
A GEOMETRIC INTERPRETATION OF COMPLEX ZEROS OF QUADRATIC FUNCTIONS

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Abstract: Most high school mathematics students learn how to determine the zeros of quadratic functions such as $f(x) = ax^2 + bx + c$, where a, b , and c are real numbers. At some point, students encounter a quadratic function of this form whose zeros are imaginary or complex-valued. Since the graph of such functions do not intersect the x -axis in the xy -plane, students may be left with the impression that complex-valued zeros of quadratics cannot be visualized. The main purpose of this manuscript is to show that if the zeros of a quadratic function with real-valued coefficients are imaginary, the zeros can be seen if we use an appropriate coordinate system. For illustrative purposes, we have used the software program GeoGebra, which allows us to create a three-dimensional Cartesian coordinate system where imaginary zeros can be viewed simultaneously with the graph of the quadratic function they correspond to. To illustrate this, we will apply geometric transformations to the function given by $f(x) = x^2 - 6x + 13$ in order to visualize its zeros, which happen to be complex-valued. Then, we will identify a particular set of complex numbers that can be used as inputs for the function f . Using this set of complex numbers, we can construct the exact image that is produced by the geometric transformations. Then, we may deem the two methods as equivalent ways to ultimately construct the geometric images of complex-valued zeros of quadratic functions with real-valued coefficients.

Keywords: quadratic functions, complex roots, GeoGebra

Quadratic Functions and their Zeros

Students are taught in high school that one way to determine the zeros of quadratic functions with real-valued coefficients is to locate the function's x -intercept(s). For example, if we wish to find the zeros of the quadratic function $g(x) = x^2 - 5x + 6$, we graph it in \mathbb{R}^2 or the xy -plane (Figure 3.1), and we see that the function $g(x)$ has intercepts at $x = 2$ and $x = 3$, which correspond to the points $(2, 0)$ and $(3, 0)$ in \mathbb{R}^2 . So, these are the zeros of the function $g(x)$ or, equivalently, the solutions of the quadratic equation $x^2 - 5x + 6 = 0$.

It is not long before students see the graph of a quadratic function such as $f(x) = x^2 - 6x + 13$ (Figure 3.2). Since the graph of f has no x -intercept(s), curious students might ask, "Where are the zeros located? How can we see them?" Indeed, the reason the graph of f has no x -intercepts is because the zeros are complex-valued, and the x -axis represents real numbers, not complex numbers. Some basic algebra allows us to verify, albeit without visualization in the xy -plane, that the zeros do exist, and $f(x) = 0$ precisely when $x = 3 \pm 2i$, where $i = \sqrt{-1}$ and $i^2 = -1$.

For ease of exposition, throughout the remainder of this manuscript we shall refer to quadratic functions with real-valued coefficients and complex-valued zeros as Type RC quadratic functions.

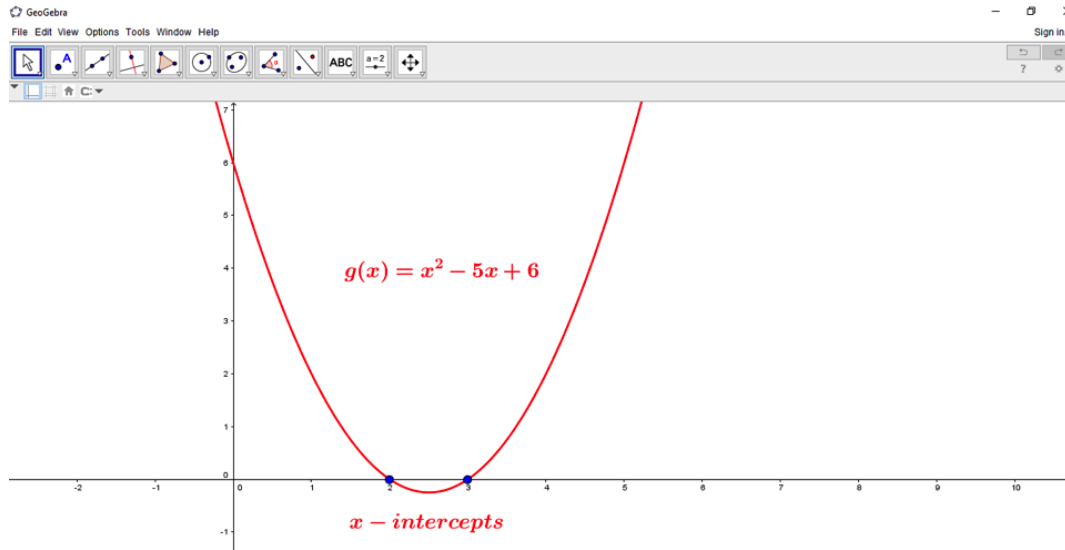


Figure 3.1: The graph of the quadratic function $g(x) = x^2 - 5x + 6$, which intersects the x -axis at the points $(2, 0)$ and $(3, 0)$. These points correspond to the real-valued zeros of $g(x)$, $x = 2$ and $x = 3$.

