USING THE TAIL OF A SEQUENCE TO EXPLORE ITS LIMIT

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Abstract: Graphing sequences is a common approach to explore limits both conceptually and computationally. In the traditional approach, the first terms of the sequence are the primary focus, however, we present a GeoGebra applet that facilitates the determination and understanding of limits by illustrating an inverted perspective that emphasizes the tail of the sequence instead. Additionally, two robust and valuable GeoGebra topics/commands will be highlighted in the applet, in particular *Lists* and *Sequences*. Finally, we conclude with a discussion of several issues surrounding the limits and GeoGebra.

Keywords: limits, GeoGebra: `List, Sequence`

Introduction

The *limit* is a key concept in mathematics in general, and one of three fundamental ideas of calculus [5], the other two being the derivative and the integral, both of which are defined in terms of limits. As such, students are expected to know how to determine a limit, develop an intuitive understanding of the mathematical limit, estimate limits from graphs or tables, and find limits at or involving infinity [1]. So, it stands to reason that much instructional time is devoted to developing this concept. A recent NSF-funded study [4] explored the relative amount of instructional time spent teaching and learning various topics in first-semester calculus and found that, on average, 15% of instructional time involves limits. Although typically in first-semester calculus, only limits of functions are considered, perhaps working with limits of sequences is more natural and easier for students [5].

In this article, we present a GeoGebra [3] applet that utilizes the tail of a sequence instead of the traditional use of the first "few" terms. Prior to this development, we include prerequisite mathematical definitions and GeoGebra syntax and commands. We provide two examples that illustrate convergence and divergence of sequences. As a bonus, the method we use will provide visual insight into a related advanced topic. Lastly, we discuss issues discovered during the preparation and writing of this manuscript.

Mathematical Sequence and Limits

We start with the definition of a real sequence as an ordered list of numbers. A **sequence** of real numbers is a function $f : \mathbb{N} \to \mathbb{R}$ defined by $f_n = f(n)$ mapping the natural numbers to the real numbers. The subscript is referred to as the **index**. According to this definition, because f is a mathematical function, it is assumed that all domain values are used, therefore, all sequences are infinite.

Formally, the limit of a sequence f_n is the number L to which the sequence converges if for every real number ϵ, there exists a natural number N such that for every natural number $n > N$, the terms f_n satisfy $|f_n - L| < \epsilon$. Literature abounds documenting the difficulty in developing a firm understanding of this definition (e.g., [2], [6], [8], [9]).

GeoGebra Lists and the `Element` and `Sequence` Commands

Lists are important objects and are used in many contexts in GeoGebra. *Lists* can contain numbers, points, polygons, circles, and even other lists. For example, a *matrix* is defined in GeoGebra as a list of lists. *Lists* are created using bracket symbols, `{` and `}`. For example, a list, `L`, of even positive integers less than or equal to 20 can be created using the syntax at the input bar as follows:

```
L={2, 4, 6, 8, 10, 12, 14, 16, 18, 20}
```

Elements of a list can be obtained using the `Element` command. In particular, the nth element of a list `L` is obtained with the command: `Element[L,n]`. For example, the number 6 in the above example is obtained using `Element[L,3]`.

A *sequence* in GeoGebra is a (special) list and can be created using the `Sequence` command. The syntax to create a list using the command line is:

```
Sequence[<expression>, <variable>, <start>, <end>]
```

For example, the sequence of even numbers from 2 to 20 (`L` above) can be generated by

```
L=Sequence[2*k, k, 1, 10]
```

The `Element` and `Sequence` commands are essential when working with *lists* in GeoGebra.

The Tail Emphasis

When considering the limit of a sequence, the "**tail**" provides all the necessary information about the convergence or divergence of the sequence as well as other details of interest. The tail is considered to be the terms of the sequence greater than or equal a certain index value, i.e., f_n for all $n \geq m$ for some m. This is sometimes called an m**-tail**. Therefore, instead of graphing the sequence $f_1, f_2, f_3, \ldots$ using the points $(1, f_1), (2, f_2), \ldots$ and attempting to view this graph "at infinity", an alternative method presented by Olson [7] suggests plotting the points

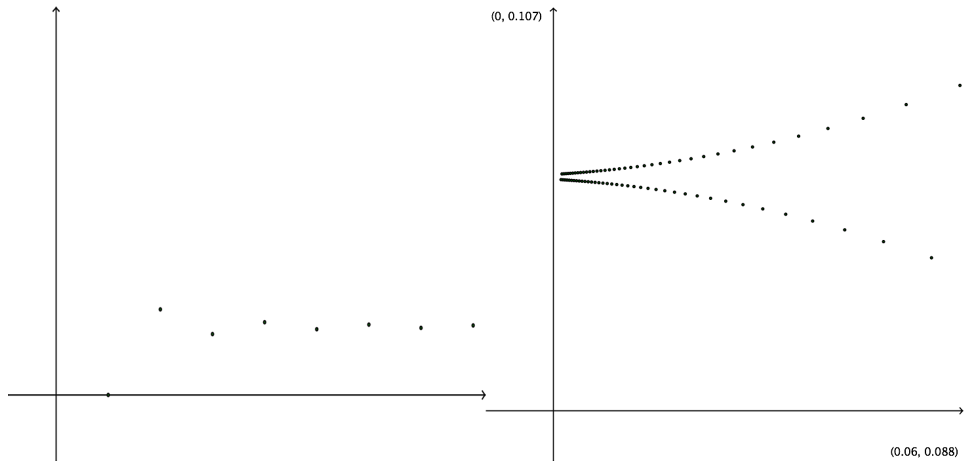
$$\left(\frac{1}{m}, f_m\right), \left(\frac{1}{m+1}, f_{m+1}\right), \left(\frac{1}{m+2}, f_{m+2}\right), \ldots$$

which represents the tail of the sequence. The successive terms are plotted from right to left approaching the $y-$axis and represents the "tail". In effect, the graph displays the ultimate behavior of the sequence. If the sequence converges, it will be visually obvious that the terms are approaching a particular value on the $y-$axis. Additionally, it is easy to add the ϵ tolerance to the limit value. Divergence is equally obvious in that there will not be one value toward which the terms are converging.

