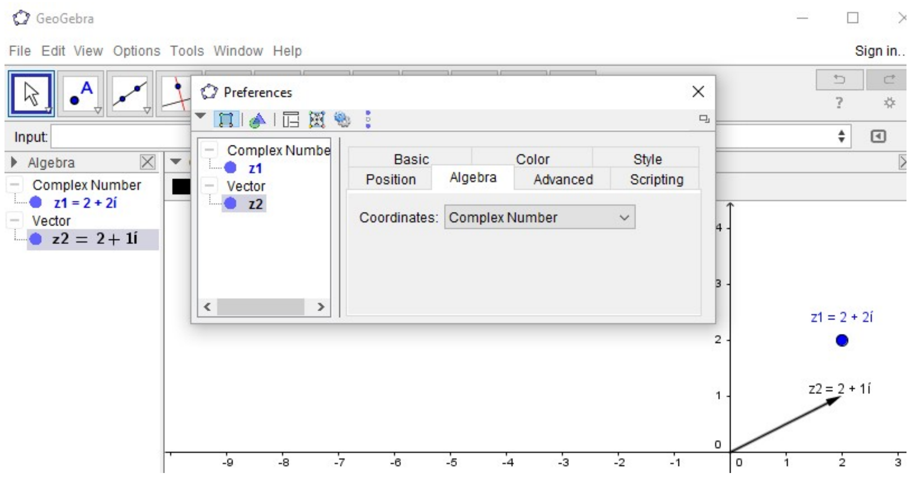


Figure 2.2: Complex numbers $z1$ as a point, and $z2$ as a vector.

```
f(z) = z^2,
```

might be expected to generate a function whose inputs may be complex numbers. However, if $f(z)$ is entered as above, followed by

```
w1 = f(z1),
```

where $z1 = 2 + 2i$, GeoGebra will process only the real part of $z1$, so the result is $w1 = 4$. Entering

```
f(2+2i)
```

results in an alert saying the argument $2 + 2i$ is illegal. The correct result can only be obtained by entering the function directly as a simple operation, not using function notation. For example, entering

```
w1 = (z1)^2,
```

yields the result $w1 = 0 + 8i$, as expected. The value of $z1$ can then be changed manually or by dragging $z1$ around the graphics view. As $z1$ updates, $w1$ will also update by the definition $w1 = (z1)^2$.

Operations With Complex Numbers

Addition and Subtraction

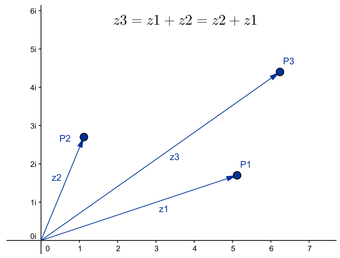
Clearly, the best way to visualize addition and subtraction is with vectors, but there are some obstacles to directly displaying vector addition with the desired parallelograms. The trick is to perform all algebra with points, then use the points to define vectors. For example, suppose we want to display the addition of 2 complex numbers $z1$ and $z2$ as $z3 = z1 + z2$. We would enter points $P1$ and $P2$ as complex numbers, then enter

```
P3 = P1 + P2.
```

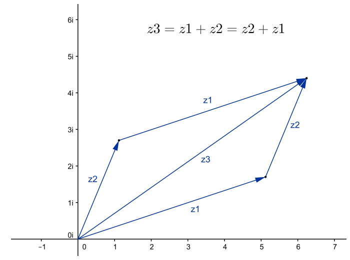
We then define our displayed complex numbers as vectors tied to each point by entering

```
z1 = Vector[P1]
z2 = Vector[P2]
z3 = Vector[P3].
```

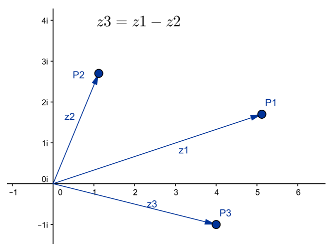
Figure 2.3a shows the result. The points defining the vectors are left labeled in Figure 2.3a for clarity, but the labels should be hidden and the points reduced to minimum size in the final version of the display. The points themselves may be hidden, but then they cannot be moved and the worksheet loses its interactive capability. Figure 2.3b shows $z3 = z1 + z2$ again, but $z1$ and $z2$ are duplicated from the tip of $z2$ and $z1$ respectively, forming a parallelogram, highlighting the commutative property. The duplicate vectors must also be defined by their corresponding points using



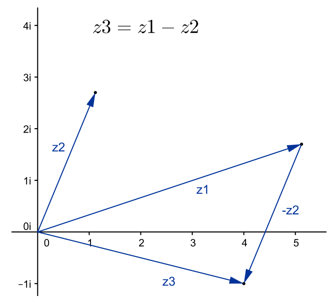
(a) Vectors $z1$, $z2$ and $z3 = z1 + z2$. Points defining vectors are shown.