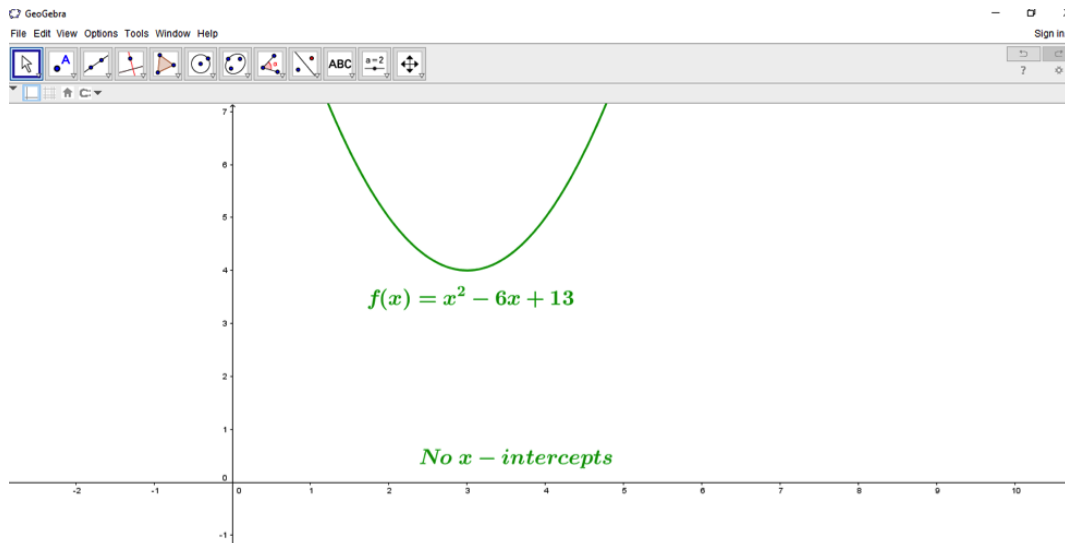


Figure 3.2: The graph of the quadratic function $f(x) = x^2 - 6x + 13$, which does not intersect the x -axis, indicating that the function f has imaginary or complex-valued zeros.

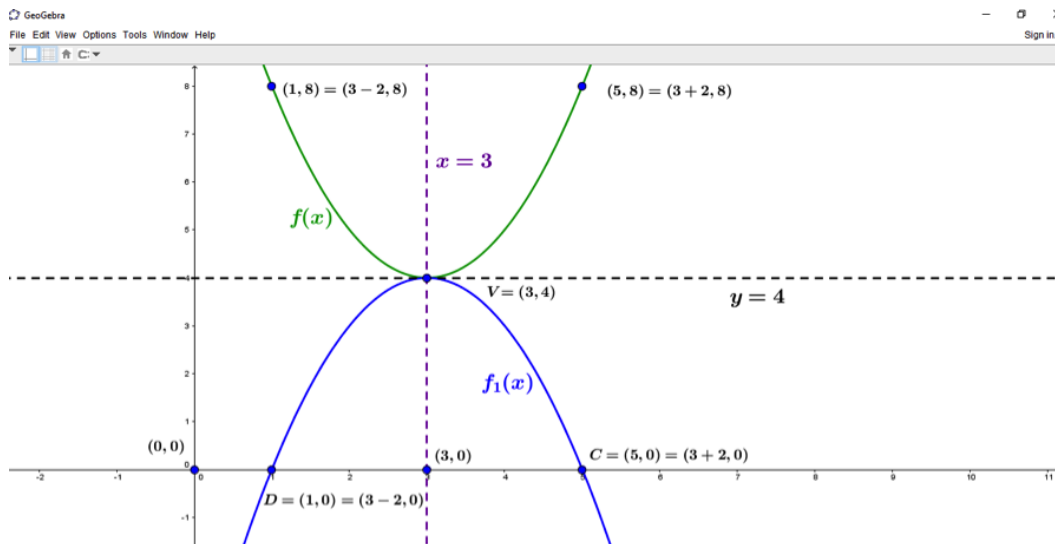


Figure 3.3: The graph of $f(x) = (x - 3)^2 + 4$ is reflected about the line $y = 4$ in the xy -plane. This produces the graph of the quadratic function given by $f_1(x) = -(x - 3)^2 + 4$, which has real zeros at the points $C = (5, 0)$ and $D = (1, 0)$.

A Process for Visualizing Complex Zeros of Type RC Quadratics

Using an appropriate coordinate system, we can apply simple geometric transformations to the graph of a Type RC quadratic function and ultimately obtain a visual image of its zeros. This process is outlined below in detail as it is applied to the function $f(x) = x^2 - 6x + 13$. The steps are accompanied by GeoGebra images that illustrate the process in a clear manner.