Modifications in the GeoGebra applet to increase n or produce (graphically) terms beyond a specified index are also easy to obtain. The general construction protocols for this method are provided below followed by two examples.

General Construction Protocols

- Define a value for `N` $\in \mathbb{N}$.
- Define a value for `m` $\in \mathbb{N}$, for the m tail.



(a) Traditional Approach (b) Alternative Approach

Figure 4.1: Plots of the sequence $f(n) = 1 + (-1)^n/n^2$

- Define a function in `n` to use in the sequence command: `f(n)` (only need to edit this function to produce other sequences)
- Define a sequence `f\_n = Sequence[f(k), k, m, N]`
- Define the tail sequence (of points) using the `Sequence` and `Element` commands: `Sequence[(1/k, Element[f_n,k]),k, m, N]`

Example 4.1 We first illustrate the tail method using the sequence defined by $f(n) = 1 + (-1)^n/n^2$ and produce the points $(1/1, f_1), (1/2, f_2), (1/3, f_3), \ldots$ and we show a side-by-side comparison with the traditional "early terms" depiction $(1, f_1), (2, f_2), (3, f_3), \ldots$. We enter at the command line the following:

```
N=50

m=1

f(n)=1+(-1)^n/n^2

f_n = Sequence[f(k), k, m, N]

Tail = Sequence[(1/k, Element[f_n,k]),k, m, N]
```

The result is shown in Figure 4.1.

Example 4.2 We illustrate divergence using the tail end of the sequence $f(n) = 2/5+\sin(3n)(n+10)/n$, redefining only $f(n)$ in Example 4.1, `f(n) = 2/5+sin(3n)(n+10)/n`. The results is shown in Figure 4.2. Observe that in this case, the tail makes it possible to view the $\limsup$ and $\liminf$ of the sequence, two important concepts in mathematical analysis.

Discussion

Interactive dynamic software is one way to foster understanding of mathematical limits. In fact, there already exist hundreds of deftly designed GeoGebra applets available on GeoGebra Tube (tube.geogebra.org) dedicated to developing limits, including one-sided limits, limits at infinity, the ϵ-δ definition, the squeeze theorem, limit approximations and calculations, etc. This immediately begs the question concerning the effectiveness of the current applets used to develop

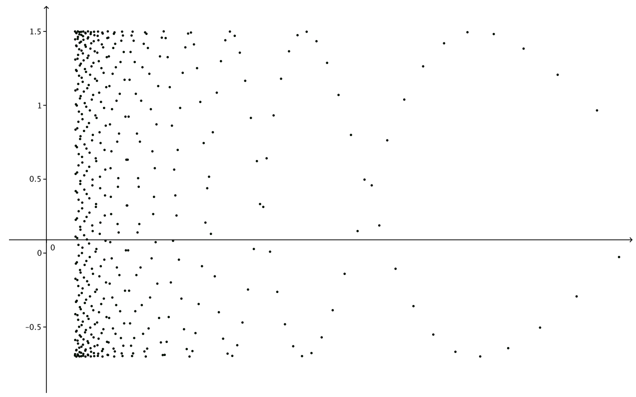


Figure 4.2: Tail of the divergent sequence $f_n = 2/5 + \sin(3n)(n+10)/n$. In this case, however, the tail makes it possible to view the $\limsup$ and $\liminf$, two important concepts in mathematical analysis.

the limit concept. With all of these applets, are students still having difficulty with understanding limits? Furthermore, there seems to be a lack of literature reporting the use and effectiveness of these apps.

In preparation for this article, the task of wading through and exploring hundreds of available applets required a considerable amount of time and was fairly overwhelming. Clearly, this would be a daunting task for any teacher searching through materials while preparing a lesson on this topic. One suggestion in this regard would be to implement some sort of user rating to aid in identifying highly effective (highly used) applets. Additionally, more detailed descriptions and better/longer list of keywords could be helpful during the selection process.

Further, there needs to be more material devoted to truly creating an intuitive understanding of the limit, not just dynamically presenting examples found in standard textbooks. One suggestion is an applet that illustrate the limit as a process of successive approximations for which materials are scarce. Examples include: approximating geometrical figures (e.g., circle) with an increasing number of sides of a polygon, approximating arc length of a curve, approximating roots of an equation, the area under a curve, or the slope of a tangent line. Although several applets exist that demonstrate the definition of derivative as the limiting process of slopes of secant lines or the integral as the limit of Riemann sums as the number of rectangles increases, these applets are not tagged with “limit" or “concept of limit" as keyword(s).

Conclusion

We developed a GeoGebra applet based on [7] for presenting the concept of the limit. Although there are many GeoGebra materials covering limits, currently none exist that emphasize the tail of a sequence. This method is robust and can also provide an introduction into the related (but more advanced) notions of limit superior and limit inferior.

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