(b) Same operation as Figure 2.3a, but point labels are off and point size is minimized.



(c) Subtraction of vectors, Points defining vectors labeled for clarity.



(d) Subtraction of $z2$ from $z1$.

Figure 2.3: Addition and Subtraction of Vectors.

the commands

```
z11 = Vector[P2, P3]
```

and

```
z22 = Vector[P1, P3].
```

The labels of $z11$ and $z22$ should be changed from *Name* to *Caption*, with captions of $z1$ and $z2$ respectively. Subtraction is essentially the same, except that $P3$ is defined as

```
P3 = P1 - P2
```

to subtract $z2$ from $z1$. Then, we label $z22$ (the vector from $P1$ to $P3$) as $-z2$ and display it, but do not display vector $z11$ (the vector from $P2$ to $P3$). The result is seen in Figures 2.3c and 2.3d.

Multiplication and Division

Multiplying two vectors whose coordinates are set to *Complex Number* will result in a point representing a complex number. If the Algebra setting of either of the vectors' coordinates are set to anything other than *Complex Number*, the result will be the scalar product of the two vectors [1]. Hence, the best way to display the multiplication of complex numbers as vectors is by using points to control the vectors, as described in Section 2. The goal is to demonstrate that the product, $z3 = z1 \cdot z2$, will have a modulus equal to the product of the modulus of $z1$ and the modulus of $z2$, and that the argument of $z3$ is the sum of the arguments of $z1$ and $z2$. Figure 2.4 shows $z3$ as the product of $z1$ and $z2$. As with addition, the algebra is done using unlabeled points $P1,\ P2$, and $P3$, where each vector is defined by its respective point. Division can be shown in a similar manner.

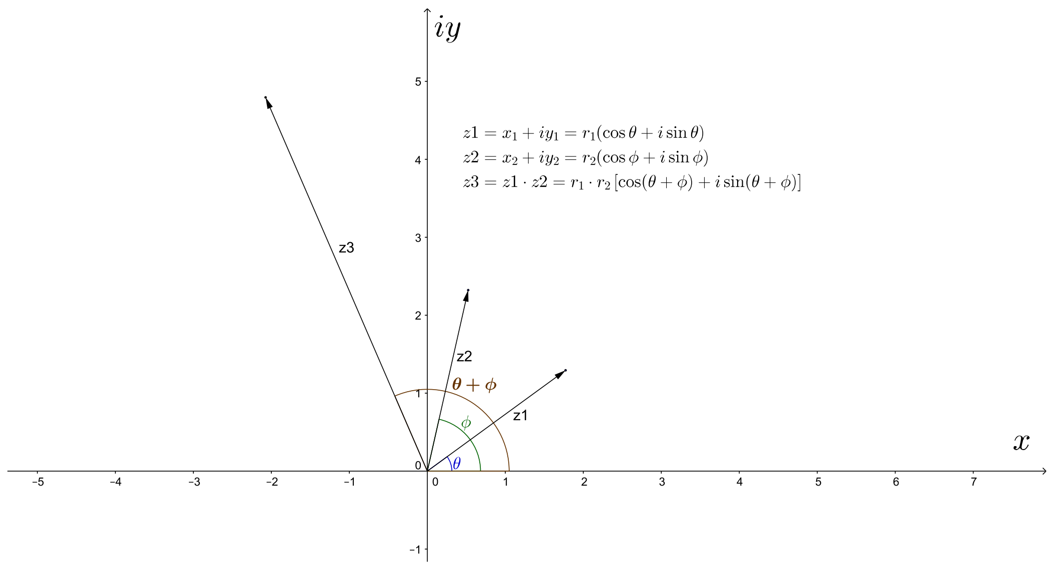


Figure 2.4: Multiplication of 2 complex numbers: $z3 = z2 \cdot z1$.