Step I: Write the Type RC quadratic function in vertex form $f(x) = a(x - h)^2 + k$, where the xy -plane vertex is the point (h, k) . For our example, we write $f(x) = (x - 3)^2 + 4$, which has xy -plane vertex $V = (3, 4)$. The graph of f is shown once again as the green parabola in Figure 3.3. We observe that the graph of f does not cross the x -axis since its zeros are $3 \pm 2i$.

Step II: In the xy -plane, reflect the graph of f about the line $y = 4$, which contains the vertex point V . The resulting image is an xy -plane parabola or the graph of the quadratic function $f_1(x) = -(x - 3)^2 + 4$ [blue parabola in Figure 3.3]. In general, reflecting an xy -plane quadratic function $f(x) = a(x - h)^2 + k$ about the line $y = k$ will yield the graph of an xy -plane parabola or quadratic function $f_1(x) = -a(x - h)^2 + k$. In our example, the quadratic function f_1 has real zeros $x = 5$ and $x = 1$ or, equivalently, x -intercepts at the two points $C = (5, 0)$ and $D = (1, 0)$. These points may also be written as $C = (3 + 2, 0)$ and $D = (3 - 2, 0)$, indicating the manner in which the zeros of a quadratic function are distributed about the axis of symmetry. For f_1 , each x -intercept is two units from the axis of symmetry, which is the line $x = 3$ in the xy -plane. Note that the graph of f contains the points $(5, 8) = (3 + 2, 8)$ and $(1, 8) = (3 - 2, 8)$, which are the preimage points of C and D , respectively, under the reflection.

Step III: Insert the zi -axis perpendicular to the xy -plane to represent the imaginary axis [blue axis in Figure 3.4], and define the resulting right-handed three-dimensional coordinate system as (x, y, zi) -space. We now denote our axis of symmetry as α since $x = 3$ represents an entire complex plane in (x, y, zi) -space. Each point (x, y) in the xy -plane may now be viewed as the point $(x, y, 0i)$ within (x, y, zi) -space. For example, our vertex point in (x, y, zi) -space is the point $V = (3, 4, 0i)$. The aforementioned points C and D are $(3 + 2, 0, 0i)$ and $(3 - 2, 0, 0i)$, respectively, and their respective preimage points are $A = (3 + 2, 8, 0i)$ and $B = (3 - 2, 8, 0i)$ [See Figure 3.5]. Although the insertion of the imaginary axis creates infinitely many complex planes within (x, y, zi) -space, the reader may want to consider that one of many ways to represent an arbitrary complex number $a + bi$ in (x, y, zi) -space is to write it as the point $(a, 0, bi)$. We choose this particular representation while reminding the reader that we want to find points that correspond to the zeros of our Type RC quadratic function f , and these points are expected to have a y -coordinate of 0.

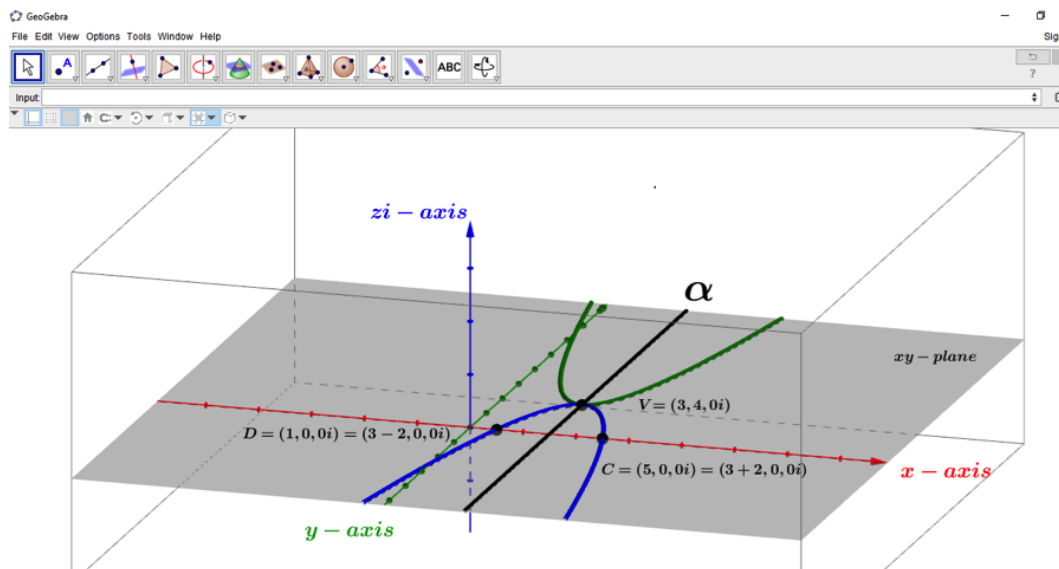


Figure 3.4: The (x, y, zi) -space image of the graphs of f , f_1 , and α , all lying in the xy -plane.

Step IV: In (x, y, zi) -space, plot the points $(3, 0, 2i)$ and $(3, 0, -2i)$. Next, we rotate the graph of f_1 by 90° in the counterclockwise direction about α , the axis of symmetry¹. We denote the resulting curve as the parabola φ , which lies entirely in the complex plane $x = 3$ [Figure 3.5]. The curves f_1 and φ must be parabolas since they are produced by applying rigid motions to f . Under this rotation, the points $C = (5, 0, 0i) = (3 + 2, 0, 0i)$ and $D = (1, 0, 0i) = (3 - 2, 0, 0i)$ are mapped to points $C' = (3, 0, 2i)$ and $D' = (3, 0, -2i)$, respectively, which are the points we plotted at the beginning of this step. Since the y -coordinate for each of C' and D' is 0, the parabola φ intersects the plane $y = 0$ precisely at two points that are equivalent to the complex zeros of f , namely $3 \pm 2i$ (see Figure 3.5). Recall that we set out to see the solutions to the equation $y = f(x) = 0$. Thus the points C' and D' appear to be the geometric images of the complex zeros of the quadratic function $f(x) = (x - 3)^2 + 4$.

To summarize what we have done thus far, we began with a Type RC quadratic function. By the procedure outlined above in Steps I – IV, the xy -plane graph of this quadratic function was reflected about the line $y = 4$, then rotated 90° counterclockwise about the axis of symmetry in (x, y, zi) -space, yielding the image of a parabolic curve that intersects the plane $y = 0$ at two points. The two points of intersection, C' and D' , correspond directly with the complex zeros of our Type RC quadratic function $f(x) = (x - 3)^2 + 4$. Next, we show that this result was not a coincidence, meaning that the points C' and D' are indeed the geometric zeros of our function f .

The connection between the parabola φ and the quadratic function rule $f(x) = (x - 3)^2 + 4$

There is an interesting relationship between the quadratic function rule $f(x) = (x - 3)^2 + 4$ and points that lie on the parabolic curve φ . To uncover this relationship, we will first analyze the effects of the rigid motions as they are applied to points on the graphs of f and f_1 in succession, ultimately producing points on the curve φ . Since (x, y, zi) -space is a Cartesian coordinate system, we can identify the results of the rigid motions very easily. Then, we will verify the accuracy of these results using algebraic means.

First, we take full advantage of the vertex form of our function rule and write our input values in terms of the x -value of the vertex. Specifically, we write x -values in the form $3 + z'$, where z' is a real number or the signed-distance from the axis of symmetry, α . Writing x -values in this form will simplify any calculations involved. For any arbitrary

¹Note that our choice of rotation above is somewhat arbitrary, as a 90° rotation in the clockwise direction would work as well. Under such a rotation, $(3 + 2, 0, 0i)$ would map to $(3, 0, -2i)$ instead of $(3, 0, 2i)$, and the point $(3 - 2, 0, 0i)$ would map to $(3, 0, 2i)$ instead of $(3, 0, -2i)$. This is not the correspondence we seek, thus the counterclockwise rotation was chosen instead.

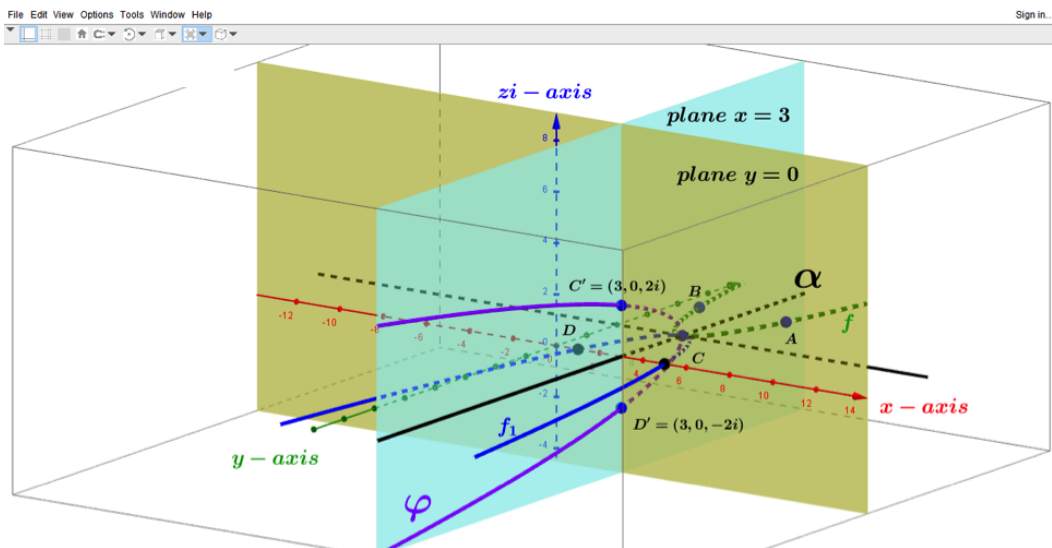


Figure 3.5: The (x, y, zi) -space image of the plane $x = 3$ (light blue), the plane $y = 0$ (yellow-grey), and the graphs of f (green), f_1 (blue), and the curve φ (purple). The complex planes $x = 3$ and $y = 0$ each contain the points C' and D' , which correspond to the complex numbers $3 \pm 2i$, the zeros of the function f . Figure 3.6 provides an alternate image of this situation.