Functions as Transformations

Basic Setup

Since complex numbers are represented in 2 dimensions, we need a plane, usually the z-plane where $z = x + iy$, for the input of our function, and a second plane, usually the w-plane, where $w = u + iv$, to display the image of z [2]. We dedicate the *Graphics* and *Graphics 2* windows as complex planes and view them side by side. All function inputs, or pre-images, will be displayed in *Graphics*, which will be labeled as the z-plane. The outputs, or images, will be displayed in *Graphics 2*, which will be labeled as the w-plane. Starting with a simple function, we create a complex number (point), $z1$, visible only in the z-plane (*Graphics*). Next, define $w1$ by entering

```
w1 = (z1)^2
```

and making $w1$ visible only in the w-plane (*Graphics 2*). Figure 2.5 shows a trace of $z1$ and $w1 = (z1)^2$ in the z-plane and w-plane respectively.

The Locus Command

The trace in Figure 2.5 is simple and somewhat informative, but much better interaction and learning potential is possible using the `Locus[ <Point Creating Locus Line>, <Point> ]` command. First set up a geometric object in the z-plane and place a point with complex coordinates on the object. As an example, create 2 points, A and B, on the z-plane (Graphics window), then create a segment named $z2Segment$ (anticipating the use of point $z2$) between A and B. Next, create a point $z2$ on $z2Segment$ by using the *Point on Object* tool or by entering

```
z2 = Point[z2Segment].
```

The algebra of point $z2$ should then be set to *Complex Number* (see Figure 2.1). We can now move $z2Segment$ anywhere on the plane by dragging the endpoints A and B. The labels for Points A and B and $z2Segment$ should be off. Now create a point $w2$ in the w-plane that is defined by a function of $z2$. For now we will continue with $f(z) = z^2$. Then we define an object to be the locus curve of $w2$ as defined by $z2$; call it *w2Locus*. Enter the following commands in the input bar:

```
w2 = (z2)^2
w2Locus = Locus[w2, z2]
```

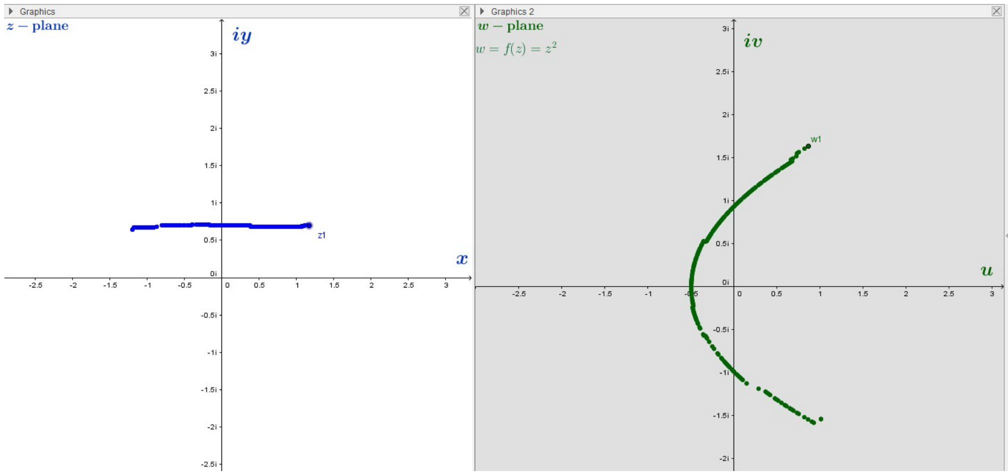


Figure 2.5: z and w planes side by side where $z = x + iy$, and $w = u + iv$.