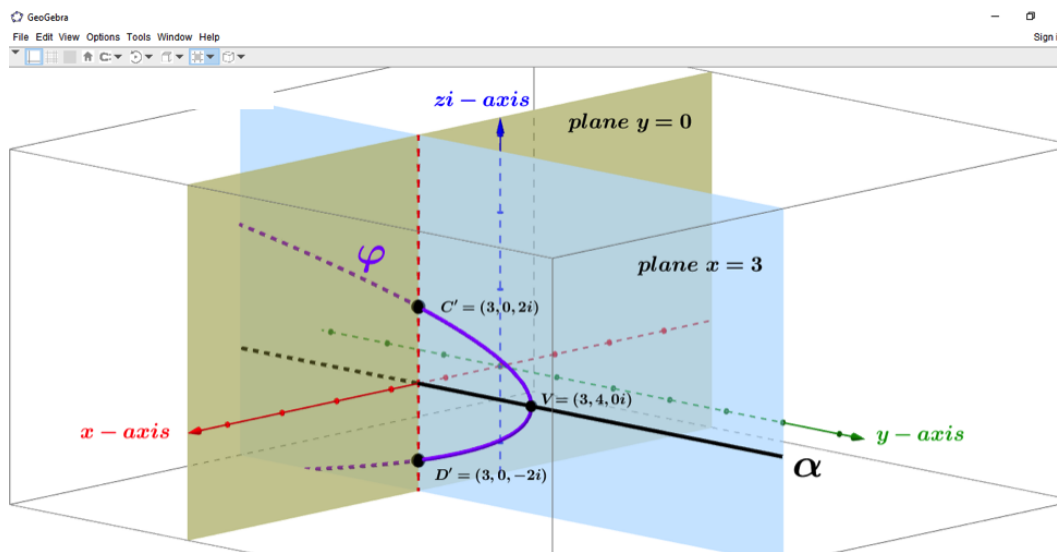


Figure 3.6: An alternate view of φ intersecting the plane $y = 0$. Note the significance of this since the complex zeros $3 \pm 2i$ correspond to the points C' and D' , where $3 \pm 2i$ are solutions of equation $y = f(x) = 0$.

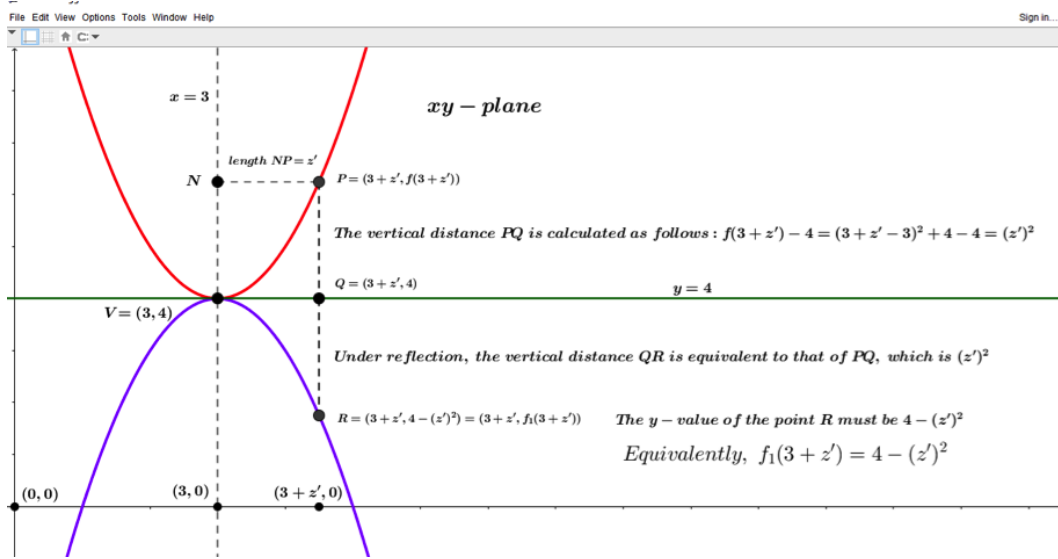


Figure 3.7: Calculation of the coordinates of an arbitrary point on the graph of f as it is mapped to a point on the graph of f_1 when f is reflected about the line $y = 4$ in the xy -plane.

$x = 3 + z'$, we have $f(x) = f(3 + z') = (3 + z' - 3)^2 + 4 = (z')^2 + 4$. So, points on the graph of f in (x, y, zi) -space take the form $(3 + z', (z')^2 + 4, 0i)_f$. Note that any such point in the xy -plane lies at the vertical distance $(z')^2$ above the line $y = 4$ since $f(3 + z') - 4 = (z')^2 + 4 - 4 = (z')^2$.