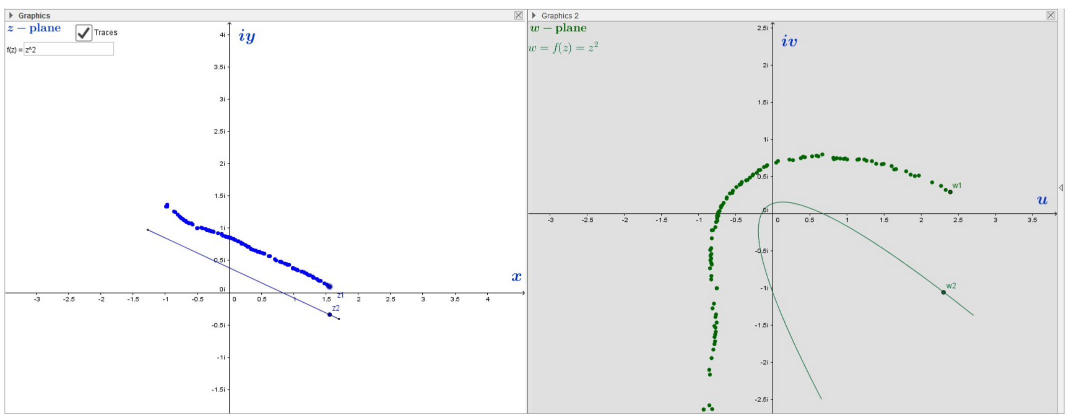


Figure 2.6: The locus of $w2$ as the image of the $z2$ along the segment.

Figure 2.6 shows the resulting locus of $w2$ compared with a similar trace. The locus is much simpler and cleaner in appearance than the trace and therefore more likely to grab the user's attention [3]. The locus is also more informative as we can see how the images of specific shapes look. In this case, the segment in the z-plane produces part of a parabola in the w-plane. Furthermore, the locus is permanent, whereas the trace will disappear as soon as an adjustment is made to the graphic. The segment $z2Segment$ can be moved arbitrarily as can the point $z2$ along $z2Segment$. $w2Locus$ and $w2$ automatically update as $z2Segment$ and/or $z2$ are updated, creating a nice interactive visualization. The process above is not limited to segments. The point $z2$ could have been attached to any geometric object. For example, Figure 2.7 explores the function $f(z) = z^5 - 1$. In the z-plane, a solid-lined fixed unit circle and a dashed-line movable circle have been created. Using the *Point on Object* tool, the point $z2$ is placed on the fixed circle and $z3$ is placed on the movable circle. In the w-plane, the corresponding image points, $w2$ and $w3$ have been created using the definitions $w2 = (z2)^5 - 1$ and $w3 = (z3)^5 - 1$. The loci were then created by entering

```
w2Locus = Locus[w2, z2]
w3Locus = Locus[w3, z3].
```

The line styles of the loci are set to match the styles of the corresponding geometric objects. The rest of the dots on the solid circle in the z-plane represent the complex roots of f. They are best entered as a list, so they are a single object in GeoGebra. By entering

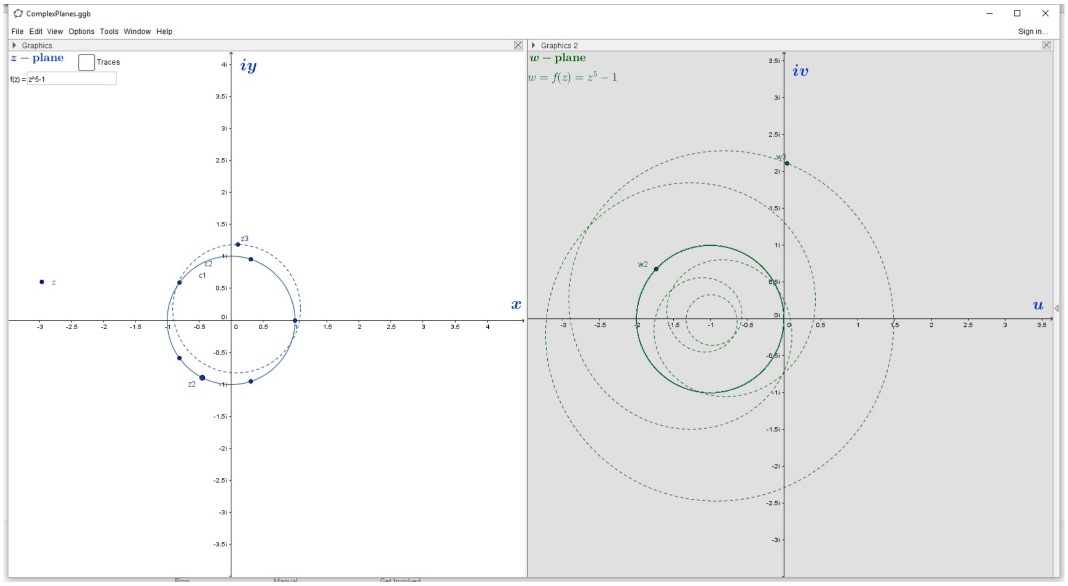


Figure 2.7: Image of $f(z) = z^5 - 1$.

```
roots = {ComplexRoot[f]},
```

the list *roots* is created whose elements are all the complex roots of $f(z) = z^5 - 1$.[4] With the roots displayed, the user should notice that when $z2$ is moved along the unit circle from one root to another, $w2$ travels all the way around the circle formed by its locus.