Upon reflecting the graph of f about the line $y = 4$ in the xy -plane, we claim that all points on f are mapped to points $(3 + z', 4 - (z')^2, 0i)_{f_1}$ on f_1 . Under the aforementioned reflection, it should be clear to the reader that the x and zi -coordinates remain fixed. With the exception of the vertex $V = (3, 4, 0i)$, all points on the graph f_1 in the xy -plane must lie below the line $y = 4$ at a vertical distance of $(z')^2$, the same vertical distance between points on f and the line $y = 4$. Consequently, points on the graph of f_1 must have a y -coordinate of $4 - (z')^2$. Equivalently, evaluating f_1 at $3 + z'$ yields $f_1(3 + z') = 4 - (z')^2$. Thus our claim is verified both geometrically and algebraically.

The results of the reflection described above are illustrated in clear detail in Figure 3.7. Since the reflection occurs in the xy -plane, for ease of exposition, points in Figure 3.7 are written in (x, y) form.

Finally, when the graph of f_1 is rotated in (x, y, zi) -space by 90° about α in the counterclockwise direction, we obtain the following results:

- The y -coordinate of points $(3 + z', 4 - (z')^2, 0i)_{f_1}$ must remain fixed under this rotation, so that points on φ must have y -coordinates of the form $4 - (z')^2$. For now, we seek points of the form $(x, 4 - (z')^2, zi)$.
- As a consequence of the 90° rotation, the parabolic curve α lies entirely in the complex plane $x = 3$. As a result, all points on φ must take the form $(3, 4 - (z')^2, zi)_\varphi$, for some real number z
- By the symmetry of the parabola f_1 , under the 90° counterclockwise rotation, the value of the zi -coordinate must be precisely that of z' given by $3 + z'$. Recall that z' is the signed-distance from the axis of symmetry, α . As a result, all points of the form $(3, 4 - (z')^2, z'i)_\varphi$ lie on the graph of φ

To lend some additional perspective to the statements above, Figure 3.8 provides an overhead view of, in particular, the points C and D and their path of rotation about α (Step IV), which occurs entirely in the plane $y = 0$. The resulting image points C' and D' are shown as well.

Summarizing thus far, if we choose a real number z' and use our function rule to generate the point $(3 + z', (z')^2 + 4, 0i)_f$ on the graph of f , the reflection and subsequent rotation described in Steps I – IV naturally gives rise to the following mapping of points from f to f_1 to φ :

$$(3 + z', (z')^2 + 4, 0i)_f \rightarrow (3 + z', 4 - (z')^2, 0i)_{f_1} \rightarrow (3, 4 - (z')^2, z'i)_\varphi \quad (3.1)$$

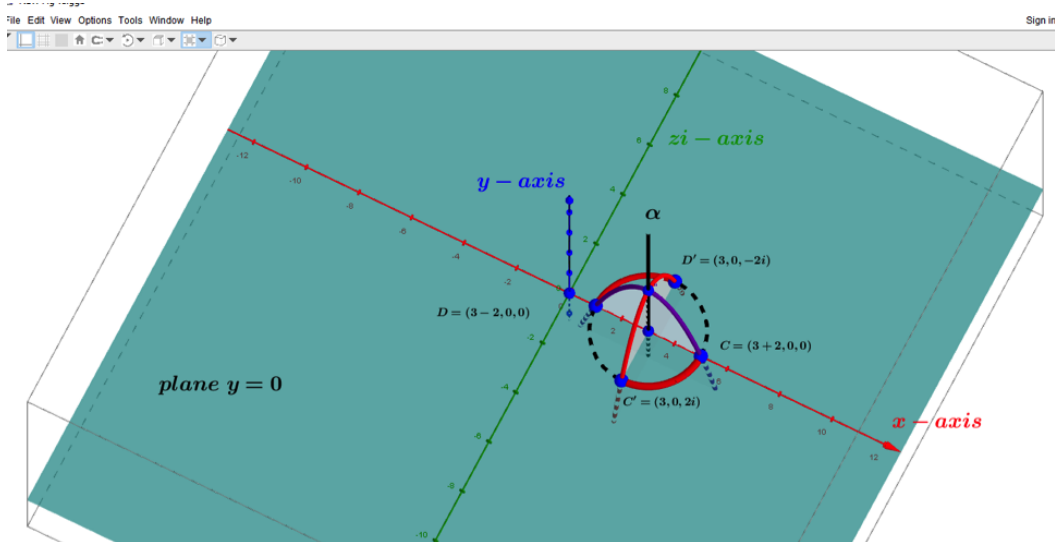


Figure 3.8: An overhead view of the plane $y = 0$, the graph of f_1 , and the parabolic curve φ , which is produced by the rotation of f_1 about α in Step IV. The graph of the xy -plane circle $(x - 3)^2 + y^2 = 4$ contains the path of rotation for the points C and D .