Returning to the function z^2, Figure 2.8 shows some more ideas for loci. The construction is created in a similar manner to that of Figure 2.7, with the addition of 2 polygons in the z-plane. For clarity, the line styles of the loci in the w-plane match the line styles of the objects in the z-plane used to create the loci. All the points in the z-plane are attached to the geometric objects as shown and all the objects are movable with the exception of the solid circle. As the objects are moved in the z-plane, the corresponding loci in the w-plane will be updated.

Advanced Interaction

Finally, we add some student interaction to the worksheet. This will require the student to calculate the real component, $u(x, y)$, and the imaginary component, $v(x, y)$, of w. The functions u and v are real-valued, multi-variable functions of x and y, where $z = x + iy$ and $w = u(x, y) + iv(x, y)$. As usual, z and w will have corresponding indices ($z1 \rightarrow w1$) on the GeoGebra worksheet. To accomplish this, we need two more functions to represent $u(z)$ and $v(z)$ in our worksheet. Since $z = x + iy$, this is easily accomplished with the two multi-variable function entries

```
u(x, y) = 1
```
and
```
v(x, y) = 1,
```

where the 1 on the right side of each equation is just a placeholder function; the student will change them later. The displays for both functions should be off. Once the functions u and v are in the worksheet, establish input boxes for each with u and v being the linked object respectively. Now the student can break a function $f(z)$ into its real and imaginary components and enter an image point, $w1$, of a complex number, $z1$, where the real coordinate is u and the imaginary coordinate is v. The student would enter the real-valued functions u and v into the input boxes and then simply create $w1$ by entering

[4]Note that the argument of the ComplexRoot[<Polynomial>] command must be a polynomial.

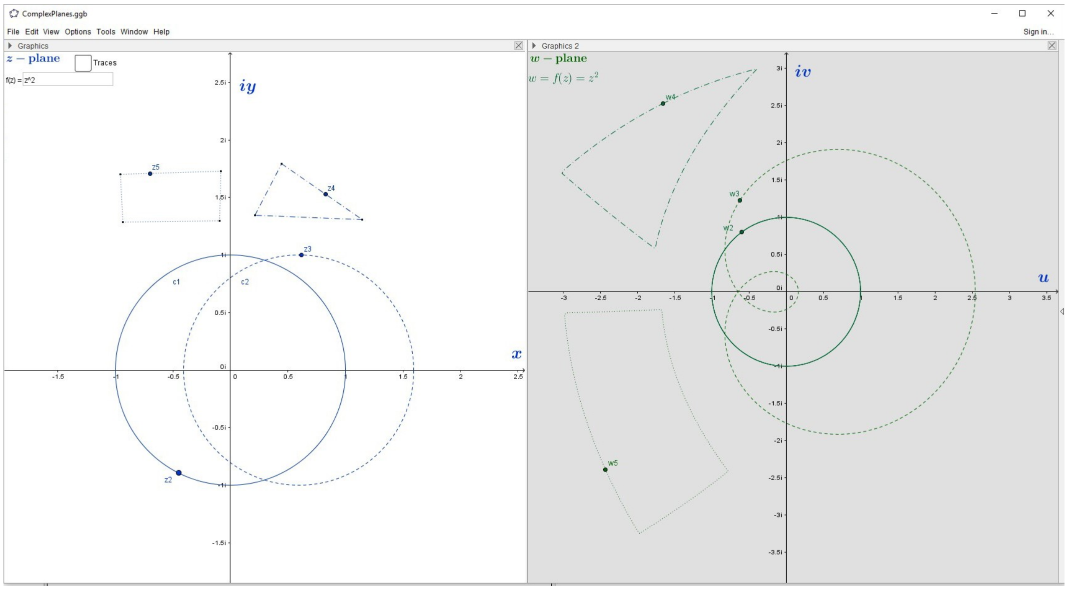


Figure 2.8: Image of $f(z) = z^2$ with multiple loci.

```
w1 = u(z1) + i * v(z1).
```