Taking $z' = 2$ gives us the mapping as it relates to the points C and D and their respective preimages and images in (x, y, zi) -space:

$$(3 \pm 2, 8, 0i)_f \rightarrow (3 \pm 2, 0, 0i)_{f_1} \rightarrow (3, 0, \pm 2i)_\varphi \tag{3.2}$$

In general, the mapping given by (3.1) suggests there are two identifiable properties of points lying on the parabolic curve φ . The first is that each has an x -coordinate of 3, which is our function's value for h in the vertex point. The second is that other than the vertex point, each point lying on φ contains a pure complex-valued zi -coordinate, meaning $z \neq 0$. Given these two properties regarding points lying on φ , perhaps pure complex numbers of the form $3 + zi$ play a role here. To establish the connection we seek between the function rule $f(x) = (x - 3)^2 + 4$ and the parabolic curve φ , we shall modify the domain of our function f , allowing for complex number inputs of this form.

To see why this makes sense, let's analyze the transformation above more closely as it relates to the points of f_1 and φ . The x -coordinate $3 + z'$ for points on f_1 transformed to produce an x -coordinate of 3 and a zi -coordinate of $z'i$ for each point on φ . Suppressing the y -coordinate of a point $(3, 4 - (z')^2, z'i)_\varphi$ lying on φ produces a point of the form $(3, z'i)$, which lies in the complex-plane $x = 3$. Equivalently, this point can be viewed as the complex number $3 + z'i$.

Following our suggestion above, if we take z' to be an arbitrary real number, evaluating f at $3 + z'i$ yields $f(3 + z'i) = (3 + z'i - 3)^2 + 4 = (z'i)^2 + 4 = 4 - (z')^2$, which is the precise value of the y -coordinate we had suppressed. Thus we can use our function rule to generate points $(3, 4 - (z')^2, z'i)_\varphi$ on the graph of φ by simply picking an arbitrary signed distance from α , say z' , and then compute $f(3 + z'i)$ to ultimately construct the points. All points produced in this manner are exactly the same points produced using the rigid motions described in Steps I - IV. Furthermore, the zeros of our function f , which happen to be the mathematical objects we have sought to visualize, can now be constructed as geometric objects using either of two methods. We may construct the zeros of our function f by evaluating $f(3 \pm 2i)$ and then constructing the points $(3, 0, \pm 2i)$ in (x, y, zi) -space. Instead, we may apply the rigid motions of Steps I - IV to the xy -plane graph of our Type RC quadratic function f and determine the points in (x, y, zi) -space where the graph of φ intersects the plane $y = 0$.

Note that if we evaluate our Type RC function f using arbitrary complex number inputs of the form $x + zi$, where $x, z \in \mathbb{R}$ and $x \neq 3$, we will not obtain points associated with the curve φ . The following calculation illustrates why this is so:

$$f(x + zi) = (x + zi - 3)^2 + 4 = [(x - 3)^2 + 4 - z^2] + 2z(x - 3)i \tag{3.3}$$

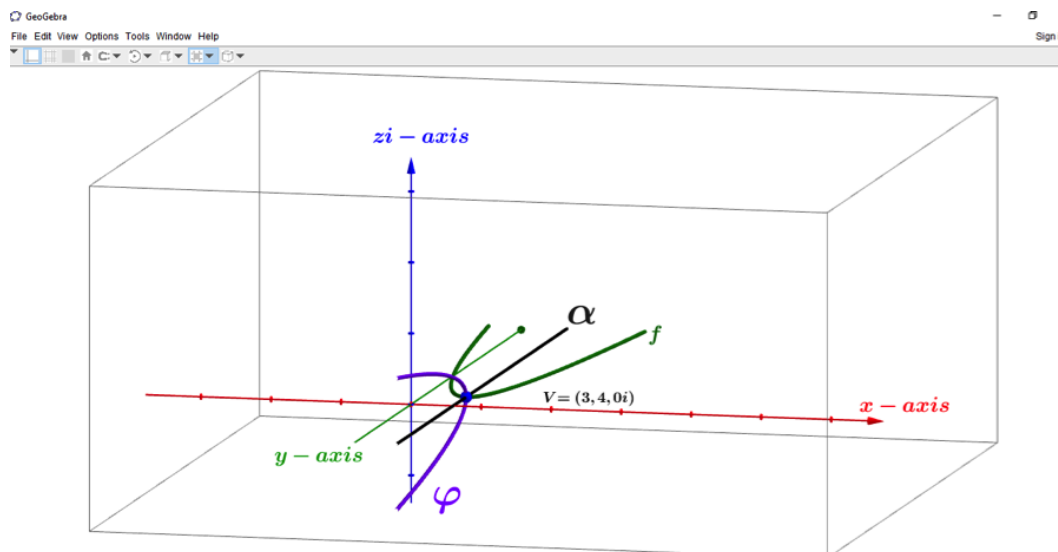


Figure 3.9: An image of the (x, y, zi) -space two-parabola system associated with $f(s) = (s-3)^2 + 4$.