The functions $u(z1)$ and $v(z1)$ are multi-variable functions that take the coordinates of $z1$ as inputs. Alternatively, $w1$ could be defined by

```
w1 = u( x(z1), y(z1) ) + i * v( x(z1), y(z1) ).
```

Figure 2.9 shows another quadratic function $(2z^2 - 3z + 4)$. In this example, however, the functions u and v are used for the mappings $z1 \rightarrow w1$ and $z2 \rightarrow w2$ and the loci are created using techniques introduced earlier. The functions u and v are determined manually and entered into the input boxes. Once the functions u and v are entered, new mappings $zk \rightarrow wk$ can easily be created by defining a new complex number, zk, then entering `wk = u(zk) + i * v(zk).`

Conclusion

While GeoGebra is not the most powerful computational tool, it is perfectly suited for early studies of complex numbers and functions involving complex variables. GeoGebra provides a unique ability to work with two graphics windows that interact with each other, which provides an engaging experience. Since GeoGebra does not handle complex numbers with functions, the user must decide whether to use direct operations or construct functions of complex variables using only real-valued functions. Both methods work equally well and can be used to check each other. Students should be encouraged to explore these and limitless other possibilities.

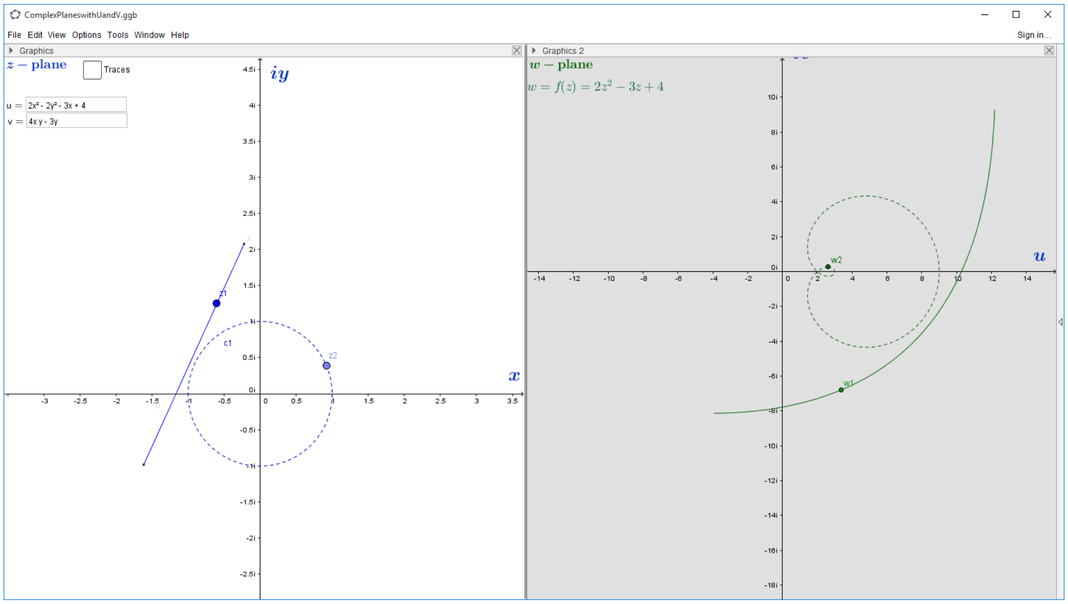


Figure 2.9: A function defined by real functions u and v entered in the input boxes shown on the z-plane. $w1 = f(z1) = u(z1) + iv(z1)$.

REFERENCES

1. Markus Hohenwarter. *GeoGebra 5.0 Manual*.
   URL: `https://www.geogebra.org/manual/en/Manual`
2. Tristan Needham. *Visual Complex Analysis*. Oxford University Press, 1997.
3. Ruth Clark. *Six Principles of Effective e-Learning: What Works and Why*. The eLearning Guild.
   URL: `http://www.elearningguild.com/pdf/2/091002DES-H.pdf`.