Since $x \neq 3$, if $z \neq 0$, then $2z(x-3)i \neq 0$, hence $f(x+zi)$ is a pure complex number². Yet all of the y -coordinates of points lying on φ are real-valued and take the form $4 - z^2$. Thus complex numbers of the form $3 + zi$ are the only complex number inputs that are associated with points on the graph of φ , as seen above by the mapping shown in (3.1).

A formal description about using the function rule for f to produce the parabolic curve φ

We now summarize in a formal yet concise manner how the function rule of our Type RC quadratic function f can be used to construct the parabolic curve denoted φ :

- Write the function rule for our Type RC quadratic function as $f(s) = (s-3)^2 + 4$, allowing s to take on all real number values as well as complex number values of the form $3 + zi$, where $z \in \mathbb{R}$.
- When s is a real number x , the output $f(x)$ is also a real number. These values of $f(x)$ produce the expected graph of a parabola in the xy -plane. Furthermore, this graph is unchanged when viewed in (x, y, zi) -space, where points on the graph of f are written as $(x, f(x), 0i)$.
- When s is any complex number of the form $3 + zi$, the output $f(3 + zi) = 4 - z^2$ is also a real number. These particular values of s and $f(s)$ can be used to construct all points lying on the graph of the parabolic curve φ , which lies entirely in the complex plane $x = 3$. Furthermore, all points lying on φ in (x, y, zi) -space take the form $(3, 4 - z^2, zi)$.
- As a result, we may construct points in (x, y, zi) -space that correspond to the zeros of f . These are the points $(3, f(3 \pm 2i), \pm 2i) = (3, 0, \pm 2i)$, which we now view as the geometric interpretation of the complex-valued zeros of the function f .

Thus we may combine the graphs of the xy -plane parabola f and the parabola φ to be one graph or a two-parabola system that is associated with a function rule of a Type RC quadratic function. Each individual parabola shares a common vertex and axis of symmetry. The two-parabola system for our function f is shown in Figure 3.9.

²Note that if $z = 0$, then $f(3 + zi) = f(3) = 4$, which corresponds to the vertex point $V = (3, 4, 0i)$.

The General Case for Type RC Quadratic Functions

Taking advantage of our hard work and computation above, addressing the general case is straightforward. We begin with a quadratic function $f(s) = a(s - h)^2 + k$, where $a, h, k \in \mathbb{R}$. If $ak > 0$, then f is a Type RC quadratic function and its zeros are the complex numbers $h \pm \sqrt{\frac{k}{a}}i$.

The rigid motions in Steps I – IV yield the following mapping of points in (x, y, zi) -space:

$$(h + z, k + az^2, 0i)_f \rightarrow (h + z, k - az^2, 0i)_{f_1} \rightarrow (h, k - az^2, zi)_\varphi \quad (3.4)$$

Taking $z = \pm\sqrt{\frac{k}{a}}$, the mapping that produces the complex-valued zeros of f is given as follows:

$$(h \pm \sqrt{\frac{k}{a}}, 2k, 0i)_f \rightarrow (h \pm \sqrt{\frac{k}{a}}, 0, 0i)_{f_1} \rightarrow (h, 0, \sqrt{\frac{k}{a}}i)_\varphi \quad (3.5)$$

Equivalently and alternatively, we can construct points on the graphs of f and φ by using the function rule and appropriate input values. If s is a real number $x = h + z$, where z is any real number and h is fixed, then $f(s) = f(h + z) = k + az^2$, and the points $(s, f(s), 0i) = (h + z, k + az^2, 0i)$ correspond to the usual xy -plane parabola.

On the other hand, if s is a complex number of the precise form $h + zi$, where h is the x -coordinate of the vertex and z is a real number, then $f(s) = f(h + zi) = k - az^2$. Hence $f(s) = f(h + zi)$ is real-valued for all such values of s and we write the points as $(h, f(h + zi), zi) = (h, k - az^2, zi)$, which correspond to points on the parabolic curve φ .

Closing comments

We would like to mention that some consideration was given to writing y as a function of x and z in an effort to approach the topic using the concept of function. Throughout this manuscript, the parabola φ was always referred to as a parabolic curve, not a function. Describing the two-parabola system of f and φ as a function is possible, and would first require us to construct what we define as the right-handed three-dimensional coordinate system (x, zi, y) -space. However, we ultimately decided not to go that route and perhaps address these ideas in future works.

As seen throughout this manuscript, GeoGebra is a great tool for illustrating the concepts under discussion. Much of the mathematical content in this article was both inspired by and deduced as a result of the authors analyzing images created by GeoGebra. We hope to inspire other practitioners of mathematics to explore mathematics using GeoGebra, and we encourage mathematics students to do the same.

There does not seem to be a lot of discussion in math-circles regarding the topic under discussion. We speculate this may be a consequence of the manner in which quadratic functions are presented at the secondary school level. Specifically, secondary school mathematics generally does not require students to evaluate Type RC quadratic functions using complex number inputs. Despite any unfamiliarity the reader may have with the content contained in this manuscript, we hope that the mathematics was presented in a clear and thorough manner, perhaps satisfying the curiosity of some mathematics instructors and students.

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